TEOREMA

$M = (m_{ij})$

Sia M una matrice $n \times n$ e fissiamo una riga la i-esima riga.

Sia M_{hk} la matrice $(n-1)\times(n-1)$ OTTENUTA DA M CANCELLANDO L'h-ESIMA RIGA E LA k-ESIMA COLONNA.

Allora

$$\mathrm{Det}_n\, M = (-1)^{i+1} m_{i1}\, \mathrm{Det}_{n-1}(M_{i1}) + (-1)^{i+2} m_{i2}\, \mathrm{Det}(M_{i2})$$

$$\cdots\cdots \quad + (-1)^{i+n} m_{in}\, \mathrm{Det}(M_{in})$$

UNA FORMULA ANALOGA VALE SE SCELGO UNA COLONNA.

ESEMPIO

$$\mathrm{Det}\begin{pmatrix} 1 & 2 & 7 & 3 & 1 \\ 6 & 3 & 4 & 4 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 4 & 5 & 1 \\ 3 & 0 & 2 & 6 & 6 \end{pmatrix} = (-1)^{3+1} \cdot 0 \cdot \mathrm{Det}(M_{31}) +$$

$$+ (-1)^{3+2} \cdot 0 \cdot \mathrm{Det}(M_{32}) +$$

$$+ (-1)^{3+3} \cdot 0 \cdot \mathrm{Det}(M_{33}) +$$

$$+ (-1)^{3+4} \cdot 2 \cdot \operatorname{Det}\left(M_{34}\right) + \cancel{(-1)^{3+5} \cdot 0} \cdot \operatorname{Det}\left(M_{35}\right)$$

$$= -2 \operatorname{Det}\begin{pmatrix} 1 & 2 & 7 & 1 \\ 6 & 3 & 4 & 2 \\ 1 & 0 & 4 & 1 \\ 3 & 0 & 2 & 6 \end{pmatrix}$$

$$= -2 \left((-1)^{1+2} \cdot 2 \cdot \operatorname{Det}\left(M_{12}\right) + (-1)^{2+2} \cdot 3 \cdot \operatorname{Det}\left(M_{22}\right) + 0 + 0 \right)$$

$$= -2 \left(-2 \operatorname{Det}\begin{pmatrix} 6 & 4 & 2 \\ 1 & 4 & 1 \\ 3 & 2 & 6 \end{pmatrix} + 3 \operatorname{Det}\begin{pmatrix} 1 & 7 & 1 \\ 1 & 4 & 1 \\ 3 & 2 & 6 \end{pmatrix} \right)$$

$$= 4 \operatorname{Det}\begin{pmatrix} 6 & 4 & 2 \\ 1 & 4 & 1 \\ 3 & 2 & 6 \end{pmatrix} - 6 \operatorname{Det}\begin{pmatrix} 1 & 7 & 1 \\ 1 & 4 & 1 \\ 3 & 2 & 6 \end{pmatrix}$$

...

<u>dim</u>

$$\begin{pmatrix} M_1 \end{pmatrix}$$

$$M = \begin{pmatrix} M_2 \\ M_i \\ M_n \end{pmatrix}$$

$$M_i = \begin{pmatrix} m_{i1} & m_{i2} & & m_{in} \end{pmatrix} =$$

$$= \begin{pmatrix} m_{i1} & 0 & & 0 \end{pmatrix} +$$
$$+ \begin{pmatrix} 0 & m_{i2} & 0 & 0 \end{pmatrix} +$$
$$\vdots$$
$$+ \begin{pmatrix} 0 & & 0 & m_{in} \end{pmatrix}$$

$$\operatorname{Det}_n M = \underbrace{\operatorname{Det}_n \begin{pmatrix} M_1 \\ m_{i1} \; 0 \; \dots \; 0 \\ M_n \end{pmatrix}} + \operatorname{Det}_n \begin{pmatrix} M_1 \\ 0 \; m_{i2} \; 0 \; \dots \; 0 \\ M_n \end{pmatrix} +$$

$$\dots + \operatorname{Det}_n \begin{pmatrix} M_1 \\ 0 \; 0 \; \; 0 \; m_{in} \\ M_n \end{pmatrix}$$

LEMMA

$$\downarrow$$

$$= (-1)^{i+1} \, m_{i1} \operatorname{Det}_{n-1} \begin{pmatrix} M_{i1} \end{pmatrix} +$$

$$+ (-1)^{i+2} \, m_{i2} \, \mathrm{Det}_{n-1} \left(M_{i2} \right) +$$

$$\dots$$

$$+ (-1)^{i+n} \, m_{in} \, \mathrm{Det}_{n-1} \left(M_{in} \right)$$

#

TEOREMA DI BINET

A, B matrici $n \times n$.

$$\mathrm{Det}(B \cdot A) = \underline{\mathrm{Det}(A) \cdot \mathrm{Det}(B)}$$

dim $$= \mathrm{Det}(B) \cdot \mathrm{Det}(A)$$

· Se $\mathrm{Det}\, A = 0$. VOGLIAMO DIMOSTRARE CHE

$$\mathrm{Det}(B \cdot A) = 0.$$

$\mathrm{Det}\, A = 0$ vuol dire A NON È INVERTIBILE. DA QUESTO SEGUE CHE $B \cdot A$ NON È INVERTIBILE. SE LO FOSSE

$$C \cdot (B \cdot A) = I.$$
$$(C \cdot B) \cdot A = I$$

CONTRO A NON INVERTIBILE.

Se $\operatorname{Det} A \neq 0$

$$F(B) = \operatorname{Det} B = \frac{\operatorname{Det}(B \cdot A)}{\operatorname{Det}(A)} = G(B)$$

Dimostro che G HA LE PROPRIETÀ CHE CARATTERIZZANO F.

1) $G(I) = 1$

$$G(I) = \frac{\operatorname{Det}(I \cdot A)}{\operatorname{Det}(A)} = 1$$

2) Se B HA DUE RIGHE UGUALI ALLORA $G(B) = 0$.

$$B = \begin{pmatrix} B_1 \\ B_i \\ \vdots \end{pmatrix} \qquad (B \cdot A) =$$

$$\begin{pmatrix} B_j = B_i \\ \\ B_n \end{pmatrix}$$

$$= \begin{pmatrix} B_1 \cdot A \\ B_2 \cdot A \\ \boxed{B_i \cdot A} \\ \boxed{B_j \cdot A} \\ \\ B_n \, A \end{pmatrix} \qquad \begin{array}{c} B_i \\ \parallel \\ B_j \end{array}$$

QUINDI $\operatorname{Det}(B \cdot A) = 0$.

3) G È LINEARE SULLE RIGHE

$$B = \begin{pmatrix} B_1 \\ \alpha B_i \\ B_n \end{pmatrix}$$

$$G(B) = \operatorname{Det}\left(B \cdot A\right) / \operatorname{Det}(A)$$

$$= \operatorname{Det}\begin{pmatrix} B_1 \cdot A \\ \underset{=}{\alpha} B_i \cdot A \\ B_n \cdot A \end{pmatrix} / \operatorname{Det} A$$

$$= \alpha \quad \mathrm{Det} \begin{pmatrix} B_1 \cdot A \\ B_i \cdot A \\ \boxed{B_n \cdot A} \end{pmatrix} / \mathrm{Det}\, A$$

$$= \alpha \quad \mathrm{Det}(B \cdot A) / \mathrm{Det}\, A = \alpha\, G(B)$$

SIMILMENTE

$$G \begin{pmatrix} B_1 \\ B_i + B_i' \\ B_n \end{pmatrix} = G \begin{pmatrix} B_1 \\ B_i \\ B_n \end{pmatrix} + G \begin{pmatrix} B_1 \\ B_i' \\ B_n \end{pmatrix} \#$$

<u>DEFINIZIONE</u> $\dim V < +\infty$.

$$F : \underline{\underline{V}} \longrightarrow \underline{\underline{V}} \quad \text{LINEARE}$$

Sia $\underline{v}$: $v_1 \dots v_n$ una base di V

$$\underline{v} = v_1, v_2, \dots, v_n$$

$\underline{A} = [F]_{\underline{v}}^{\underline{v}}$ È LA STESSA BASE IN PARTENZA E

IN ARRIVO.

E DEFINISCO $\det(F) = \underline{Det(A)}$

DEVO FAR VEDERE CHE QUESTA DEFINIZIONE DEL DETERMINANTE DI F NON DIPENDE DALLA BASE CHE HO SCELTO.

PROPOSIZIONE

SIA A UNA MATRICE $n \times n$ INVERTIBILE

$$Det(A^{-1}) = \frac{1}{Det(A)}$$

DIM

$$A \cdot A^{-1} = I.$$

$$Det(A \cdot A^{-1}) = Det(I) = 1.$$

$$\|$$

$$Det(A) \cdot Det(A^{-1}) = 1$$

$$1$$

DA CUI $\quad \mathrm{Det}\, A^{-1} = \dfrac{\quad}{\mathrm{Det}\, A}$ #

$F: V \longrightarrow V$

$\underline{v}$: $v_1 \dots v_n$ una base di V

$\underline{w}$: $w_1 \dots w_n$ una base di V

$$\underline{A = [F]_{\underline{v}}^{\underline{v}}} \qquad \underline{B = [F]_{\underline{w}}^{\underline{w}}}$$

e voglio far vedere che $\mathrm{Det}\, A = \mathrm{Det}\, B$.

$$[F]_{\underline{w}}^{\underline{w}} = \underset{\substack{\parallel \\ G^{-1}}}{[Id]_{\underline{w}}^{\underline{v}}} \; [F]_{\underline{v}}^{\underline{v}} \; \underset{\substack{\parallel \\ G}}{[Id]_{\underline{v}}^{\underline{w}}}$$

$$B = \underline{G^{-1}} \cdot \underline{A} \cdot \underline{G}$$

$$\mathrm{Det}\, B = \underline{\mathrm{Det}\, G^{-1}} \cdot \mathrm{Det}\, A \cdot \mathrm{Det}\, G$$

$$= \frac{1}{\cancel{\mathrm{Det}\, G}} \cdot \mathrm{Det}\, A \cdot \cancel{\mathrm{Det}\, G}$$

$$= \mathrm{Det}\, A$$ #

$$\mathrm{Det}\begin{pmatrix} a_1 & & & * \\ 0 & a_2 & & \\ 0 & & a_3 & \\ & 0 & & \ddots & \\ 0 & & & & a_n \end{pmatrix} = a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

$$\mathrm{Det}\left(\begin{array}{cc|ccc} 1 & 2 & & & \\ 3 & 4 & & * & \\ \hline 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 & 8 \\ 0 & 0 & 7 & 6 & 5 \end{array}\right) = \mathrm{Det}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \mathrm{Det}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 8 \\ 7 & 6 & 5 \end{pmatrix}$$

LEMMA

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \qquad \begin{array}{ll} A & n \times n \\ B & m \times m \\ & \\ M & (n+m) \times (n+m) \end{array}$$

$$\mathrm{Det}_{n+m} M = \mathrm{Det}_n A \cdot \mathrm{Det}_m B.$$

dim

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$$

$$\operatorname{Det} M = \underline{\operatorname{Det} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}} \cdot \operatorname{Det} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$$

$$\begin{array}{cc} \parallel & \parallel \\ \operatorname{Det} A & \operatorname{Det} B \end{array}$$

$$\operatorname{Det} \begin{pmatrix} A & 0 \\ \underline{0} & \underline{I_m} \end{pmatrix}$$

SVILUPPIAMO RISPETTO ALL'ULTIMA RIGA $(\underbrace{0 \quad \cdots \quad 0}_{n} \; \underbrace{0 \quad \cdots \quad 0}_{m-1} \; 1)$

RIMANE $\quad (-1)^{n+m+n+m} \cdot 1 \cdot \operatorname{Det} \begin{pmatrix} A & 0 \\ 0 & I_{m-1} \end{pmatrix}$

$$\begin{array}{cc} \parallel & \parallel \\ 1 & 1 \end{array}$$

$$\begin{pmatrix} e & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \end{pmatrix}$$

$$= \operatorname{Det}\begin{pmatrix} 0 & \underline{I}_{m-1} \end{pmatrix} = \quad = \operatorname{Det}(A)$$

#

$$\begin{array}{ccc} & n & m \\ n & \begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix} \\ m & & \end{array} \qquad A \quad n \times n$$

<u>LEMMA</u>

A $n \times n$ $\quad B$ $m \times m$

X $n \times m$

$$\operatorname{Det}_{n+m}\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = \operatorname{Det}_n(A) \cdot \operatorname{Det}_m(B)$$

<u>dim</u>

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = \begin{pmatrix} I_n & Y \\ 0 & I_m \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$$= \begin{pmatrix} I_n \cdot A + Y \cdot 0 & Y \cdot B \\ 0 & B \end{pmatrix}$$

$$\begin{pmatrix} \underline{A} & X \\ \underline{0} & \underline{B} \end{pmatrix} = \begin{pmatrix} \underline{A} & YB \\ \underline{0} & \underline{B} \end{pmatrix}$$

SE B È INVERTIBILE

POSSO SCEGLIERE $Y = X \cdot B^{-1}$

$$Y \cdot B = X$$

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = \begin{pmatrix} I_n & XB^{-1} \\ 0 & I_m \end{pmatrix} \underline{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}}$$

$$\mathrm{Det} \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = \underline{\mathrm{Det} \begin{pmatrix} I_n & XB^{-1} \\ 0 & I_m \end{pmatrix}} \, \mathrm{Det}\, A \cdot \mathrm{Det}\, B$$

$$\mathrm{Det} \begin{pmatrix} I_n & XB^{-1} \\ 0 & I_m \end{pmatrix} \underset{*}{=} \mathrm{Det} \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} = 1$$

$$\mathrm{Det} \begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 5 & 6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \underline{\underline{R_{25}(-6)}}$$

PERCHE POSSO ANNULLARE LE ENTRATE DEL BLOCCO IN ALTO A DESTRA CON OPERAZIONI DEL TIPO $R_{ij}(\alpha)$

CHE NON CAMBIANO IL DETERMINANTE.

Se B è invertibile abbiamo dimostrato

$$\operatorname{Det}\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = \operatorname{Det} A \cdot \operatorname{Det} B$$

SE B NON È INVERTIBILE ALLORA $\operatorname{Det} B = 0$. E VOGLIO DIMOSTRARE

$$\operatorname{Det}\begin{pmatrix} A & X \\ 0 & \boxed{B} \end{pmatrix} = 0$$

SE B NON È INVERTIBILE ALLORA RANGO $(B) < m$.

QUINDI LE RIGHE SONO LIN. DIP.

$$\begin{array}{|cc|} A & X \\ 0 & B \end{array}$$

ANCHE LE RIGHE DI $\begin{matrix} A & X \\ 0 & B \end{matrix}$ SONO LIN. DIP. E QUINDI

$\begin{pmatrix} A & X \end{pmatrix}$

$$\mathrm{Det}\begin{pmatrix} A & \\ 0 & B \end{pmatrix} = 0$$

#

$$\begin{aligned} a x + b y &= e \\ c x + d y &= f \end{aligned}$$

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\parallel \atop A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

SE A È INVERTIBILE

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} e \\ f \end{pmatrix}$$

$\mathrm{Det}\, A \neq 0$

$ad - bc \neq 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{de - bf}{ad-bc} \\[2ex] \dfrac{-ce + af}{ad-bc} \end{pmatrix}$$

$$\begin{cases} x = \dfrac{de - bf}{ad - bc} \\[2ex] y = \dfrac{-ce + af}{ad - bc} \end{cases}$$

$$x = \frac{\operatorname{Det}\begin{pmatrix} e & b \\ f & d \end{pmatrix}}{\operatorname{Det}(A)}$$

$$y = \frac{\operatorname{Det}\begin{pmatrix} a & e \\ c & f \end{pmatrix}}{\operatorname{Det}(A)}$$

$$\begin{matrix} \cancel{a} & \cancel{b} & e \\ \cancel{c} & \cancel{d} & f \end{matrix}$$

TEOREMA (CRAMER)

A è $n \times n$ invertibile

$$A = \begin{pmatrix} A_1 & \dots & A_n \end{pmatrix}$$ le colonne di A.

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in K^n$$

$$A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = b$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$x_i = \mathrm{Det}\begin{pmatrix} A_1 & \dots & \underset{\cancel{A_i}}{b} & \dots & A_n \end{pmatrix} \Big/ \mathrm{Det}\, A$$