$$M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\cdot 1]{R_{31}(-1)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow{R_{23}(5)}$$

$$\to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -9 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow[\cdot(-1)]{R_{23}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -9 \end{pmatrix}$$

$$\operatorname{Det} M = -\operatorname{Det} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -9 \end{pmatrix} = -(1)\cdot(-1)(-9) = -9.$$

$$\operatorname{Det} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 5 & 1 \\ 1 & 2 & 1 \end{pmatrix} = -1 \operatorname{Det} \begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix}$$

$$- 0 \cdot \operatorname{Det} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$+ 1 \operatorname{Det} \begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix} =$$

$$= (-1)\left(5 \cdot 1 - 2 \cdot 1\right) - 0 + 1\left(1 \cdot 2 - 5 \cdot 1\right) =$$

$= -3 - 3 = -6$

ESERCIZIO 9.3

$\exists f, g$ $\boxed{f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2}$ $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$g \circ f$ sia biniettiva?

$$f : \underline{\underline{\mathbb{R}^3}} \longrightarrow \mathbb{R}^2$$

$$3 = \underline{\dim \operatorname{Im} f} + \underline{\dim N(f)}$$

$$\operatorname{Im} f \subset \mathbb{R}^2 \qquad \dim \operatorname{Im} f \leq 2$$

QUINDI $\dim N f = 3 - \dim \operatorname{Im} f \geq 1$.

quindi $N(f) \neq 0$ ovvero f non è iniettiva.

$$\underline{g \circ f} : \underline{\underline{\mathbb{R}^3}} \longrightarrow \mathbb{R}^3$$

$g \circ f$ non può essere iniettiva.

Infatti: $\exists\ x \in \mathbb{R}^3$ tale che $x \neq 0$
e $f(x) = 0$.

Allora $(g \circ f)(x) = g(\underline{f(x)}) = g(0) = 0$

quindi $N(g \circ f) \neq 0$ ovvero $g \circ f$ non è iniettiva.

ESERCIZIO 9.5

$$V = \mathbb{C}[t]_{\leq 3} \qquad F: V \longrightarrow V$$

$$F(p(t)) = p'(t) + p(\underline{t+1})$$

$$F(\underline{t^2+t}) = 2t+1 + \underline{(t+1)^2 + (t+1)}$$

$$= t^2 + 5t + 3$$

$\mathcal{B}$: $1, t, t^2, t^3$ è la base standard di V.

$$p(t) = \underline{a_0} \cdot 1 + \underline{a_1}\, t + \underline{a_2}\, t^2 + \underline{a_3}\, t^3$$

$$[p]_{\mathcal{B}} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$F(\underline{1}) = 0 + 1 = 1$$

$$F(\underline{t}) = 1 + (t+1) = t + 2$$

$$F(\underline{t^2}) = 2t + (t+1)^2 = \underline{t^2 + 4t + 1}$$

$$F(\underline{t^3}) = \underline{3t^2} + (t+1)^3 = t^3 + 3t^2 + 3t + 1 + 3t^2$$
$$= t^3 + 6t^2 + 3t + 1$$

1
t
t^2
t^3

$$[F(1)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad [F(t)]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad [F(t^2)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 0 \end{pmatrix}$$

$$[F(t^3)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 3 \\ 6 \\ 1 \end{pmatrix} \quad A = [F]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A \cdot [p]_{\mathcal{B}} = [F(p)]_{\mathcal{B}}$$

$$p = \underline{a_0} + \underline{a_1}\, t + \underline{a_2}\, t^2 + \underline{a_3}\, t^3 \qquad [p]_{\mathcal{B}} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{pmatrix} b_0 \end{pmatrix}$$

$$(F_p) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 \qquad (F_p) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = A \cdot \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$V \qquad \underline{v} : v_1, \dots, v_n$ BASE DI V

$\qquad \underline{w} : w_1, \dots, w_n$ BASE DI V

$$[v]_{\underline{v}} \qquad\qquad [v]_{\underline{w}}$$

$$Id_V(v) = v \qquad\qquad Id_V : V \longrightarrow V$$

$$[Id_V]^{\underline{v}}_{\underline{w}} \, [v]_{\underline{v}} = [Id_V(v)]_{\underline{w}} = [v]_{\underline{w}}$$

ESEMPIO $\qquad V = \mathbb{R}^2$

$\underline{w} \qquad e_1 \; ; \; e_2$

$\underline{v}$ $\quad v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \qquad [v]_{e_1 e_2} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{[v]_{v_1 v_2}} = \underline{[\,\underline{Id}\,]^{e_1 e_2}_{v_1 v_2}} \cdot [v]_{e_1 e_2} =$$

$$Id(e_1) = e_1 = a\, v_1 + c\, v_2 \qquad *$$

$$Id(e_2) = e_2 = b\, v_1 + d\, v_2$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c \begin{pmatrix} 2 \\ 2 \end{pmatrix} =$$

$$\begin{cases} a + 2c = 1 \\ 3a + 2c = 0 \end{cases}$$

$$\begin{cases} c = -\frac{3a}{2} \\ -2a = 1 \end{cases} \qquad a = -1/2 \qquad c = 3/4$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = b \begin{pmatrix} 1 \\ 3 \end{pmatrix} + d \begin{pmatrix} 2 \\ 2 \end{pmatrix} =$$

$$\begin{cases} b + 2d = 0 \\ 3b + 2d = 1 \end{cases} \qquad \begin{cases} b = -2d \\ -4d = 1 \end{cases}$$

$$d = -1/4 \qquad b = 1/2$$

$$Id(e_1) = e_1 = -\frac{1}{2} v_1 + \frac{3}{4} v_2$$

$$Id(e_2) = e_2 = \frac{1}{2} v_1 - \frac{1}{4} v_2$$

$$\boxed{[Id]^{e_1 e_2}_{v_1 v_2}} = \begin{pmatrix} -1/2 & 1/2 \\ 3/4 & -1/4 \end{pmatrix}$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = [v]_{e_1 e_2} \qquad [v]_{v_1 v_2} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underline{\begin{pmatrix} -1/2 & 1/2 \\ 3/4 & -1/4 \end{pmatrix}} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$F : V \longrightarrow W$ Lineare

$v_1 \quad v_n$ base di V $\qquad x_1 \dots x_n$ base di V

$w_1 \quad w_n$ base di W $\qquad y_1 \dots y_n$ base di W.

$$\boxed{[F]^{\underline{v}}_{\underline{w}}} \qquad\qquad \boxed{[F]^{\underline{X}}_{\underline{Y}}} \leftarrow$$

$$[F]^{\underline{X}}_{\underline{Y}} = [Id_W \circ F \circ Id_V]^{\underline{X}}_{\underline{Y}} = [Id_W]^{\underline{w}}_{\underline{Y}} \cdot \underbrace{[F]^{\underline{v}}_{\underline{w}} \cdot [Id_V]^{\underline{X}}_{\underline{v}}}$$

<u>RICORDO</u>

$$[H \circ G]^{\underline{\alpha}}_{\underline{\gamma}} = [H]^{\underline{\beta}}_{\underline{\gamma}} \cdot [G]^{\underline{\alpha}}_{\underline{\beta}}$$

$$[Id_W]^{\underline{w}}_{\underline{Y}} \cdot [F \circ Id_V]^{\underline{X}}_{\underline{w}} =$$

$$= [Id \circ F \circ Id]^{\underline{X}}_{\underline{Y}} = [F]^{\underline{X}}_{\underline{Y}} .$$

$$[F]^{\underline{X}}_{\underline{Y}} = [Id]^{\underline{w}}_{\underline{Y}} \cdot [F]^{\underline{v}}_{\underline{w}} \cdot [Id]^{\underline{X}}_{\underline{v}}$$

$$A = [F]^{\underline{v}}_{\underline{w}}$$

$$[v]_{\underline{v}} \quad K^n \xrightarrow{L_A} K^m \quad [w]_{\underline{w}}$$

$$
\begin{tikzcd}
 & V \arrow[r, "F"] \arrow[dl, "{[\ ]_X}"'] & W \arrow[dr, "{[\ ]_Y}"] & \\
{[v]_{\underline{X}}} \quad K^n \arrow[rrr, "L_B"] & & & K^m \quad {[w]_{\underline{Y}}}
\end{tikzcd}
$$

$[\ ]_{\underline{v}}$ $[\ ]_{\underline{w}}$ v Fv

$$B = [F]^{X}_{\underline{Y}}$$

$$[Id]^{\underline{v}}_{\underline{Y}} \cdot A \cdot [I]^{\underline{X}}_{\underline{v}} = B.$$