$\boxed{A x = b}$ con A invertibile

A matrice $n \times n$ rango $(A) = n$.

$A = \left(A_{(1)} \dots A_{(n)} \right)$ $b \in K^n$

$x = \begin{pmatrix} x_1 \\ \\ x_n \end{pmatrix}$ /

$$\mathrm{Det} \left(A_{(1)} \dots \underbrace{b}_{\text{j-esima colonna}} \; A_{(n)} \right) =$$

$Ax = b$ $Ax = \underline{x_1 A_{(1)}} + x_2 A_{(2)} + \dots + x_n A_n$

Possiamo sdrazzare alla COLONNA j-ESIMA b x_1 volte $A_{(1)}$, x_2 volte $A_{(2)}$

$$= \mathrm{Det} \left(A_{(1)} \; A_{(2)} \dots x_j A_{(j)} \dots A_{(n)} \right) =$$

$$= \boxed{= x_j \ \mathrm{Det}(A).}$$

TEOREMA Se A è $n \times n$ e $\det A \neq 0$

$Ax = b$ con

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ALLORA

$$x_j = \frac{\mathrm{Det}(A_{(1)} \ldots \overset{\text{colonna } j}{\overset{\downarrow}{b}} \ldots A_{(n)})}{\mathrm{Det}(A)}.$$

ESEMPIO

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Det}\ A = 1 \cdot \mathrm{Det}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} - 0 \cdot \mathrm{Det}\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$+ 1 \ \mathrm{Det}\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} =$$

$$= 1 \cdot 1 + 1 \cdot 3 = 4.$$

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{matrix}$$

$$x_1 = \frac{\operatorname{Det}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}}{4} \longleftarrow \operatorname{Det} A$$

$$= \frac{-1}{4}$$

$$x_2 = \frac{\operatorname{Det}\left(A_{(1)} \ \ b \ \ A_{(3)}\right)}{\operatorname{Det} A} =$$

$$= \frac{\operatorname{Det}\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}}{4} \qquad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \frac{\operatorname{Det}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}{4}$$

$$= \frac{1 \cdot \operatorname{Det}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}{4} = \frac{1}{4}.$$

CALCOLO DELLA MATRICE INVERSA

$A \quad n \times n \qquad \text{Det } A \neq 0$

$$\underline{A \cdot B = I_n} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \leftarrow \\ 0 \\ \vdots \\ \cdot \end{matrix}$$

$$B = \left(B_{(1)} \; \dots \; B_{(n)} \right)$$

$$\boxed{\underline{A \cdot B_{(j)} = e_j}}$$

È LA COLONNA j-ESIMA DI $A \cdot B$.

$$\text{Sia} \quad B = (b_{ij}) \qquad B_{(j)} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} \leftarrow i \text{ ESIMA}$$

Se APPLICO LA FORMULA PRECEDENTE CON $x = B_{(j)}$ e $b = e_j$

colonna i-esima

$$b_{ij} = \frac{\boxed{\text{Det}\left(A_{(1)} \quad e_j \quad A_{(n)}\right)}}{\text{Det}(A)}$$

$$\begin{array}{cccc} & & 0 & \\ A_{(1)} & A_{(2)} & 0 & A_{(n)} \\ & & 1 & \\ & & 0 & \\ & & \vdots & \\ & & 0 & \end{array}$$

RIGA j-ESIMA.

COLONNA i-ESIMA

QUINDI

$$b_{ij} = (-1)^{i+j} \frac{\mathrm{Det}(X)}{\mathrm{Det}\, A}$$

X la matrice ottenuta da A togliendo la j-RIGA E LA i-ESIMA COLONNA.

OVVERO

$$b_{ij} = (-1)^{i+j} \frac{\mathrm{Det}(A_{ji})}{\mathrm{Det}\, A}$$

<u>ESEMPIO</u>

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc \neq 0$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$b_{11} = (-1)^{1+1} \frac{d}{\text{Det } A} = \frac{d}{ad-bc}$$

$$b_{12} = (-1)^{1+2} \frac{b}{\text{Det } A.} = \frac{-b}{ad-bc}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$b_{21} = (-1)^{2+1} \frac{c}{\text{Det } A} = \frac{-c}{ad-bc}$$

$$b_{22} = (-1)^{2+2} \frac{a}{\text{Det}(A)} = \frac{a}{ad-bc}$$

$$1 \qquad (\; d \qquad -b \;)$$

$$B = \frac{\qquad}{ad-bc}\begin{pmatrix} & \\ -c & a \end{pmatrix}$$

ESERCIZIO 9.6

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \quad x+y+z = 0 \right\}$$

$$V = \left\{ p(t) \in \mathbb{R}[t]_{\leq 3} \quad : \quad \underline{\underline{p(1) = 0}} \right\}$$

$$F : \underline{\underline{W}} \longrightarrow V$$

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{x + 2yt + zt^2 + (x+z)t^3}_{= f(t)}$$

$$F\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = x + 4t - 3t^2 - 2t^3$$

$$f(1) = x + 2y + z + x + z = 2\underline{(x+y+z)} = 0$$

base di W $\qquad x + y + z = 0 . \qquad x = -y - z$, $\quad y = 1, \ z = 0$

$$w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

base di V $\qquad p(1)$

$$\underbrace{f_1 = t - 1 \qquad f_2 = t^2 - 1 \qquad f_3 = t^3 - 1}$$

$$f(t) = a + bt + ct^2 + dt^3$$

$$f(1) = a + b + c + d = 0 \qquad a = -\underline{b} - \underline{c} - \underline{d}$$

$$[F]^{w_1\ w_2}_{f_1\ f_2\ f_3}$$

$$F(w_1) = F\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -1 + 2t - t^3 = \underline{a} f_1 + \underline{b} f_2 + \underline{c} f_3$$

$$= \underline{2} \cdot f_1 + \underline{0} \cdot f_2 \underline{-} f_3$$

$$F(w_2) = F\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1 + t^2 = f_2$$

$$= \underline{0} \cdot f_1 + \underline{1} \cdot f_2 + \underline{0} \cdot f_3$$

$$[F]^{w_1\ w_2}_{f_1\ f_2\ f_3} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$G : \mathbb{R}^3 \longrightarrow \mathbb{R}[t]_{\leq 3} \quad \text{LINEARE}$$

$$G\begin{pmatrix}x\\y\\z\end{pmatrix} = F\begin{pmatrix}x\\y\\z\end{pmatrix} \quad \text{SE} \quad \begin{pmatrix}x\\y\\z\end{pmatrix} \in W$$

$$\longrightarrow G\begin{pmatrix}1\\0\\0\end{pmatrix} = 1 + t \qquad G\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$G\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \boxed{G\begin{pmatrix}0\\1\\0\end{pmatrix}} \qquad G\begin{pmatrix}0\\0\\1\end{pmatrix}$$

- $[G]^{\underline{w_1}, \underline{w_2}, \underline{e_1}}_{1, t, t^2, t^3}$

$$G(w_1) = F(w_1) = -1 + 2t - t^3$$

$$G(w_2) = F(w_2) = t^2 - 1 = \underline{\underline{-1}} \cdot 1 + \underline{\underline{0}} \cdot t + \underline{\underline{1}} \cdot t^2 + \underline{\underline{0}} \cdot t^3$$

$$G(e_1) = 1 + t$$

$$[G]^{w_1\ w_2\ e_1}_{1, t, t^2, t^3} = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$[G]^{e_1\ e_2\ e_3}_{1,\ t,\ t^2,\ t^3} = [G]^{w_1\ w_2\ e_1}_{1,\ t,\ t^2,\ t^3}\ [Id]^{e_1\ e_2\ e_3}_{w_1\ w_2\ e_1}$$

$$[Id]^{w_1\ w_2\ e_1}_{e_1\ e_2\ e_3} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[Id]^{e_1\ e_2\ e_3}_{w_1\ w_2\ e_1} = \left([Id]^{w_1\ w_2\ e_1}_{e_1\ e_2\ e_3}\right)^{-1}$$

$$= \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$[G]^{e_1\ e_2\ e_3}_{1,\ t,\ t^2,\ t^3} = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1+e_1 & 1 & 1 & 1 & \\ 1 & 1+e_2 & 1 & 1 & \\ 1 & 1 & 1+e_3 & 1 & \\ 1 & 1 & 1 & 1+e_4 & \\ & \cdot & & & \ddots \end{pmatrix}$$

$e_1, \dots, e_n \neq 0.$

$$\begin{array}{cccc} 1+e_1 & 1-1-e_1 & 1-1-e_1 & \\ 1 & 1+e_2-1 & 0 & \\ 1 & 0 & 1+e_3-1 & \\ 1 & 0 & 0 & 1+e_4-1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{cccc} 1+e_1 & -e_1 & -e_1 & \end{array}$$

$\frac{1}{e_2} \quad 0 \quad \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \qquad \begin{array}{c} e_2 \\ 0 \\ 0 \\ 0 \end{array} \qquad \begin{array}{c} 0 \\ e_3 \\ 0 \\ 0 \end{array} \qquad \begin{array}{c} 0 \\ 0 \\ e_4 \end{array}$

$$\begin{array}{c} 1 + e_1 + \frac{e_1}{e_2} \\ 0 \\ 1 \\ 1 \\ 1 \end{array} \qquad \begin{array}{c} -e_1 \\ e_2 \\ 0 \\ \vdots \end{array}$$