

$$F: V \rightarrow W$$

⋮

ES EPIOM

$$V = \mathbb{C}[t]_{\leq 2}$$

$$F(p(t)) = p'(t) + p(t+1)$$

$$\mathcal{B}: 1, t, t^2$$

$$[F]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{C}: 1+t \quad t^2-1 \quad t^2$$

$$[F]_{\mathcal{C}}^{\mathcal{B}}$$

USO LA DEFINIZIONE

$$\cdot \frac{F(1)}{\quad} = 1 = \frac{a_1}{\quad} \cdot (1+t) + \frac{b_1}{\quad} (t^2-1) + \frac{c_1}{\quad} t^2$$

$$\frac{F(t)}{\quad} = 1+t+t = t+2 = \frac{a_2}{\quad} (1+t) + \frac{b_2}{\quad} (t^2-1) + c_2 t^2$$

$$\frac{F(t^2)}{\quad} = 2t + (t+1)^2 = t^2 + 4t + 1 = \frac{a_3}{\quad} (1+t) + \frac{b_3}{\quad} (t^2-1) + c_3 t^2$$

$$[F]_e^B = \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ b_1 & b_2 & b_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

calcolo $\bar{e}_1, b_1, \bar{c}_1$

$$\begin{aligned} 1 &= \bar{e}_1 + t\bar{e}_1 + t^2 b_1 - b_1 + t^2 \bar{c}_1 \\ &= \underbrace{t^2(b_1 + \bar{c}_1)} + \underbrace{t\bar{e}_1} + \underbrace{\bar{e}_1 - b_1} \end{aligned}$$

$$b_1 + \bar{c}_1 = 0 \quad \bar{e}_1 = 0 \quad \bar{e}_1 - b_1 = 1$$

$$\bar{e}_1 = 0 \quad b_1 = -1 \quad \bar{c}_1 = 1$$

Similmente si calcolano

$$\bar{e}_2, b_2, \bar{c}_2 \quad e \quad \bar{e}_3, b_3, \bar{c}_3.$$

SECONDO PODO

$$\begin{aligned} [F]_e^B &= [Id \circ F]_e^B & [F]_B^B \\ &= [Id]_e^B \cdot [F]_B^B \end{aligned}$$

$$= [Id]_e^B \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

calcolo $[Id]_e^B$.

$$\cdot \quad \text{Id}(1) = 1 = a_1 \cdot (t+1) + b_1 (t^2-1) + c_1 t^2$$

$$\cdot \quad \text{Id}(t) = t = a_2 (t+1) + b_2 (t^2-1) + c_2 t^2$$

$$\cdot \quad \text{Id}(t^2) = t^2 = a_3 (t+1) + b_3 (t^2-1) + c_3 t^2$$

$$\bullet \quad \begin{aligned} 1 &= \underbrace{(b_1+c_1)}_{=} t^2 + \underbrace{a_1}_{=} t + \underbrace{a_1-b_1}_{=} \\ t &= \underbrace{(b_2+c_2)}_{=} t^2 + \underbrace{a_2}_{=} t + \underbrace{a_2-b_2}_{=} \\ t^2 &= \underbrace{(b_3+c_3)}_{=} t^2 + \underbrace{a_3}_{=} t + \underbrace{a_3-b_3}_{=} \end{aligned} \quad \begin{cases} e_1=0 & a_1=0 \\ b_1+c_1=0 & b_1=-1 \\ a_1-b_1=1 & c_1=1 \end{cases} \quad \begin{cases} e_2=1 \\ e_2-b_2=0 & b_2=1 \\ b_2+c_2=0 & c_2=-1 \end{cases} \quad \begin{cases} e_3=0 \\ a_3-b_3=0 & b_3=0 \\ b_3+c_3=1 & c_3=1 \end{cases}$$

$$[\text{Id}]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{aligned} [F]_{\mathcal{E}}^{\mathcal{B}} &= \underbrace{[\text{Id}]_{\mathcal{E}}^{\mathcal{B}}}_{\text{matrix}} \cdot \underbrace{[F]_{\mathcal{B}}^{\mathcal{B}}}_{\text{matrix}} = \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & 1 & 4 \\ -1 & -1 & 3 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\boxed{[Id]_{\mathcal{B}}^{\mathcal{E}}} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$t+1 = \underline{1 \cdot 1} + \underline{1 \cdot t} + \underline{0 \cdot t^2}$$

$$t^2 - 1 = -1 \cdot 1 + 0 \cdot t + 1 \cdot t^2$$

$$t^2 = 0 \cdot 1 + 0 \cdot t + 1 \cdot t^2$$

$$[Id]_{\mathcal{B}}^{\mathcal{E}} \cdot [Id]_{\mathcal{E}}^{\mathcal{B}}$$

$$[Id]_{\mathcal{B}}^{\mathcal{E}} \cdot [Id]_{\mathcal{E}}^{\mathcal{B}} = [Id]_{\mathcal{B}}^{\mathcal{B}} = I$$

PROPOSIZIONE

V uno spazio vettoriale $\dim V = n$

$\underline{\sigma} : \sigma_1, \dots, \sigma_n$ una base di V

$\underline{\omega} : \omega_1, \dots, \omega_n$ una base di V

$$1) \quad [Id]_{\underline{\sigma}}^{\underline{\sigma}} = Id.$$

$$2) \quad [Id]_{\underline{\omega}}^{\underline{\omega}} \cdot [Id]_{\underline{\omega}}^{\underline{\sigma}} = Id.$$

cioè sono una l'inversa dell'altra.

dim 1) l' i -esima colonna di $[Id]_{\underline{\sigma}}^{\underline{\sigma}}$

$$[Id(\sigma_i)]_{\underline{\sigma}} = [\sigma_i]_{\underline{\sigma}} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-esimo}$$

$$\sigma_i = 0 \cdot \sigma_1 + 0 \cdot \sigma_2 + \dots + 1 \cdot \sigma_i + 0 \cdot \sigma_{i+1} + \dots + 0 \cdot \sigma_n$$

2)

$$\begin{aligned} [Id]_{\underline{\omega}}^{\underline{\omega}} \cdot [Id]_{\underline{\omega}}^{\underline{\sigma}} &= [Id \circ Id]_{\underline{\omega}}^{\underline{\sigma}} = \\ &= [Id]_{\underline{\sigma}}^{\underline{\sigma}} = Id \end{aligned}$$

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E S E R C I O

$$M = \begin{pmatrix} -5 & 12 \\ -4 & 9 \end{pmatrix}$$

$$M^{1017}$$

$$F: L_M: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

1) Esiste un vettore $v_1 \neq 0$: $F(v_1) = v_1$

$$M \cdot v_1 = v_1 \quad M \cdot v_1 - v_1 = 0$$

$$\underline{(M - I_d)} \cdot v_1 = 0 \quad v_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{\begin{pmatrix} -6 & 12 \\ -4 & 8 \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 12 \\ -4 & 9 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} -6x + 12y = 0 \\ -4x + 8y = 0 \end{cases} \quad \begin{cases} x - 2y = 0 \\ x - 2y = 0 \end{cases}$$

$$x = 2y \quad y = 1 \quad e \quad x = 2$$

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

2) Esiste un vettore $v_2 \neq 0$ con $F(v_2) = 3v_2$

è come dire $M \cdot v_2 = 3v_2$ ovvero

$$\underline{(M - 3I_d) v_2 = 0}$$

$$\begin{pmatrix} -5 & 12 \\ -4 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -8 & 12 \\ -4 & 6 \end{pmatrix}$$

$$\begin{pmatrix} -8 & 12 \\ -4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -8x + 12y = 0 \\ -4x + 6y = 0 \end{cases}$$

$$\begin{cases} 2x - 3y = 0 \\ 2x - 3y = 0 \end{cases}$$

$$2x - 3y = 0 \quad x = 3 \quad y = 2$$

$$v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$[F]_{\substack{\sigma_1 & \sigma_2 \\ \sigma_1 & \sigma_2}} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \underline{\underline{N}}$$

$$F(\sigma_1) = \sigma_1 = 1 \cdot \sigma_1 + 0 \cdot \sigma_2$$

$$F(\sigma_2) = 3\sigma_2 = 0 \cdot \sigma_1 + 3 \cdot \sigma_2$$

$$N^{1017} = \begin{pmatrix} 1^{1017} & 0 \\ 0 & 3^{1017} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3^{1017} \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}$$

$$\bullet \quad N^2 = [F]_{\substack{\sigma \\ \sigma}} \cdot [F]_{\substack{\sigma \\ \sigma}} = [F^2]_{\substack{\sigma \\ \sigma}}$$

$$N^{1017} = [F^{1017}]_{\substack{\sigma \\ \sigma}} \quad \bullet$$

$$M^{1017} = [F^{1017}]_{\substack{e_1, e_2 \\ e_1, e_2}} \quad \bullet$$

$$\begin{bmatrix} F^{1017} \\ e_1 & e_2 \end{bmatrix} = \begin{bmatrix} I_d \\ e \end{bmatrix}^{-1} \begin{bmatrix} F^{1017} \\ e \end{bmatrix} \begin{bmatrix} I_d \\ e \end{bmatrix}^{-1}$$

$$\begin{bmatrix} I_d \\ e \end{bmatrix}^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$v_1 = 2 \cdot e_1 + 1 e_2$$

$$v_2 = 3 \cdot e_1 + 2 e_2$$

$$\begin{bmatrix} I_d \\ e \end{bmatrix}^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{-1} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$ad-bc \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$M^{1017} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \cdot N^{1017} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} \textcircled{1} & 0 \\ 0 & 3^{1017} \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ -3^{1017} & 2 \cdot 3^{1017} \end{pmatrix} =$$

$$= \begin{pmatrix} 4 - 3^{1018} & -6 + 2 \cdot 3^{1018} \\ 2 - 2 \cdot 3^{1017} & -3 + 4 \cdot 3^{1017} \end{pmatrix}$$

• SECONDA SPIEGAZIONE

$$N^{1017}$$

$$M^{1017}$$

$$M = [F]_{e_1, e_2}^{e_1, e_2}$$

$$N = [F]_{\sigma_1, \sigma_2}^{\sigma_1, \sigma_2}$$

$$M = \underset{\parallel}{[Id]}_{e_1}^{\sigma_1} \cdot N \cdot \underset{\parallel}{[Id]}_{\sigma_1}^{e_1}$$

$$M^{1017} = (G \cdot N \cdot G^{-1})^{1017}$$

$$= \overbrace{G \cdot N \cdot G^{-1} \cdot \cancel{G \cdot N \cdot G^{-1}} \cdot \cancel{G \cdot N \cdot G^{-1}} \cdot \dots} \\ \dots G \cdot N \cdot G^{-1} \cdot \cancel{G \cdot N \cdot G^{-1}}$$

$$= G \cdot N^{1017} \cdot G^{-1}$$

DEFINIZIONE

SE A e B SONO MATRICI $n \times n$
 SI DICONO CONIUGATE SE
 ESISTE UNA MATRICE G INVERTIBILE
 $n \times n$ TALE CHE

$$A = G \cdot B \cdot G^{-1}$$

TEOREMA Se $X \in \mathbb{R}^{n \times n}$

$$\underline{\text{Det}(X) = \text{Det}(X^t)}$$

COROLLARIO

Tutte le proprietà che abbiamo detto l'altra volta per le righe valgono anche per le colonne.

PER ESEMPIO

$$\bullet \text{Det} \begin{pmatrix} C_1 & \dots & C_i & C_j & C_n \end{pmatrix} =$$

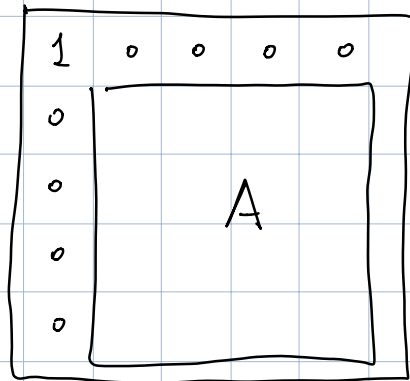
$$= - \text{Det} \begin{pmatrix} C_1 & C_j & C_i & C_n \end{pmatrix}$$

$$\bullet \text{Det} \begin{pmatrix} C_1 & C_i & C_j & C_n \end{pmatrix} =$$

$$\text{Det} \begin{pmatrix} C_1 & C_i + \alpha C_j & C_j & C_n \end{pmatrix}$$

FORMULA DI LAPLACE

$$A = (n-1) \times (n-1)$$



\bar{A}

LEMMA

$$\text{Det}_n \bar{A} = \text{Det}_{n-1} A$$

dim

$$F: \text{Mat}_{n-1 \times n-1}(K) \longrightarrow K$$

$$G: \text{Mat}_{n-1 \times n-1}(K) \longrightarrow K$$

$$F(A) = \text{Det}_{n-1}(A) \quad \bullet$$

$$G(A) = \text{Det}_n(\bar{A})$$

F è caratterizzata da queste proprietà

$$1) F(I) = 1$$

$$2) F(A) = 0 \quad \text{Se } A \text{ ha DUE RIGHE UGUALI}$$

$$3) F \text{ è lineare nelle righe}$$

SE DIMOSTRO CHE G HA LE
STESSE PROPRIETÀ ALLORA $G = F$.

$$1) G(I) = \text{Det}_n \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & I & \\ 0 & & & n-1 \end{pmatrix} = 1$$
$$\bar{I} = I_{d_n}$$

2) Se A ha due righe uguali
allora \bar{A} ha due righe uguali:

$$A = \begin{pmatrix} A_1 \\ A_i \\ A_i \\ A_{n-1} \end{pmatrix} \quad \bar{A} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & A_i & & \\ 0 & A_i & & \end{pmatrix}$$

$$G(A) = \text{Det}(\bar{A}) = 0.$$

3) Sia

$$\begin{pmatrix} A_i \end{pmatrix}$$

$$A = \begin{pmatrix} A_i + B_i \\ \vdots \\ A_{n-1} \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & A_i \\ \hline 0+0 & & & A_i + B_i \\ 0 & & & A_{n-1} \end{pmatrix}$$

$$(0 \ A_i) + (0 \ B_i) = (0 \ A_i + B_i)$$

$$\text{Det}(\bar{A}) = \text{Det} \begin{pmatrix} 1 & 0 & & 0 \\ 0 & & & A_i \\ 0 & & & A_i \\ 0 & & & A_{n-1} \end{pmatrix} + \text{Det} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & A_i \\ \vdots & & & B_i \\ 0 & & & A_{n-1} \end{pmatrix}$$

$$\parallel$$

$$G \begin{pmatrix} A_i \\ A_i + B_i \\ A_{n-1} \end{pmatrix} = G \begin{pmatrix} A_i \\ A_i \\ A_{n-1} \end{pmatrix} + G \begin{pmatrix} A_i \\ B_i \\ B_{n-1} \end{pmatrix}$$

$$G \begin{pmatrix} A_i \\ \alpha A_i \\ A_{n-1} \end{pmatrix} = \alpha G \begin{pmatrix} A_i \\ A_i \\ A_{n-1} \end{pmatrix}$$

SI VERIFICA ALLO STESSO MODO #

LEMMA

$$\text{Det} \begin{pmatrix} \boxed{A} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{B} \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \boxed{C} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{D} \end{pmatrix} \stackrel{\substack{\text{c-erie i+j} \\ \leftarrow}}{=} (-1)^{i+j} \text{Det} \begin{pmatrix} \boxed{A} & \boxed{B} \\ \boxed{C} & \boxed{D} \end{pmatrix}$$

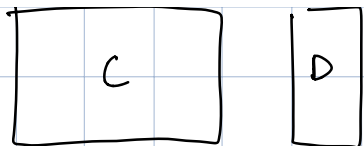
↓ j-column

dim

.	1	2	3	0	4		0	0	0	1	0	
.	5	6	7	0	8		5	6	7	0	8	
	0	0	0	1	0		1	2	3	0	4	
	9	8	6	0	4		
	2	3	7	0	9		

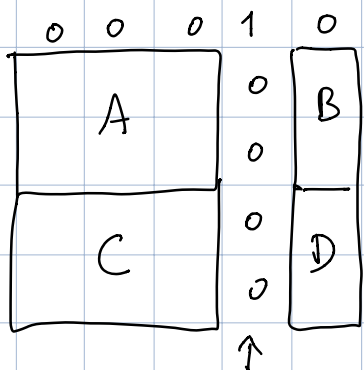
0	0	0	1	0
1	2	3	0	4
5	6	7	0	8
.
.

$$\text{Det} \begin{pmatrix} \boxed{A} & \boxed{B} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \dots$$



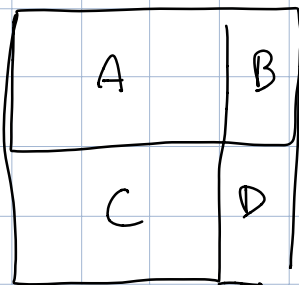
HO FATTO
 $i-1$ SCAMBII DI
 RIGHE.

$$= (-1)^{i-1} \text{Det}$$



$$= (-1)^{j-1} (-1)^{i-1} \text{Det} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \\ 0 & & \end{pmatrix} =$$

$$= (-1)^{i+j-2} \text{Det}$$



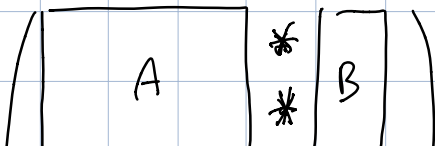
$$\parallel$$

$$(-1)^{i+j}$$

$$(-1)^2 = 1$$

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LEPNA



$$\text{Det} \begin{vmatrix} 0 & 0 & 0 & a & 0 \\ \hline & C & & & \\ \hline & & & & D \end{vmatrix} =$$

$$= (-1)^{i+j} a \text{ Det} \begin{vmatrix} & A & & B \\ \hline & C & & D \end{vmatrix}$$

glim

Se $a = 0$ è vero $0 = 0$

Se $a \neq 0$

$$\text{Det} \begin{vmatrix} & A & & B \\ \hline 0 & 0 & 0 & a & 0 \\ \hline & C & & & D \end{vmatrix} = a \text{ Det} \begin{vmatrix} & A & & B \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline & C & & & D \end{vmatrix}$$

$R_{ij}(\alpha)$ sommare e alle righe h un multiplo della riga i . non cambia il determinante.

operando in questo modo

$$\begin{vmatrix} & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \end{vmatrix}$$

$$= e \operatorname{Det} \begin{array}{|ccc|c|c|} \hline & A & & 0 & D \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline & C & & 0 & D \\ \hline & & & 0 & \\ \hline \end{array} =$$

APPLICO IL LEMMA PRECEDENTE E OTTENGO LA TESI.

#

$$\begin{array}{cccc} \left(\begin{array}{ccc} 1 & & 1 \\ 2 & & 2 \\ 3 & & \end{array} \right) & & & \begin{array}{c} i \\ 1 \\ 2 \end{array} \\ \left(\begin{array}{ccc} \dots & & \\ \dots & & \\ \dots & & \end{array} \right) & \dots & \dots & \dots \\ \left(\begin{array}{ccc} \dots & & i-2 \\ \dots & & i \\ \dots & & i-1 \end{array} \right) & \dots & \dots & \dots \\ \left(\begin{array}{ccc} \dots & i-1 & \\ \dots & i & \end{array} \right) & \dots & \dots & \dots \\ \left(\begin{array}{ccc} \dots & & \\ \dots & & \end{array} \right) & \dots & \dots & \dots \end{array}$$

SONO $i-1$ SCARBI.