$$A\,x = b$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}\left\}\begin{array}{|cccccc|} \hline 1 & 3 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}\right. \qquad \begin{array}{c} \textcircled{3} \\ \textcircled{4} \\ 5 \\ 7 \end{array} \qquad \begin{array}{|c|} \hline x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \hline \end{array} \qquad A x \quad 4 \times 1$$

$$A \qquad\qquad b \qquad\qquad x$$

$$\begin{aligned} x_1 + 3x_2 + 2x_4 + 3x_6 &= 3 \\ x_3 + 4x_4 + 5x_6 &= 4 \\ x_5 + 4x_6 &= 5 \\ 0x_1 + 0 \cdot x_2 + 0x_3 + \cdots + 0 \cdot x_6 &= 0 = 7 \quad \ast \end{aligned}$$

- NON CI
- SONO SOLUZIONI
-

$$\left.\begin{array}{|cccccc|} \hline 1 & 3 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}\right. \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 \cdot x_1 + 3 \cdot x_2 + 0 \cdot x_3 + 2 \cdot x_4 + 0 \cdot x_5 + 3x_6 \\ x_4 + 4x_4 + 5x_6 \\ x_5 + 4x_6 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 7 \end{pmatrix}$$

$$\left(\begin{array}{cccccc|c} \textcircled{1} & 3 & 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & \textcircled{1} & 4 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

x_1 x_3 x_5 saranno le var. dip. y

x_2 x_4 x_6 sono le var. libere. Z

Le soluzioni del sistema sono le $x=(x_1, \ldots x_6)$ t.c.

$$\boxed{\begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} = -\underbrace{\begin{pmatrix} 3 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}}_{C} \underset{\uparrow}{\begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix}} + \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{d}}$$

$$\begin{cases} \textcircled{x_1} + 3x_2 + 2x_4 + 3x_6 = 1 \\ \textcircled{x_3} + 4x_4 + 5x_6 = 2 \\ \textcircled{x_5} + 4x_6 = 3 \end{cases} \qquad \boxed{\begin{cases} x_1 = -3x_2 - 2x_4 - 3x_6 + 1 \\ x_3 = -4x_4 - 5x_6 + 2 \\ x_5 = -4x_6 + 3 \end{cases}}$$

$$-\begin{pmatrix} 3 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\begin{pmatrix} 3x_2 + 2x_4 + 3x_6 \\ 4x_4 + 5x_6 \\ 4x_6 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 - 3x_6 + 1 \\ -4x_4 - 5x_6 + 2 \\ -4x_6 + 3 \end{pmatrix}$$

$$\begin{array}{ccccccc|c}
 & & & A\,x & & & & b \\
\textcircled{2} & 3 & 3 & 6 & 6 & 10 & 8 & 1 \\
0 & 0 & \textcircled{4} & 8 & 12 & 16 & 4 & 2 \\
0 & 0 & 0 & 0 & \textcircled{3} & 5 & 4 & 3 \\
\boxed{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0} & & & & & & & \beta
\end{array} \qquad 0 = \beta$$

$x_1 \quad x_3 \quad x_5$ DIP.

$x_2 \quad x_4 \quad x_6 \quad x_7$

Se $\beta \neq 0$ NON C'È SOLUZIONE

Se $\beta = 0$ C'È SOLUZIONE E LE SOLUZIONI SONO TUTTE LE $x = (x_1 \dots x_7)$ TALI

$$\begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} = \underline{\underline{C}} \begin{pmatrix} x_2 \\ x_4 \\ x_6 \\ x_7 \end{pmatrix} + \underline{\underline{\overset{d_{11}}{\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}}}}$$

$$\begin{array}{ccccccc|c|c}
\textcircled{2} & 3 & 3 & 6 & 6 & 10 & 8 & 1 & 1/2 \\
0 & 0 & \textcircled{4} & 8 & 12 & 16 & 4 & 2 & 1/4 \\
0 & 0 & 0 & 0 & \textcircled{3} & 5 & 4 & 3 & 1/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}$$

$R_3(1/3) \downarrow \quad R_1(1/2) \quad R_2(1/4)$

$$\begin{array}{ccccccc|cl}
1 & 3/2 & \textcircled{3/2} & 3 & \textcircled{3} & 5 & 4 & 1/2 & \leftarrow \\
0 & 0 & 1 & 2 & \textcircled{3} & 4 & 1 & 1/2 & \leftarrow 3/2 \\
0 & 0 & 0 & 0 & 1 & 5/3 & 4/3 & 1 &
\end{array}$$

$$\downarrow \quad R_{12}\left(-\frac{3}{2}\right)$$

$$\begin{array}{ccccccccc}
1 & 3/2 & 0 & 0 & 3-\frac{9}{2} & 5-\frac{12}{2} & 4-\frac{3}{2} & & \frac{1}{2}-\frac{3}{4} \\
0 & 0 & 1 & 2 & 3 & 4 & 1 & & 1/2 \\
0 & 0 & 0 & 0 & 1 & 5/3 & 4/3 & & 1
\end{array}$$

$$\begin{array}{ccccccccc}
\text{\textcircled{1}} & 3/2 & \text{\textcircled{0}} & 0 & -3/2 & -1 & 5/2 & & -1/4 \\
0 & 0 & \text{\textcircled{1}} & 2 & 3 & 4 & 1 & & 1/2 \\
0 & 0 & 0 & 0 & \text{\textcircled{1}} & 5/3 & 4/3 & & 1
\end{array}$$

$$\downarrow \quad R_{23}(-3)$$

$$\begin{array}{ccccccccc}
\text{\textcircled{1}} & 3/2 & 0 & 0 & -3/2 & -1 & 5/2 & & -1/4 \\
0 & 0 & \text{\textcircled{1}} & 2 & 0 & -1 & -3 & & 1/2 - 3 = -5/2 \\
0 & 0 & 0 & 0 & \text{\textcircled{1}} & 5/3 & 4/3 & & 1
\end{array}$$

$$\downarrow \quad R_{13}\ (3/2)$$

$$\begin{array}{ccccccccc}
1 & 3/2 & 0 & 0 & 0 & -1+\frac{5}{2} & \frac{5}{2}+2 & & -1/4 + 3/2 \cdot 1 = 5/4 \\
0 & 0 & 1 & 2 & 0 & -1 & -3 & & -5/2 \\
0 & 0 & 0 & 0 & 1 & 5/3 & 4/3 & & 1
\end{array}$$

$$\boxed{\begin{array}{ccccccccc}
1 & 3/2 & 0 & 0 & 0 & 3/2 & 9/2 & & 5/4 \\
0 & 0 & 1 & 2 & 0 & -1 & -3 & & -5/2 \\
0 & 0 & 0 & 0 & 1 & 5/3 & 4/3 & & 1
\end{array}}$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = -\begin{pmatrix} 3/2 & 0 & 3/2 & 9/2 \\ 0 & 2 & -1 & -3 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + \begin{pmatrix} 5/4 \\ -5/2 \end{pmatrix}$$

$$\begin{matrix} \big(\; x_5 \; \big) \\ /\!/ \\ \uparrow \end{matrix} \qquad \underbrace{\underbrace{\big(\; 0 \;\; 0 \;\; 5/3 \;\; 4/3 \; \big)} \begin{pmatrix} x_6 \\ x_7 \end{pmatrix} \underbrace{\big(\; 1 \; \big)}}_{\uparrow}$$

RIDUZIONE A SCALINI DI UNA MATRICE QUALSIASI

$$\begin{cases} 3x_1 + 4x_2 + x_3 + x_4 = 1 \\ 3x_1 + 2x_2 + 2x_3 + x_4 = 2 \\ 2x_2 - x_3 = \textcircled{c} \\ x_1 + 7x_2 + x_3 + x_4 = \textcircled{d} \end{cases}$$

$$\underbrace{\underbrace{\begin{matrix} 3 & 4 & 1 & 1 \\ 3 & 2 & 2 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & 7 & 1 & 1 \end{matrix}}_{\text{MAT ASSOCIATA}} \qquad \begin{matrix} 1 \\ 2 \\ c \\ d \end{matrix}}_{\text{MAT. COMPLETA ASS}}$$

$$\Big\downarrow \; R_{21}(-1)$$

$$\begin{matrix} 3 & 4 & 1 & 1 & & 1 \\ 0 & -2 & 1 & 0 & & 1 \\ 0 & 2 & -1 & 0 & & c \\ 1 & 7 & 1 & 1 & & d \end{matrix}$$

$$\Big\downarrow \; R_{32}(1)$$

$$\begin{array}{ccccc} 3 & 4 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & c+1 \\ 1 & 7 & 1 & 1 & d \end{array}$$

$$\downarrow R_{34}$$

$$\begin{array}{ccccc} 3 & 4 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 & 1 \\ 1 & 7 & 1 & 1 & d \\ 0 & 0 & 0 & 0 & c+1 \end{array}$$

$$\downarrow R_{31}\left(-\tfrac{1}{3}\right)$$

$$\begin{array}{ccccc} 3 & 4 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & 7-\frac{4}{3} & 1-\frac{1}{3} & 1-\frac{1}{3} & d-\frac{1}{3} \\ 0 & 0 & 0 & 0 & c+1 \end{array}$$

$$\begin{array}{ccccc} 3 & 4 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & 17/3 & 2/3 & 2/3 & d-1/3 \\ 0 & 0 & 0 & 0 & c+1 \end{array}$$

$$\downarrow R_3(3)$$

$$\begin{array}{cccccc} & 3 & 4 & 1 & 1 & 1 \\ 17/2 \rightarrow & 0 & -2 & 1 & 0 & 1 \\ & 0 & 17 & 2 & 2 & 3d-1 \\ & 0 & 0 & 0 & 0 & c+1 \end{array}$$

$0 \quad -17 \quad 17/2 \quad 0 \quad 17/2$

17

$$\downarrow R_{32}\left(\frac{17}{2}\right)$$

$$\begin{matrix} 3 & 4 & 1 & 1 & & 1 \\ 0 & -2 & 1 & 0 & & 1 \\ 0 & 0 & 2+\frac{17}{2} & 2 & & 3\alpha-1+\frac{17}{2} \\ 0 & 0 & 0 & 0 & & c+1 \end{matrix}$$

$$\begin{matrix} & \boxed{3} & 4 & 1 & 1 & & 1 \\ & 0 & \boxed{-2} & 1 & 0 & & 1 \\ & 0 & 0 & \boxed{21/2} & 2 & & 3\alpha+15/2 \\ \rightarrow & 0 & 0 & 0 & 0 & & c+1 \end{matrix}$$

IL SISTEMA HA SOLUZIONE SE E SOLO SE

$c+1=0.$ CIOÈ $c=-1$

SE $c=-1$ $(x_1, x_2\ x_3\ x_4) =$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} x_4 + \begin{pmatrix} \delta \\ \varepsilon \\ \eta \end{pmatrix}$$

OGNI SISTEMA LO POTETE RIDURRE A SCALINI IN FORMA FORTE UTILIZZANDO LE MOSSE R_{ij} $R_i(\alpha)$ $R_{ij}(\alpha)$

DEFINIZIONE

Sia A una matrice $m \times n$ e sia B una riduzione a scalini di A ovvero una matrice a scalini ottenuta da A con le mosse R_{ij} $R_i(\alpha)$ $R_{ij}(\alpha)$

definisco rango(A) = rango(B) =
= n° di righe di B non nulle.

$$\text{rango}\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2$$

$$\text{rango}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 1$$

$$\left\downarrow \begin{array}{l} R_{31}(-1) \\ R_{21}(-1) \end{array}\right.$$

$$\text{rango}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 1$$

LA DEFINIZIONE COSÌ FORMULATA NON È COMPLETA PERCHÉ NON ABBIAMO DIMOSTRATO CHE NON DIPENDE DA B

RIMANDIAMO QUESTA DIMOSTRAZIONE

TEOREMA DI ROUCHE - CAPELLI

DATO UN SISTEMA $Ax = b$

A $m \times n$ x $n \times 1$ b $m \times 1$

A Ab -

1) IL SISTEMA HA SOLUZIONE SE E SOLO SE rango (A) = rango (Ab)

2) SE R = rango A = rango (Ab) ALLORA POSSIAMO SCEGLIERE
   R VARIABILI DIPENDENTI y (ALCUNE DELLE x)
   $n-R$ VARIABILI LIBERE z (LE RIMANENTI x)
   E ESISTONO UNA MATRICE $C: R \times (n-R)$
   E UN VETTORE d $R \times 1$
   E LE SOLUZIONI DEL SISTEMA SONO TUTTE LE y, z TALI CHE

$$y = Cz + d.$$

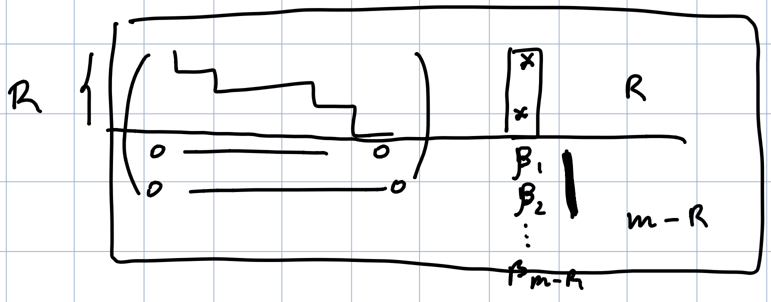
dim $\quad Ax = b$

E LO RIDUCIAMO A SCALINI IN FORMA FORTE

$$\begin{array}{c} A \quad b \\ \downarrow \\ A' \quad b' \end{array}$$

CON A' A SCALINI

rango (A) = rango (A')



IL SISTEMA HA SOLUZIONE SE $\beta_i = 0$.

OSSERVIAMO CHE SE $\beta_i \neq 0$ PER QUALCHE i ALLORA RANGO $(A'b') >$ RANGO (A')

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & * \\ 0 & 0 & 1 & 2 & * \\ 0 & 0 & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & \boxed{\beta_2} \end{array} \qquad \beta_1 \neq 0$$

$$\downarrow \quad R_3 \left(1/\beta_1\right)$$

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & * \\ 0 & 0 & 1 & 2 & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \beta_2 \end{array}$$

$$\downarrow \quad R_{43} (-\beta_2$$

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & * \\ 0 & 0 & 1 & 2 & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \leftarrow$$

$\text{RANGO}(A'b') = 3 \qquad \text{rango } A' = 2$

SE INVECE $\beta_i = 0 \quad i = 1 \ldots. \; m - R$
LA MATRICE COMPLETA È GIÀ RIDOTTA
A SCALINI $\underline{\text{RANGO}(A'b') = \text{rango}(A')}$

$\text{rango}(Ab) = \text{rango}(A)$

2) $A x = b$

$$\left|\begin{array}{ccccccc|c} 0 & \cdots & \textcircled{1} & 2 & 0 & & & 1 \\ 0 & & & 0 & \textcircled{1} & & & 2 \\ 0 & & & & 0 & \textcircled{1} & 3 & 3 \\ 0 & \multicolumn{5}{c}{\hrulefill} & 0 & 0 \\ 0 & \multicolumn{5}{c}{\hrulefill} & 0 & 0 \end{array}\right|$$

CI SONO R VAR. DIP. E $n-2$ LIBERE.

#