Esercizi su Affinità

Titolo nota 23/05/2017

4.13 f, g rotaz. di $\pi/2$ intorno a 2 punti del piano – Mostrare che $f \circ g$ è una riflessione intorno a un punto R

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x - x_P \\ y - y_P \end{pmatrix} + \begin{pmatrix} x_P \\ y_P \end{pmatrix} = \begin{pmatrix} -y + y_P + x_P \\ x - x_P + y_P \end{pmatrix}$$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x - x_Q \\ y - y_Q \end{pmatrix} + \begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = \begin{pmatrix} \boxed{-y + y_Q + x_Q} \\ \boxed{x - x_Q + y_Q} \end{pmatrix}$$

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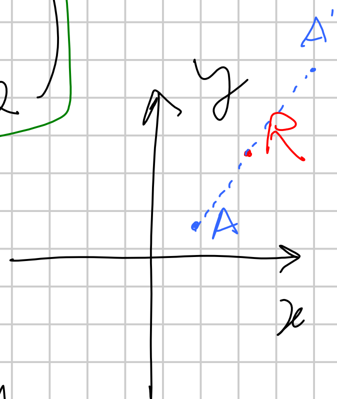
$$(f \circ g)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \boxed{-x + x_Q - y_Q} + y_P + x_P \\ -y + y_Q + x_Q - x_P + y_P \end{pmatrix} =$$

$$\boxed{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix} + \boxed{\begin{pmatrix} x_P + x_Q + y_P - y_Q \\ -x_P + x_Q + y_P + y_Q \end{pmatrix}}$$

$$\frac{x_A + x_{A'}}{2} = x_R \qquad \frac{y_A + y_{A'}}{2} = y_R$$



$$x_{A'} = 2x_R - x_A \qquad y_{A'} = 2y_R - y_A$$

$$S_R\begin{pmatrix} x \\ y \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix} + \boxed{\begin{pmatrix} 2x_R \\ 2y_R \end{pmatrix}}$$

$$x_R = \frac{x_P + x_Q + y_P - y_Q}{2} \qquad y_R = \frac{-x_P + x_Q + y_P + y_Q}{2}$$

In questo modo, abbiamo anche determinato R

$$R_P = \left(\begin{array}{cc|c} 0 & -1 & x_P + y_P \\ 1 & 0 & -x_P + y_P \\ \hline 0 & 0 & 1 \end{array}\right) \qquad R_Q = \left(\begin{array}{cc|c} 0 & -1 & x_Q + y_Q \\ 1 & 0 & -x_Q + y_Q \\ \hline 0 & 0 & 1 \end{array}\right)$$

$f \circ g$ ha come matrice associata $R_P \cdot R_Q$

$$\begin{pmatrix} 0 & -1 & x_P + y_P \\ 1 & 0 & -x_P + y_P \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{cc|c} 0 & -1 & x_Q + y_Q \\ 1 & 0 & -x_Q + y_Q \\ \hline 0 & 0 & 1 \end{array}\right) =$$

$$\begin{pmatrix} -1 & 0 & +x_Q - y_Q + x_P + y_P \\ 0 & -1 & x_Q + y_Q - x_P + y_P \\ 0 & 0 & 1 \end{pmatrix}$$

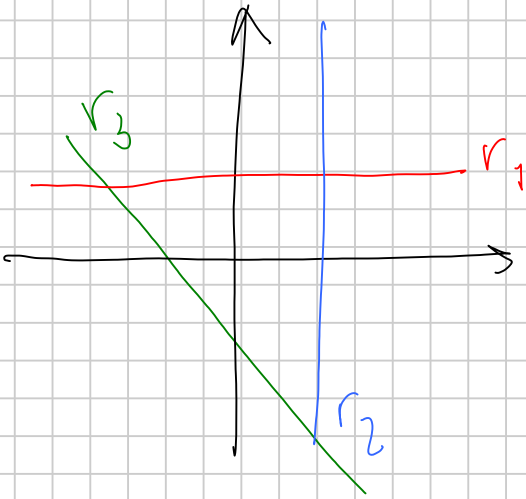
$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \boxed{\begin{pmatrix} x_Q + y_Q \\ -x_Q + y_Q \end{pmatrix}}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \end{pmatrix} \qquad \left(\begin{array}{c|c} A & b \\ \hline 0 \; 0 & 1 \end{array}\right)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

4.15 $\quad f(x) = Ax + b \qquad r_1 \quad r_2 \quad r_3$

$f(r_1) = r_2 \qquad f(r_2) = r_3 \qquad f(r_3) = r_1$

$r_1 : \; y = 1 \qquad r_2 : \; x = 1 \qquad r_3 : \; x + y = -1$



$$r_1: \begin{cases} x = t \\ y = 1 \end{cases} \qquad r_2: \begin{cases} x = 1 \\ y = u \end{cases} \qquad \begin{matrix} t \in \mathbb{R} \\ u \in \mathbb{R} \end{matrix}$$

$\forall t \in \mathbb{R}\ \exists u$ tale che

$$* \quad f\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ u \end{pmatrix} \qquad f(r_1) = r_2$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ax + by + e \\ cx + dy + f \end{pmatrix}$$

$\boxed{a, b, c, d, e, f \text{ costanti}}$

$$\begin{cases} a \cdot t + b \cdot 1 + e = 1 \\ ct + d \cdot 1 + f = u \end{cases} \qquad \forall t$$

devo trovare u dato t

$$at + (b + e - 1) = 0 \quad \forall t \quad \Rightarrow \quad \begin{matrix} a = 0 \\ b + e = 1 \end{matrix}$$

$\forall u \in \mathbb{R}$ devo trovare s tale che

$$f\begin{pmatrix} 1 \\ u \end{pmatrix} = \begin{pmatrix} s \\ -1-s \end{pmatrix} \qquad r_3: \begin{cases} x = s \\ y = -1-s \end{cases}$$

$$\begin{cases} a + bu + e = s \\ c + du + f = -1-s \end{cases} \quad \forall s \quad \Rightarrow \quad c + du + f = -1 - bu - e - a$$

$$(d+b)u + c + f + a + e + 1 = 0 \quad \forall u$$

$$d + b = 0 \qquad c + f + a + e = -1$$

$\forall s$ devo trovare t tale che

$$f\begin{pmatrix} s \\ -1-s \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix} \qquad \forall s \begin{cases} as + b(-1-s) + e = t \\ cs + d(-1-s) + f = 1 \end{cases}$$

$$(c-d)s + f - d = 1 \quad \forall s \qquad \begin{matrix} c - d = 0 \\ f - d = 1 \end{matrix}$$

$$\begin{cases} a=0 \\ b+e=1 \\ b+d=0 \\ c+f+a+e=-1 \\ c-d=0 \\ f-d=1 \end{cases}$$

$a=0$

$c=d$

$b=-d$

$e=1+d$

$f=1+d$

$d+1+d+0+1+d=-1$

$3d=-3 \qquad d=-1$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \cancel{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \begin{pmatrix} y \\ -x-y \end{pmatrix}$$

$$\underbrace{f\begin{pmatrix} t \\ 1 \end{pmatrix}}_{\in r_1} = \underbrace{\begin{pmatrix} 1 \\ -1-t \end{pmatrix}}_{\in r_2} \; u \qquad \underbrace{f\begin{pmatrix} 1 \\ u \end{pmatrix}}_{\in r_2} = \underbrace{\begin{pmatrix} u \\ -1-u \end{pmatrix}}_{\in r_3}$$

$$\underbrace{f\begin{pmatrix} s \\ -1-s \end{pmatrix}}_{\in r_3} = \underbrace{\begin{pmatrix} -1-s \\ 1 \end{pmatrix}}_{\in r_1}$$

4.16

$f(x)=Ax+b$

$f(r)=r'$

senza punti fissi

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$r: \; y=x \qquad r': \; y=-x$

$p=\begin{pmatrix} t \\ t \end{pmatrix} \qquad p'=\begin{pmatrix} u \\ -u \end{pmatrix}$

$\forall t \in \mathbb{R} \quad \exists \, (;) \; u$ tale che $\begin{cases} at+bt+e=u \\ ct+dt+f=-u \end{cases}$

$(a+b+c+d)t+e+f=0 \quad \forall t$

$f = -e \qquad a+b+c+d=0$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & -(a+b+c) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ -e \end{pmatrix}$$

$-a(a+b+c) - bc \neq 0$ (invertibilità)

$F = \begin{pmatrix} x_F \\ y_F \end{pmatrix} \qquad f\begin{pmatrix} x_F \\ y_F \end{pmatrix} = \begin{pmatrix} x_F \\ y_F \end{pmatrix}$

$$\begin{cases} a x_F + b y_F + e = x_F \\ c x_F - (a+b+c) y_F - e = y_F \end{cases}$$

NON DEVE AVERE SOLUZIONI!

$$\begin{cases} (a-1) x_F + b y_F = -e \\ c x_F - (a+b+c+1) y_F = e \end{cases}$$

$rk(A) = 1$

$$\begin{cases} -(a-1)(a+b+c+1) - bc = 0 \\ (a-1)e + ce \neq 0 \end{cases}$$

$rk(A') = 2$

Rouché Capelli

$e(a+c-1) \neq 0$

$e \neq 0 \qquad e = 1$ ad esempio $\qquad b = 0$

$a + c \neq 1 \qquad a = 1$

$c = 1$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ x-2y-1 \end{pmatrix} \qquad f\begin{pmatrix} t \\ t \end{pmatrix} = \begin{pmatrix} t+1 \\ -(t+1) \end{pmatrix}$$

$$\begin{cases} x+1 = x \\ x-2y-1 = y \end{cases}$$ effettivamente non ha soluzioni

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4.17 $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x-1 \\ y+1 \\ z-2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$

$$= \begin{pmatrix} (x-1)\cos\alpha - (y+1)\sin\alpha + 1 \\ (x-1)\sin\alpha + (y+1)\cos\alpha - 1 \\ z+b \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} +$$

$$+ \begin{pmatrix} 1-\cos\alpha-\sin\alpha \\ -1+\cos\alpha-\sin\alpha \\ b \end{pmatrix}$$

Imponiamo $f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1-\cos\alpha-\sin\alpha \\ -1+\cos\alpha-\sin\alpha \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} 1-\cos\alpha-\sin\alpha = 2 \\ -1+\cos\alpha-\sin\alpha = -2 \\ (\cos^2\alpha+\sin^2\alpha=1) \\ b+1 = 2 \end{cases}$$

$b=1$ $\quad \begin{cases} \cos\alpha + \sin\alpha = -1 \\ \cos\alpha - \sin\alpha = -1 \end{cases}$ $\quad \begin{array}{l} \sin\alpha = 0 \\ \cos\alpha = -1 \end{array}$ OK! $\alpha = \pi$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

4.18

$x=0 \quad \pi_1$

$y=0 \quad \pi_2$

Isometria senza punti fissi t.c. $f(\pi_1) = \pi_2$

rotaz. di $\frac{\pi}{2}$ intorno all'asse z + traslaz. lungo z

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -y \\ x \\ z+1 \end{pmatrix}$$

$P \in \pi_1$ $\begin{pmatrix} 0 \\ t \\ u \end{pmatrix}$ $P' \in \pi_2$ $\begin{pmatrix} v \\ 0 \\ w \end{pmatrix}$

$$f\begin{pmatrix} 0 \\ t \\ u \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ u+1 \end{pmatrix} \nwarrow \in \pi_2$$

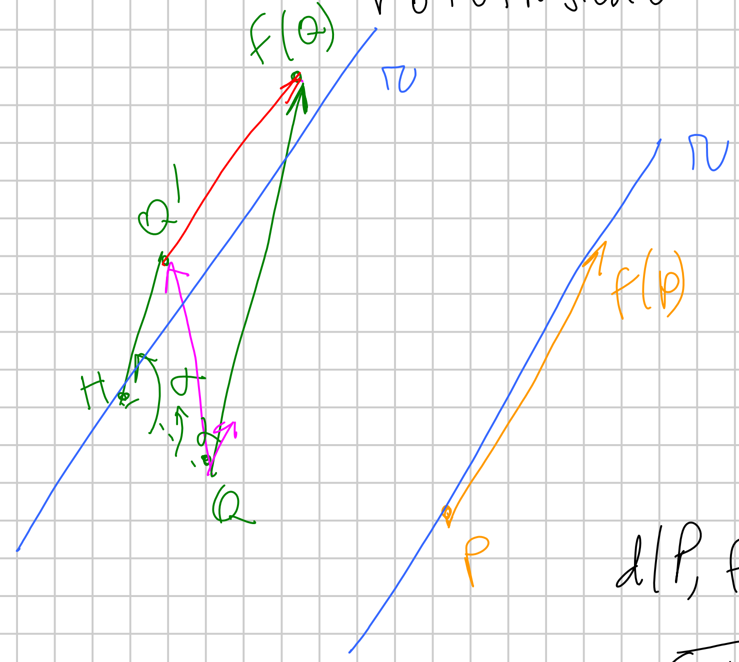
$v = -t$

$w = u + 1$

$$\begin{cases} x = -y \\ y = x \\ z = z+1 \end{cases}$$

impossibile!

4.19 $f : \mathbb{R}^3 \to \mathbb{R}^3$ rototraslazione asse r



$d(P, f(P)) =$

$= d(Q', f(Q))$ diseg. triangolare

$d(P, f(P)) = d(Q', f(Q)) \leq$

$\underbrace{d(Q, Q')}_{>0} + d(Q, f(Q))$

$d(P, f(P)) < d(Q, f(Q))$

TEOREMA SPETTRALE

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ simmetrica verificare che $\exists$ una base ortogonale di autovettori

$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ non è simm. la base di autovettori (se $\exists$) non è ortogonale

$$\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{3 \pm \sqrt{5}}{2}\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} 2x + y = \frac{3 \pm \sqrt{5}}{2} x \\ x + y = \frac{3 \pm \sqrt{5}}{2} y \end{cases}$$

$$\begin{cases} \left(2 - \frac{3}{2} \mp \frac{\sqrt{5}}{2}\right) x + y = 0 \\ x + \left(1 - \frac{3}{2} \mp \frac{\sqrt{5}}{2}\right) y = 0 \end{cases} \qquad \begin{cases} (1 \mp \sqrt{5})\, x + 2y = 0 \\ 2x + (-1 \mp \sqrt{5})\, y = 0 \end{cases}$$

$\begin{pmatrix} -2 \\ 1 - \sqrt{5} \end{pmatrix}$ $\begin{pmatrix} -2 \\ 1 + \sqrt{5} \end{pmatrix}$ ORTOGONALI

OK

$$\begin{pmatrix} -2 & 1-\sqrt{5} \end{pmatrix}\begin{pmatrix} -2 \\ 1+\sqrt{5} \end{pmatrix} = 4 + 1 - 5 = 0$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{3 \pm 1}{2} \begin{cases} 1 \\ 2 \end{cases}$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ autovettore con $\lambda = 1$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ autovettore con $\lambda = 2$

non sono ortogonali

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot 1 + 0 \cdot 1 = 1 \neq 0$$

5.2 $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

$$(1-\lambda)^2 - 1 = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 2$$

$\begin{pmatrix} 1 \\ i \end{pmatrix}$ autovettore con $\lambda_1 = 0$

$\begin{pmatrix} 1 \\ -i \end{pmatrix}$ autovettore con $\lambda_2 = 2$

ortogonali

$$\begin{pmatrix} 1 & i \end{pmatrix}^* \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = 1 - 1 = 0$$