DETERMINANTE

$A \in Mat_{n\times n}(K)$

A è invertibile se verifica una delle seg. condiz.

1) $\exists B: \quad AB = I_n$ $\qquad B = C = A^{-1}$
2) $\exists C: \quad CA = I_n$
3) Il sistema $Ax = 0$ ha una sola sol.
4) Il sistema $Ax = b$ ha sempre soluzione
5) rango $A = n$.

- $L_A : K^n \longrightarrow K^n$ è iniettiva
- $L_A : K^n \longrightarrow K^n$ è suriettiva
- $L_A : K^n \longrightarrow K^n$ è bigettiva

VOGLIAMO DEFINIRE UNA FUNZIONE

$$F: Mat_{n\times n}(K) \longrightarrow K.$$

TALE CHE $F(A) \neq 0$ SE E SOLO A È INVERTIBILE.

Esempio $n = 2$

$$\det: Mat_{n\times n}(K) \longrightarrow K$$

$$\det_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{ad - bc}$$

TEOREMA

Esiste una unica funzione

$$F : Mat_{n\times n}(K) \longrightarrow K$$

TALE CHE

1) $F(I_n) = 1$

2) $F(M) = 0$

se $M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}$ e $M_i = M_j$ con $i \neq j$

3) F È LINEARE NELLE RIGHE

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix} \quad e \quad M_i = \lambda M_i'$$

$$\begin{pmatrix} M_1 \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} M_1 \\ \vdots \end{pmatrix}$$

$$F\begin{pmatrix} \lambda M_i' \\ M_n \end{pmatrix} = \lambda F\begin{pmatrix} M_i' \\ M_n \end{pmatrix}$$

$$F\begin{pmatrix} M_1 \\ P_i + Q_i \\ M_n \end{pmatrix} = F\begin{pmatrix} M_1 \\ P_i \\ M_n \end{pmatrix} + F\begin{pmatrix} M_1 \\ Q_i \\ M_n \end{pmatrix}$$

DEFINIZIONE con le proprietà del teorema

L'unica funzione $F : Mat_{n \times n} \longrightarrow K$
Si chiama il determinante.

$$\det = \underline{\underline{\det_n}}$$

Esempio

$$\mathrm{Det}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{Det}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \overset{0}{\overset{\shortparallel}{\mathrm{Det}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}}$$

$$(0 \ \ 2 \ \ 1) = (0 \ \ 2 \ \ 0) + (0 \ \ 0 \ \ 1)$$

$$= \mathrm{Det}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{Det}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \overset{0}{\overset{\shortparallel}{\mathrm{Det}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}}$$

$$(1 \ \ 0 \ \ 1) = (1 \ \ 0 \ \ 0) + (0 \ \ 0 \ \ 1)$$

$$= \mathrm{Det}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 1 \end{pmatrix} = 2\,\mathrm{Det}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2$$

$$\mathrm{Det}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

$$(0 \; 2 \; 0) = 2(0 \; 1 \; 0)$$

P R O P O S I Z I O N E

1) $$\mathrm{Det}\begin{pmatrix} M_1 \\ M_i \\ M_j \\ M_n \end{pmatrix} = -\,\mathrm{Det}\begin{pmatrix} M_1 \\ M_j \\ M_i \\ M_n \end{pmatrix}$$

2) $$\mathrm{Det}\begin{pmatrix} M_1 \\ M_i + \alpha M_j \\ M_j \\ M_n \end{pmatrix} = \mathrm{Det}\begin{pmatrix} M_1 \\ M_i \\ M_j \\ M_n \end{pmatrix}$$

$$\mathrm{Det}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 6 & 5 \\ \pi & 3 & 4 & 1 \\ 1 & 0 & \sqrt{2} & 1 \end{pmatrix} = -\,\mathrm{Det}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & \sqrt{2} & 1 \\ \pi & 3 & 4 & 1 \\ 3 & 7 & 6 & 5 \end{pmatrix}$$

$$\mathrm{Det}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

$$\begin{pmatrix} M_3 \\ M_4 \end{pmatrix}$$

$$\operatorname{Det}\begin{pmatrix} M_1 \\ M_2 + M_4 \\ M_3 \\ M_2 + M_4 \end{pmatrix} = 0$$

$$\|$$

$$\operatorname{Det}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_2 + M_4 \end{pmatrix} + \operatorname{Det}\begin{pmatrix} M_1 \\ M_4 \\ M_3 \\ M_2 + M_4 \end{pmatrix} =$$

$$= \underbrace{\operatorname{Det}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_2 \end{pmatrix}}_{\substack{\| \\ 0}} + \operatorname{Det}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} + \operatorname{Det}\begin{pmatrix} M_1 \\ M_4 \\ M_3 \\ M_2 \end{pmatrix} + \underbrace{\operatorname{Det}\begin{pmatrix} M_1 \\ M_4 \\ M_3 \\ M_4 \end{pmatrix}}_{\substack{\| \\ 0}}$$

$$0 = \operatorname{Det}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \end{pmatrix} + \operatorname{Det}\begin{pmatrix} M_1 \\ M_4 \\ M_3 \\ \end{pmatrix}$$

$$M_4 \qquad\qquad M_2$$

$$\mathrm{Det}\begin{pmatrix} M_1 \\ M_4 \\ M_3 \\ M_2 \end{pmatrix} = -\,\mathrm{Det}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix}$$

2)

$$\mathrm{Det}\begin{pmatrix} M_1 \\ M_2 + \alpha M_4 \\ M_3 \\ M_4 \end{pmatrix} = \mathrm{Det}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix}$$

$$\|$$

$$\mathrm{Det}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} + \alpha\,\underbrace{\mathrm{Det}\begin{pmatrix} M_1 \\ M_4 \\ M_3 \\ M_4 \end{pmatrix}}_{=\,0} = \mathrm{Det}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix}$$

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<u>Esempio</u>

$$\mathrm{Det}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$\Big\downarrow R_{21}(-1)$$

Det non cambia.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 3 & 1 & 1 \end{pmatrix}$$

$$\Big\downarrow$$

Det non cambia.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -5 & -8 \end{pmatrix}$$

$5/2$

$$\Big\downarrow$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\Big\downarrow$$

$$\begin{matrix} 1 & 2 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{matrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\mathrm{Det}\left(\text{matrice iniziale}\right) = \mathrm{Det}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} =$$

$$= (-2)\,\mathrm{Det}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \underline{-3} \end{pmatrix} = (-2)(-3)\,\mathrm{Det}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{1}$$

$$= 6.$$

PROPOSIZIONE

1) $$\mathrm{Det}\begin{pmatrix} e_1 & * & & & & \\ 0 & e_2 & * & & * & \\ 0 & 0 & e_3 & & & \\ 0 & 0 & 0 & \ddots & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & & & 0 & e_n \end{pmatrix} = e_1 e_2 e_3 \quad e_n$$

2) Se rango $A < n$ allora $\operatorname{Det} A = 0$.

dim

2) SE UNA RIGA $\overset{\text{di } M}{\vee}$ È ZERO ALLORA $\operatorname{Det} M = 0$

$$\operatorname{Det}\begin{pmatrix} M_1 \\ 0 \\ M_n \end{pmatrix} = \operatorname{Det}\begin{pmatrix} M_1 \\ 0 \cdot 0 \\ M_n \end{pmatrix} = 0 \cdot \operatorname{Det}\begin{pmatrix} M_1 \\ 0 \\ M_n \end{pmatrix} = 0.$$

Sia rango $(M) < n$

ovvero le righe sono lin. dipendenti

$$M_i = \alpha_1 M_1 + \alpha_2 M_2 + \alpha_{i-1} M_{i-1} + \alpha_{i+1} M_{i+1} + \dots + \alpha_n M_n$$

$$\operatorname{Det}\begin{pmatrix} M_1 \\ M_2 \\ \alpha_1 M_1 + \alpha_2 M_2 + \alpha_4 M_4 \\ M_4 \end{pmatrix} = \operatorname{Det}\begin{pmatrix} M_1 \\ M_2 \\ \alpha_2 M_2 + \alpha_4 M_4 \\ M_4 \end{pmatrix} =$$

$$= \ \dots \ = \operatorname{Det}\begin{pmatrix} r_1 \\ r_2 \\ 0 \\ r_4 \end{pmatrix} = 0$$

1)

$$\operatorname{Det}\begin{pmatrix} a_1 & & & & * \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & 0 & & & a_n \end{pmatrix}$$

Se $a_i = 0$ allora rango $A < n$

$$\boxed{\begin{pmatrix} a_1 & & & \\ 0 & 0 & & \\ 0 & 0 & a_3 & \\ 0 & 0 & 0 & a_4 \end{pmatrix}} \qquad \begin{matrix} \bullet & & & \\ 0 & 0 & \bullet & \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \end{matrix}$$

e quindi per il punto 2) il

Determinante è zero.

$$\operatorname{Det}\begin{pmatrix} a_1 & x & x & \\ 0 & 0 & 1 & \\ 0 & 0 & a_3 & \\ 0 & 0 & 0 & a_4 \end{pmatrix} = a_1 \cdot 0 \cdot a_3 \cdot a_4 = 0 .$$

Se $a_1, a_2, a_3, \dots a_n \neq 0$

$$\operatorname{Det}\begin{pmatrix} a_1 & x & x & x \\ 0 & a_2 & x & x \\ 0 & 0 & a_3 & x \\ 0 & 0 & 0 & a_4 \end{pmatrix} = a_1 \operatorname{Det}\begin{pmatrix} 1 & x & x & x \\ 0 & a_2 & x & x \\ 0 & 0 & a_3 & x \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

$$= a_1 a_2 a_3 a_4 \operatorname{Det}\begin{pmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ \boxed{0 \quad 0 \quad 1 \quad x} \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow$$

$$= a_1 a_2 a_3 a_4 \operatorname{Det}\begin{pmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= a_1 a_2 a_3 a_4 \operatorname{Det}\begin{pmatrix} 1 & & \\ & 1 & 0 \\ & & \ddots \end{pmatrix} =$$

$$\begin{pmatrix} & 0 & 1 & \\ & & & 1 \end{pmatrix}$$

$$= a_1\, a_2\, a_3\, a_4$$

#

Esempio

$$\mathrm{Det}\begin{pmatrix} 0 & 1 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 1 \end{pmatrix} = -\,\mathrm{Det}\begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 6 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= -\,\mathrm{Det}\begin{pmatrix} 1 & 3 & 1 \\ 0 & -2 & 4 \\ 0 & 1 & 3 \end{pmatrix} \leftarrow \mathrm{III} + \frac{1}{2}\,\mathrm{II}$$

$$= -\,\mathrm{Det}\begin{pmatrix} 1 & 3 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 5 \end{pmatrix} = -1\cdot(-2)\cdot 5 = 10.$$

PROPOSIZIONE

$\mathrm{Det}(M) \neq 0$ SE E SOLO SE M È INV.

DIM.

$\operatorname{Det}(M) \neq 0$ SE E SOLO SE rango $M = n$.

Se rango $M < n$ ALLORA $\operatorname{Det}(M) = 0$.

Devo far vedere che se il rango è n
ALLORA $\operatorname{Det}(M) \neq 0$.

Sia M di RANGO n.

SE RIDUCO M A SCALINI HO
TRE MOSSE

- SCABIARE 2 RIGHE CHE CAMBIA IL SEGNO
- SOMMARE AD UNA RIGA UN MULTIPLO DI UN'ALTRA RIGA NON CAMBIA IL DET.
- MOLTIPLICARE UNA RIGA PER $\lambda \neq 0$ MOLTIPLICA IL DET. PER λ.

quindi se M' è la matrice ridotta a scalini

$$\underline{\det M} = \pm \lambda_1 \dots \lambda_n \underline{\operatorname{Det} M'}$$

con $\lambda_1 \dots \lambda_n \neq 0$.

$$M \rightsquigarrow M' = \begin{pmatrix} e_1 & & & \\ & e_2 & & \\ & & \ddots & \\ & 0 & & e_n \end{pmatrix} \leftarrow \text{PERCHÈ IL RANGO } = n.$$

con $e_1, e_2, \dots e_n \neq 0$.

$$\operatorname{Det} M = \pm \underline{\lambda_1} \dots \underline{\lambda_n} \cdot \underline{e_1} \cdot \dots \cdot \underline{e_n} \neq 0.$$

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SVILUPPO DI LAPLACE.

$$\operatorname{Det} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} =$$

$$(1,2,3) = (1,0,0) + (0,2,0) + (0,0,3)$$

$$= \operatorname{Det} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \operatorname{Det} \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$+ \ \mathrm{Det}\begin{pmatrix} 0 & 0 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} =$$

$$\mathrm{Det}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} + 2 \ \mathrm{Det}\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$+ \ 3 \ \mathrm{Det}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{matrix} + & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{matrix}$$

$$\boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & \boxed{\begin{matrix} 0 & 1 \\ 1 & 2 \end{matrix}} \end{matrix}} \qquad \begin{matrix} 0 & 1 & 0 \\ \boxed{\begin{matrix} 3 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 1 \\ 2 \end{matrix}} \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 \\ \boxed{\begin{matrix} 3 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \end{matrix}$$

$$\mathrm{Det}\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$F : Mat_{2\times 2} \longrightarrow Det_3 \begin{pmatrix} 0 & a & b \\ 0 & c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$F \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Det_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

$$F \begin{pmatrix} a & b \\ a & b \end{pmatrix} = Det_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & a & b \end{pmatrix} = 0$$

$$F \begin{pmatrix} a+\alpha & b+\beta \\ c & d \end{pmatrix} =$$

$$= Det_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a+\alpha & b+\beta \\ 0 & c & d \end{pmatrix}$$

$$= Det_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} + Det_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & c & d \end{pmatrix}$$

$$= F \begin{pmatrix} a & b \\ c & d \end{pmatrix} + F \begin{pmatrix} \alpha & \beta \\ c & d \end{pmatrix}$$

quindi: $F = Det_{2\times 2}$

$$\mathrm{Det} = \mathrm{Det}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} + 2\,\mathrm{Det}\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} + 3\,\mathrm{Det}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \mathrm{Det}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} - 2\,\mathrm{Det}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} + 3\,\mathrm{Det}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

Sia M una matrice $n \times n$

$$M = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

e sia N_i la matrice ottenuta da M cancellando la prima riga e la i-esima colonna

Allora $\mathrm{Det}_n\, M =$

$$a_1 \,\mathrm{Det}_{n-1}(N_1) - a_2 \,\mathrm{Det}_{n-1}(N_2) + a_3 \,\mathrm{Det}_{n-1}(N_3)$$

$$----- + (-1)^{n+1} \mathrm{Det}_{n-1}(N_n)$$

Esempio 1 $n=2$ e $n=3$

$n=1$ $\quad \mathrm{Det}_1 : \mathrm{Mat}_{1\times 1}(K) \longrightarrow K.$

$$\mathrm{Det}_1(a) = a$$

$n=2$ $$\mathrm{Det}\left(\begin{array}{cc} a & b \\ \hline c & d \end{array}\right) = a \cdot d - b \cdot c$$

$n=3$

$$\mathrm{Det}\left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right) =$$

$$= a_1 \,\mathrm{Det}\begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - \underline{a_2 \,\mathrm{Det}\begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix}} +$$

$$+ a_3 \ \mathrm{Det} \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}$$

$$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1$$

$$+ a_3 b_1 c_2 - a_3 b_2 c_1$$

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}$$