ESERCIZI GEOMETRIA

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$$A=\begin{pmatrix} k-1 & 2k & k \\ 0 & k-1 & 0 \\ 2 & k+2 & 1 \end{pmatrix}$$

- Pol. car. $\det(A-\lambda I_3)$

3° grado in λ $\to$ 3 soluz in $\mathbb{C}$

in $\mathbb{R}$ ✓

- Autov. distinti o no?

$$\det\begin{pmatrix} k-1-\lambda & 2k & k \\ 0 & k-1-\lambda & 0 \\ 2 & k+2 & 1-\lambda \end{pmatrix} = (k-1-\lambda)\det\begin{pmatrix} k-1-\lambda & k \\ 2 & 1-\lambda \end{pmatrix}$$

$\to \lambda_1 = k-1 \in \mathbb{R}$

$$=(k-1-\lambda)\cdot[(k-1-\lambda)(1-\lambda)-2k] = (k-1-\lambda)[\lambda^2-\lambda k+\cancel{\lambda}-\cancel{\lambda}+k-1-2k]=$$

$$=(k-1-\lambda)(\lambda^2-\lambda k-k-1)=0 \qquad \lambda_{2,3}=\frac{k\pm\sqrt{k^2+4k+4}}{2}$$

$k+1 \in \mathbb{R}$

$-1 \in \mathbb{R}$

$k+1 \neq k-1 \quad \forall k \in \mathbb{R}$

$k-1 \neq -1$ se $k \neq 0$

$k+1 \neq -1$ se $k \neq -2$

Se $k\neq 0 \wedge k \neq -2$, allora A è diag.

se $k=0$ $\quad \lambda_1=\lambda_3=-1$ $\quad$ m.a.$(-1)=\boxed{2}$

$$m.g.(-1) = 3-rg\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix} = 3-1=\boxed{2}$$

A è diag

Se $k=-2$ $\quad \lambda_2=\lambda_3=-1$ $\quad$ m.a.$(-1)=\boxed{2}$

$$m.g.(-1) = 3-rg\begin{pmatrix} -2 & -4 & -2 \\ 0 & -2 & 0 \\ 2 & 0 & 2 \end{pmatrix} = 3-2=\boxed{1}$$

A non è diag.

1.6 $\quad A=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad T: M(2)\to M(2)$ $\quad T(X)=A\cdot X$

prodotto

$$\mathcal{B}=\left\{\underset{e_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{e_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}\right\}$$

$$\mathcal{B}'=\left\{\underset{e_1'}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{e_2'}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{e_3'}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{e_4'}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}\right\}$$

Usiamo $\mathcal{B}$ $\quad T(e_1)=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = 2e_1+e_3$

$T(e_2)=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}=2e_2+e_4$ $\quad T(e_3)=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}=e_1+e_3$

$T(e_4)=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}=e_2+e_4$

$$T_{\mathcal{B}\mathcal{B}}=\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$T(e_1')=2e_1'+e_2'$ $\quad T(e_2')=e_1'+e_2'$

$T(e_3')=2e_3'+e_4'$ $\quad T(e_4')=e_3'+e_4'$

$$T_{\mathcal{B}'\mathcal{B}'}=\left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\det(T_{\mathcal{B}'\mathcal{B}'}-\lambda I_4) = [\det(A-\lambda I_2)]^2$$

se $\bar\lambda$ visto come autov. di A ha m.a. m, lo stesso $\bar\lambda$ visto come autov di T ha m.a. 2m

$$\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 1 = 0 \qquad \lambda_1 = \frac{3-\sqrt{5}}{2} \quad \lambda_2 = \frac{3+\sqrt{5}}{2} \quad \in \mathbb{R}$$

distinti

$f : V \to V \qquad \mathcal{B}$ base di $V \qquad \{v_1 \dots v_n\}$

$[f]_{\mathcal{B}}^{\mathcal{B}}$ è diag $\iff$ $\mathcal{B}$ è formata da autovettori di f

$\Leftarrow \quad f(v_1) = \lambda_1 v_1 \quad \dots\dots\dots \quad f(v_n) = \lambda_n v_n$

$$[f]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & 0 & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$, cioè $[f]$ è diag.

$$\Rightarrow \begin{bmatrix} f_{11} & 0 & \dots & 0 \\ 0 & f_{22} & & \vdots \\ 0 & \vdots & & \vdots \\ 0 & & & 0 \\ 0 & 0 & \dots & f_{nn} \end{bmatrix} \qquad f(v_1) = \begin{pmatrix} f_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = f_{11} v_1 \quad v_1 \text{ è autovett. di } f$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$f(v_n) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_{nn} \end{pmatrix} = f_{nn} v_n \qquad v_n \text{ è autovett di } f$$

1.13 Se $T : V \to V$ è diagonalizzabile, allora

$$V = \ker(T) \oplus \operatorname{Im}(T)$$

$\exists\ \mathcal{B} = \{v_1 \dots v_n\}$ base di V formata da autovettori di T

$\{\lambda_1 \dots\dots \lambda_n\} = \{\underbrace{0 \dots 0}_{m}, \underbrace{\lambda_{m+1} \dots \lambda_n}_{n-m}\} \qquad 0 \le m \le n$

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & \dots & 0 & 0 & & 0 \\ \vdots & & \vdots & \vdots & & \\ \vdots & & \vdots & \lambda_{m+1} & & \\ \vdots & & \vdots & & \ddots & \\ 0 & \dots & 0 & 0 & & \lambda_n \end{pmatrix}$$

$\underbrace{\qquad}_{m \text{ colonne}}$

$$rg\left([T]_{\mathcal{B}}^{\mathcal{B}}\right) = n - m$$
$$= \dim(\operatorname{Im}(T))$$
$$\dim(\ker(T)) = m$$

$$\operatorname{Im}(T) \cap \ker(T) = \{0_V\}$$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$v = \alpha_1 e_1 + \alpha_2 e_2$

$v = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_e$

$f_1 = e_1 + e_2 \qquad f_2 = e_1 - 2e_2$

$v = \beta_1 f_1 + \beta_2 f_2 = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}_f$

$= \beta_1(e_1 + e_2) + \beta_2(e_1 - 2e_2) =$

$(\beta_1 + \beta_2) e_1 + (\beta_1 - 2\beta_2) e_2$

$$\alpha_1 = \beta_1 + \beta_2$$
$$\alpha_2 = \beta_1 - 2\beta_2$$

$$\boxed{\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}} \leftarrow \text{DA f a e}$$

$$\alpha_1 - \alpha_2 = 3\beta_2$$
$$2\alpha_1 + \alpha_2 = 3\beta_1$$

$$\beta_2 = \frac{\alpha_1 - \alpha_2}{3}$$
$$\beta_1 = \frac{2\alpha_1 + \alpha_2}{3}$$

$$\boxed{\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}}$$

$\uparrow$ DAe a f