$$F : V \longrightarrow W$$

$\vdots \qquad \vdots$

ESEMPIO $\quad V = \mathbb{C}[t]_{\leq 2}$

$$F(p(t)) = p'(t) + p(t+1)$$

$\mathcal{B}: \quad 1, \ t, \ t^2$

$$[F]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{C}: \quad 1+t \quad t^2-1 \quad t^2$

$$[F]^{\mathcal{B}}_{\mathcal{C}}$$

USO LA DEFINIZIONE

$\cdot \quad \underline{F(1)} = 1 = \underline{a_1} \cdot (1+t) + \underline{b_1} (t^2-1) + \underline{c_1} \, t^2$

$\underline{F(t)} = 1 + t + = t + 2 = \underline{a_2} (1+t) + \underline{b_2} (t^2-1) + c_2 t^2$

$\underline{F(t^2)} = 2t + (t+1)^2 = t^2 + 4t + 1 = a_3 (1+t) + b_3 (t^2-1) + c_3 t^2$

$\vdots$

$$[F]_{\mathcal{E}}^{\mathcal{B}} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ b_1 & b_2 & b_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

calcolo $\bar{a}_1, b_1, \bar{c}_1$

$$1 = \bar{a}_1 + t\bar{a}_1 + t^2 b_1 - b_1 + t^2 \bar{c}_1$$

$$= t^2(b_1 + \bar{c}_1) + t\bar{a}_1 + \bar{a}_1 - b_1$$

$b_1 + \bar{c}_1 = 0 \qquad \bar{a}_1 = 0 \qquad \bar{a}_1 - b_1 = 1$

$\bar{a}_1 = 0 \qquad b_1 = -1 \qquad \bar{c}_1 = 1$

SimiLMENTE SI CALCOLANO
$\bar{a}_2\ b_2\ \bar{c}_2$ e $\bar{a}_3\ b_3\ \bar{c}_3$.

SECONDO MODO

$$[F]_{\mathcal{E}}^{\mathcal{B}} = [Id \circ F]_{\mathcal{E}}^{\mathcal{B}} \qquad\qquad [F]_{\mathcal{B}}^{\mathcal{B}}$$

$$= [Id]_{\mathcal{E}}^{\mathcal{B}} \cdot [F]_{\mathcal{B}}^{\mathcal{B}}$$

$$= [Id]_{\mathcal{E}}^{\mathcal{B}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

CALCOLO $[Id]_{\mathcal{E}}^{\mathcal{B}}$.

$\cdot \quad Id(1) = 1 = a_1 \cdot (t+1) + b_1 (t^2-1) + c_1 t^2$

$\cdot \quad Id(t) = t = a_2 (t+1) + b_2 (t^2-1) + c_2 t^2$

$\cdot \quad Id(t^2) = t^2 = a_3 (t+1) + b_3 (t^2-1) + c_3 t^2$

$\bullet \quad 1 = (b_1+c_1) t^2 + \underset{=}{a_1} t + a_1 - b_1 \qquad \begin{cases} a_1 = 0 \\ b_1 + c_1 = 0 \\ a_1 - b_1 = 1 \end{cases} \qquad \begin{matrix} a_1 = 0 \\ b_1 = -1 \\ c_1 = 1 \end{matrix}$

$t = \underline{(b_2+c_2)} t^2 + \underline{a_2} t + \underline{a_2 - b_2} \qquad \begin{cases} a_2 = 1 \\ a_2 - b_2 = 0 \\ b_2 + c_2 = 0 \end{cases} \qquad \begin{matrix} \\ b_2 = 1 \\ c_2 = -1 \end{matrix}$

$t^2 = \underline{(b_3+c_3)} t^2 + \underset{=}{a_3} t + \underline{a_3 - b_3} \qquad \begin{cases} a_3 = 0 \\ a_3 - b_3 = 0 \\ b_3 + c_3 = 1 \end{cases} \qquad \begin{matrix} \\ b_3 = 0 \\ c_3 = 1 \end{matrix}$

$$[Id]_{\mathcal{E}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$[F]_{\mathcal{E}}^{\mathcal{B}} = \underline{[Id]_{\mathcal{E}}^{\mathcal{B}} \; [F]_{\mathcal{B}}^{\mathcal{B}}} =$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 4 \\ -1 & -1 & 3 \\ 1 & 1 & -2 \end{pmatrix}$$

$$[Id]_{\underline{\underline{B}}}^{\underline{\mathcal{E}}} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$t+1 = \underline{1} \cdot 1 + \underline{1} \cdot t + \underline{0} \cdot t^2$$

$$t^2 - 1 = -1 \cdot 1 + 0 \cdot t + 1 \cdot t^2$$

$$t^2 = 0 \cdot 1 + 0 \cdot t + 1 \cdot t^2$$

$$[Id]_{B}^{\mathcal{E}} \quad e \quad [Id]_{\mathcal{E}}^{B} .$$

$$[Id]_{B}^{\mathcal{E}} \cdot [Id]_{\underline{\mathcal{E}}}^{B} = [Id]_{B}^{B} = I$$

PROPOSIZIONE

V uno spazio vettoriale $\dim V = n$

$\underline{v}: v_1 \dots v_n$ una base di V

$\underline{w}: w_1 \dots w_n$ una base di V

1) $$[Id]_{\underline{v}}^{\underline{v}} = Id.$$

2) $$[Id]_{\underline{v}}^{\underline{w}} \cdot [Id]_{\underline{w}}^{\underline{v}} = Id.$$

cioè sono una l'inversa dell'altra.

dim 1)

l'i-esima colonna di $[Id]_{\underline{v}}^{\underline{v}}$

$$[Id(v_i)]_{\underline{v}} = [v_i]_{\underline{v}} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow \text{i-esimo}$$

$$v_i = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 1 \cdot v_i + 0 \cdot v_{i+1} + \dots + 0 \cdot v_n$$

2)

$$[Id]_{v}^{\underline{w}} \cdot [Id]_{\underline{w}}^{v} = [Id \circ Id]_{\underline{v}}^{\underline{v}} =$$

$$= [Id]_{\underline{v}}^{\underline{v}} = Id$$

#

ESERCIZIO

$$M = \begin{pmatrix} -5 & 12 \\ -4 & 9 \end{pmatrix}$$

$$M^{1017} \qquad F : L_M : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

1) Esiste un vettore $v_1 \neq 0$: $F(v_1) = v_1$

$$M \cdot v_1 = v_1 \qquad M \cdot v_1 - v_1 = 0$$

$$\underline{(M - Id)} \cdot v_1 = 0 \qquad v_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -6 & 12 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} -5 & 12 \\ -4 & 9 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} -6x + 12y = 0 \\ -4x + 8y = 0 \end{cases} \qquad \begin{cases} x - 2y = 0 \\ x - 2y = 0 \end{cases}$$

$$x = 2y \qquad y = 1 \quad e \quad x = 2$$

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

2) Esiste un vettore $v_2 \neq 0$ con $F(v_2) = 3v_2$

è come dire $M \cdot v_2 = 3 v_2$ ovvero

$$\underline{(M - 3Id)\, v_2 = 0}$$

$$\begin{pmatrix} -5 & 12 \\ -4 & 9 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -8 & 12 \\ -4 & 6 \end{pmatrix}$$

$$\begin{pmatrix} -8 & 12 \\ -4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -8x + 12y = 0 \\ -4x + 6y = 0 \end{cases}$$

$$\begin{cases} 2x - 3y = 0 \\ 2x - 3y = 0 \end{cases}$$

$2x - 3y = 0$ $\qquad x = 3 \qquad y = 2$

$$v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$[F]^{v_1 \; v_2}_{\underline{v_1 \; v_2}} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \underline{\underline{N}}$$

$$F(v_1) = v_1 = 1 \cdot v_1 + 0 \cdot v_2$$

$$F(v_2) = 3v_2 = 0 \cdot v_1 + 3 \cdot v_2$$

$$N^{1017} \underset{*}{=} \begin{pmatrix} 1^{1017} & 0 \\ 0 & 3^{1017} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3^{1017} \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}$$

•

$$N^2 = [F]^{\underline{v}}_{\underline{v}} \cdot [F]^{\underline{v}}_{\underline{v}} = [F^2]^{\underline{v}}_{\underline{v}}$$

$$N^{1017} = [F^{1017}]^{\underline{v}}_{\underline{v}} \quad •$$

$$M^{1017} = [F^{1017}]^{e_1 \; e_2}_{e_1 \; e_2} \quad •$$

$$\left[F^{1017}\right]_{e_1\, e_2}^{e_1\, e_2} = \underline{\left[Id\right]_{\underline{e}}^{\underline{v}}}\ \underline{\underline{\left[F^{1017}\right]_{\underline{v}}^{\underline{v}}}}\ \underline{\left[Id\right]_{\underline{v}}^{\underline{e}}}$$

$$\left[Id\right]_{\underline{e}}^{\underline{v}} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$v_1 = 2 \cdot e_1 + 1\, e_2$$
$$v_2 = 3 \cdot e_1 + 2\, e_2$$

$$\left[Id\right]_{\underline{v}}^{\underline{e}} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \underline{\frac{1}{1}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}}$$

$ad-bc \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$M^{1017} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \cdot N^{1017} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 3^{1017} \end{pmatrix} \cdot \underline{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ -3^{1017} & 2 \cdot 3^{1017} \end{pmatrix} =$$

$$= \begin{pmatrix} 4 - 3^{1018} & -6 + 2 \cdot 3^{1018} \\ 2 - 2 \cdot 3^{1017} & -3 + 4 \cdot 3^{1017} \end{pmatrix}$$

- SECONDA SPIEGAZIONE

N^{1017} $\qquad$ M^{1017}

$M = [F]_{e_1\, e_2}^{e_1\, e_2}$ $\qquad$ $N = [F]_{v_1\, v_2}^{v_1\, v_2}$

$$M = \underset{\underset{G}{\parallel}}{[Id]_{\underline{e}}^{\underline{v}}} \cdot N \cdot \underset{\underset{G^{-1}}{\parallel}}{[Id]_{\underline{v}}^{\underline{e}}}$$

$$M^{1017} = (G \cdot N \cdot G^{-1})^{1017}$$

$$= G \cdot N \cdot \cancel{G^{-1}} \cdot \cancel{G} \cdot N \cdot \cancel{G^{-1}} \cdot \cancel{G} \cdot N \cdot \cancel{G^{-1}} \cdot \dots$$

$$\dots \cancel{G} \cdot N \cdot \cancel{G^{-1}} \cdot \cancel{G} \cdot N \cdot G^{-1}$$

$$= G \cdot N^{1017} G^{-1}$$

DEFINIZIONE

SE A e B SONO MATRICI $n \times n$ SI DICONO CONIUGATE SE ESISE UNA MATRICE G INVERTIBILE $n \times n$ TALE CHE

$$A = G \cdot B \cdot G^{-1}$$

TEOREMA Se X è $n \times n$

$$\operatorname{Det}(X) = \operatorname{Det}(X^t)$$

COROLLARIO

TUTTE LE PROPRIETÀ CHE ABBIAMO DETTO L'ALTRA VOLTA PER LE RIGHE VALGONO ANCHE PER LE COLONNE.

PER ESEMPIO

- $$\operatorname{Det}\begin{pmatrix} C_1 \dots C_i & C_j & C_n \end{pmatrix} = -\operatorname{Det}\begin{pmatrix} C_1 & C_j & C_i & C_n \end{pmatrix}$$

- $$\operatorname{Det}\begin{pmatrix} C_1 & C_i & C_j & C_n \end{pmatrix} = \operatorname{Det}\begin{pmatrix} C_1 & C_i + \alpha C_j & C_j & C_n \end{pmatrix}$$

FORMULA DI LAPLACE

$A = (n-1) \times (n-1)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & A & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}$$

$$\bar{A}$$

LEMMA

$$\mathrm{Det}_n \bar{A} = \mathrm{Det}_{n-1} A$$

dim

$$F : \mathrm{Mat}_{n-1 \times n-1}(K) \longrightarrow K$$

$$G : \mathrm{Mat}_{n-1 \times n-1}(K) \longrightarrow K$$

$$F(A) = \mathrm{Det}_{n-1}(A)$$

$$G(A) = \mathrm{Det}_n(\bar{A})$$

F è caratterizzato da queste proprietà

1) $F(I) = 1$

2) $F(A) = 0$ Se A ha DUE RIGHE UGUALI

3) F è lineare nelle righe

SE DIMOSTRO CHE G HA LE STESSE PROPRIETÀ ALLORA $G = F$.

1) $$G(I_{n-1}) = Det_n \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{pmatrix} = 1$$

$$\bar{I} = Id_n$$

2) Se A ha due righe uguali allora $\bar{A}$ ha due righe uguali

$$A = \begin{pmatrix} A_1 \\ A_i \\ A_i \\ A_{n-1} \end{pmatrix} \quad \bar{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & A_i & \\ 0 & & A_i & \end{pmatrix}$$

$$G(A) = Det(\bar{A}) = 0.$$

3) Sia

$$\begin{pmatrix} A_1 \\ \end{pmatrix}$$

$$A = \begin{pmatrix} A_i + B_i \\ A_{n-1} \end{pmatrix}$$

$$\overline{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & A_1 & \\ 0+0 & & A_i + B_i & \\ \hline 0 & & A_{n-1} & \end{pmatrix}$$

$$(0 \quad A_i) + (0 \quad B_i) = (0 \quad A_i + B_i)$$

$$\mathrm{Det}(\overline{A}) = \mathrm{Det}\begin{pmatrix} 1 & 0 & & 0 \\ 0 & & A_1 & \\ 0 & & A_i & \\ 0 & & A_{n-1} & \end{pmatrix} + \mathrm{Det}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & A_1 & \\ \vdots & & B_i & \\ 0 & & A_{n-1} & \end{pmatrix}$$

$$\parallel$$

$$G\begin{pmatrix} A_1 \\ A_i + B_i \\ A_{n-1} \end{pmatrix} = G\begin{pmatrix} A_1 \\ A_i \\ A_{n-1} \end{pmatrix} + G\begin{pmatrix} A_1 \\ B_i \\ B_{n-1} \end{pmatrix}$$

$$G\begin{pmatrix} A_1 \\ \alpha A_i \\ A_{n-1} \end{pmatrix} = \alpha \, G\begin{pmatrix} A_1 \\ A_i \\ A_{n-1} \end{pmatrix}$$

SI VERIFICA ALLO STESSO MODO #

LEMMA

$$
\begin{array}{c}
\qquad\qquad\qquad\qquad j\text{-colonna} \qquad\qquad i\text{-esima riga}\\[2pt]
\mathrm{Det}\begin{pmatrix} A & \begin{matrix}0\\0\end{matrix} & B \\ \begin{matrix}1 & 0 & 0 & 0\end{matrix} & 1 & 0 \\ C & \begin{matrix}0\\0\end{matrix} & D \end{pmatrix} = (-1)^{i+j}\,\mathrm{Det}\begin{pmatrix} A & B \\ C & D \end{pmatrix}
\end{array}
$$

dim

$$
\begin{matrix}
1 & 2 & 3 & 0 & 4\\
5 & 6 & 7 & 0 & 8\\
0 & 0 & 0 & 1 & 0\\
9 & 8 & 6 & 0 & 4\\
2 & 3 & 7 & 0 & 9
\end{matrix}
\qquad\qquad
\begin{matrix}
0 & 0 & 0 & 1 & 0\\
5 & 6 & 7 & 0 & 8\\
1 & 2 & 3 & 0 & 4\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{matrix}
$$

$$
\begin{matrix}
0 & 0 & 0 & 1 & 0\\
1 & 2 & 3 & 0 & 4\\
5 & 6 & 7 & 0 & 8\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{matrix}
$$

$$
\mathrm{Det}\begin{matrix} A & & B \\ 0\ 0\ 0 & 1 & \end{matrix} =
$$

$$\begin{matrix} C & & D \end{matrix}$$

HO FATTO
$i-1$ SCAMBI DI
RIGHE.

$$= (-1)^{i-1} \operatorname{Det} \begin{array}{ccc} 0\ 0\ 0 & 1 & 0 \\ \hline A & \begin{matrix}0\\0\end{matrix} & B \\ C & \begin{matrix}0\\0\end{matrix} & D \end{array}$$

$$= (-1)^{j-1}(-1)^{i-1} \operatorname{Det} \begin{pmatrix} 1 & 0\ 0\ 0\ 0 \\ \begin{matrix}0\\0\\0\\0\end{matrix} & \begin{array}{c|c} A & B \\ \hline C & D \end{array} \end{pmatrix} =$$

$$= (-1)^{i+j-2} \operatorname{Det} \begin{array}{c|c} A & B \\ \hline C & D \end{array}$$

$$\| $$

$$(-1)^{i+j}$$

$$(-1)^2 = 1$$

#

LEMMA

$$\begin{pmatrix} A & \begin{matrix}*\\*\end{matrix} & B \end{pmatrix}$$

$$\operatorname{Det}\begin{pmatrix} & & & & \\ 0 & 0 & 0 & a & 0 \\ & C & & * & D \\ & & & * & \end{pmatrix} =$$

$$= (-1)^{i+j} a_{ij} \operatorname{Det}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

<u>elim</u>

Se $a = 0$ è vero $0 = 0$

Se $a \neq 0$

$$\operatorname{Det}\begin{pmatrix} & A & & * & B \\ & & & * & \\ 0 & 0 & 0 & a & 0 \\ & C & & * & D \\ & & & * & \end{pmatrix} = a \operatorname{Det}\begin{pmatrix} & A & & * & B \\ & & & * & \\ 0 & 0 & 0 & 1 & 0 \\ & C & & * & D \\ & & & * & \end{pmatrix}$$

$R_{hk}(\alpha)$ sommare alla riga h un multiplo della riga h. non cambia il determinante.

operando in questo modo

$$\begin{pmatrix} A & 0 & B \end{pmatrix}$$

$$= e \ \mathrm{Det} \begin{pmatrix} A & \begin{matrix} 0 \end{matrix} & B \\ 0 \quad 0 \quad 0 & 1 & 0 \\ C & \begin{matrix} 0 \\ 0 \end{matrix} & D \end{pmatrix} =$$

APPLICO IL LEMMA PRECEDENTE E OTTENGO LA TESI.

#

$$\begin{array}{ccccc} 1 & & 1 & & i \\ 2 & & 2 & & 1 \\ 3 & & & & 2 \\ & & & \dashrightarrow & \\ & & i-2 & & \\ i-1 & & i & & \\ i & & i-1 & & i-1 \end{array}$$

SONO $i-1$ SCAMBI.