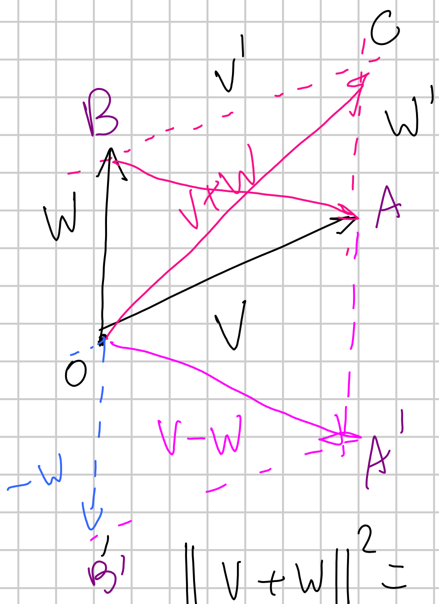
ESERCIZI VARI SU BASI ORTOGONALI

Note Title 02/05/2017

3,22 $\quad \|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right)$



BA ∥ OA'

la somma dei quadrati delle diagonali di un parallelogramma è = alla somma dei quadrati dei lati

$$\|v+w\|^2 = (v+w)\cdot(v+w) = \|v\|^2 + \|w\|^2 + 2\, v\cdot w$$

$$\|v-w\|^2 = (v-w)\cdot(v-w) = \|v\|^2 + \|w\|^2 - 2\, v\cdot w$$

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2 + \cancel{2\, v\cdot w} - \cancel{2\, v\cdot w}$$

$$= 2\left(\|v\|^2 + \|w\|^2\right)$$

3,25 $\quad \mathbb{R}^3, \cdot \qquad U = \operatorname{span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right) =$

$$= \alpha\begin{pmatrix}1\\1\\0\end{pmatrix} + \beta\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}\alpha+\beta\\ \alpha\\ \beta\end{pmatrix} \qquad \alpha, \beta \in \mathbb{R}, \text{ qualsiasi}$$

$$u_1 = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\\ 0\end{pmatrix} \qquad \begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}\alpha+\beta\\ \alpha\\ \beta\end{pmatrix} = 0 \Rightarrow \alpha + (\beta+\alpha) + 0 = 0$$

$$2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha$$

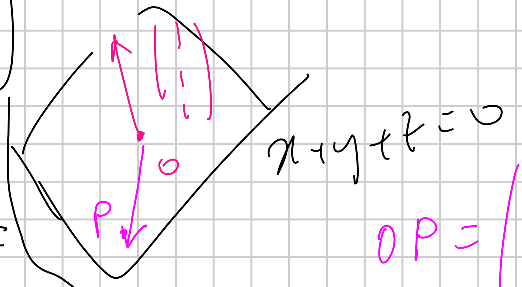
$$\begin{pmatrix}\alpha + (-2\alpha)\\ \alpha\\ -2\alpha\end{pmatrix} = \begin{pmatrix}-\alpha\\ \alpha\\ -2\alpha\end{pmatrix} = \alpha\begin{pmatrix}-1\\ 1\\ -2\end{pmatrix}$$

$$\left\|\alpha\begin{pmatrix}-1\\1\\-2\end{pmatrix}\right\|^2 = \alpha^2\left((-1)^2+(+1)^2+(-2)^2\right) = 6\alpha^2$$

Impongo $6\alpha^2 = 1 \to \alpha = \pm\frac{1}{\sqrt{6}}$

$$u_2 = \begin{pmatrix}1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\end{pmatrix}$$

$$W = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 : x+y+z=0 \right\}$$



$$g = \begin{pmatrix}1\\1\\1\end{pmatrix} \perp W$$

$$OP = \begin{pmatrix}x_p\\y_p\\z_p\end{pmatrix}$$

$$\vec{OP}\cdot g = \begin{pmatrix}x_p\\y_p\\z_p\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} =$$

$= x_p + y_p + z_p = 0$ perché $P \in$ piano

$$\hat{g} = \frac{g}{\|g\|} = \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$l = m = n = 1/\sqrt{3}$$

$$V_\alpha = \left(\frac{1}{3}\sin\alpha + \frac{1}{\sqrt{3}}\cos\alpha\right)e_1 + \left(\frac{1}{3}\sin\alpha - \frac{1}{\sqrt{3}}\cos\alpha\right)e_2$$
$$\left(-2/3 \sin\alpha\right)e_3$$

$$\|V_\alpha\|^2 = \frac{2}{9}\sin^2\alpha + \frac{4}{9}\sin^2\alpha + \frac{2}{3}\cos^2\alpha = 2/3$$

$$\|V_\alpha\| = \sqrt{2/3}$$

$$\hat{V}_\alpha = \frac{V_\alpha}{\|V_\alpha\|} = \begin{pmatrix}\frac{1}{\sqrt{6}}\sin\alpha + \frac{1}{\sqrt{2}}\cos\alpha\\ \frac{1}{\sqrt{6}}\sin\alpha - \frac{1}{\sqrt{2}}\cos\alpha\\ -\sqrt{\frac{2}{3}}\sin\alpha\end{pmatrix}$$

Basta scegliere 2 valori di α che differiscono di $\pi/2$

$\alpha_1 = 0$ $\quad w_1 = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{pmatrix}$ $\quad \alpha_2 = \frac{\pi}{2}$ $\quad w_2 = \begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix}$

$v_1 \in U \cap W \qquad v_2 \in U \qquad v_3 \in W$ base di $\mathbb{R}^3$ ortonormale

$$\alpha = \alpha'/\sqrt{2} \qquad \beta = \beta'/\sqrt{6}$$

$$w \in W \quad \alpha w_1 + \beta w_2 = \alpha' \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta' \begin{pmatrix} 1 \\ +1 \\ -2 \end{pmatrix}$$

$$u \in U \quad \gamma u_1 + \delta w_2 = \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha' + \beta' \\ -\alpha' + \beta' \\ -2\beta' \end{pmatrix} = \begin{pmatrix} \gamma + \delta \\ \gamma \\ \delta \end{pmatrix} \qquad \delta = -2\beta'$$

$$\begin{cases} \alpha' + \beta' = \gamma - 2\beta' \\ -\alpha' + \beta' = \gamma \end{cases}$$

$$\begin{cases} \alpha' + 3\beta' = \gamma \\ -\alpha' + \beta' = \gamma \end{cases} \qquad 4\beta' = 2\gamma \qquad \gamma = 2\beta'$$

$$\alpha' = 2\beta' - 3\beta' = -\beta'$$

$$\begin{pmatrix} -\beta' + \beta' \\ \beta' + \beta' \\ -2\beta' \end{pmatrix} = \begin{pmatrix} 0 \\ 2\beta' \\ -2\beta' \end{pmatrix} = c \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \in U \cap W$$

primo vettore della base di $\mathbb{R}^3$

$v_2 \in U \qquad v_2 \perp v_1 \qquad \|v_2\| = 1$

$$\left(\delta' \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma' \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$\begin{pmatrix} \delta' + \gamma' \\ \delta' \\ \gamma' \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0 \Rightarrow \delta' - \gamma' = 0 \Rightarrow \delta' = \gamma'$$

$\delta'\begin{pmatrix}2\\1\\1\end{pmatrix}$ è la forma richiesta del vettore richiesto, basta normalizzare

$$\left\|\delta'\begin{pmatrix}2\\1\\1\end{pmatrix}\right\|^2=1 \quad = \quad \delta'^2(4+1+1)=1 \Rightarrow \delta'=\pm\frac{1}{\sqrt6}$$

secondo vettore di base

$$V_2=\begin{pmatrix}2/\sqrt6\\1/\sqrt6\\1/\sqrt6\end{pmatrix}$$

$$\left[\alpha'\begin{pmatrix}1\\-1\\0\end{pmatrix}+\beta'\begin{pmatrix}1\\1\\-2\end{pmatrix}\right]\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0$$

$$\left[\alpha'\begin{pmatrix}1\\-1\\0\end{pmatrix}+\beta'\begin{pmatrix}1\\1\\-2\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}=0$$

$$\begin{pmatrix}\alpha'+\beta'\\-\alpha'+\beta'\\-2\beta'\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0 \Rightarrow -\alpha'+\beta'+2\beta'=0$$

$$\begin{pmatrix}\alpha'+\beta'\\-\alpha'+\beta'\\-2\beta'\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}=0 \Rightarrow 2\alpha'+\cancel{2\beta'}-\alpha'+\beta'-\cancel{2\beta'}=0$$

$$\begin{cases}-\alpha'+3\beta'=0\\ \alpha'+\beta'=0\end{cases}$$

soluz. banale non accettabile

Proposta di una base

$$\frac{1}{\sqrt2}\begin{pmatrix}0\\-1\\1\end{pmatrix} \qquad \begin{pmatrix}2\\1\\1\end{pmatrix}\frac{1}{\sqrt6} \qquad \begin{pmatrix}-1\\1\\1\end{pmatrix}\frac{1}{\sqrt3}$$

$x+y+z=0$

$-1+1+1=1\neq0$

$\in U\cap W$ $\qquad$ $\in U$ $\qquad$ $\notin W$

$\dim(U\cap W)=1$

3.28 $\mathbb{R}^4$ $U = \text{Span}\left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right) \subset \mathbb{R}^4$

- determinare $U^\perp$ $\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix} \in \mathbb{R}^4$, qualsiasi

$\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix} \perp u \in U$, qualsiasi - Basta imporre questa condizione sui vettori che generano lo span

$\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = 0 \Rightarrow \alpha + \beta + \gamma + \delta = 0$

da mettere insieme in un sistema

$\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\\0\end{pmatrix} = 0 \Rightarrow \alpha + \gamma = 0$

$$\begin{cases}\alpha + \beta + \delta + \gamma = 0\\ \alpha + \gamma = 0\end{cases} \Rightarrow \begin{cases}\beta + \delta = 0\\ \alpha + \gamma = 0\end{cases}$$

$\delta = -\beta \quad \gamma = -\alpha$

$w \in U^\perp$ è del tipo $\begin{pmatrix}\alpha\\\beta\\-\alpha\\-\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$

$U^\perp = \text{Span}\left(\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right)$

base di $U = \left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\}$ che è una base ortogonale base di $U^\perp = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

base di $\mathbb{R}^4$ ortonormale

$$\left\{ \underset{f_1}{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{f_2}{\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}}, \underset{f_3}{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}, \underset{f_4}{\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}} \right\}$$

$$e_1 = \sum_{i=1}^{4} (e_1 \cdot f_i) f_i = \frac{1}{\sqrt{2}} f_1 + 0 f_2 + \frac{1}{\sqrt{2}} f_3 + 0 \cdot f_4$$
$$= \frac{f_1 + f_3}{\sqrt{2}}$$

$$e_3 = \sum_{i=1}^{4} (e_3 \cdot f_i) f_i = \frac{1}{\sqrt{2}} f_1 + 0 f_2 - \frac{1}{\sqrt{2}} f_3 + 0 f_4 =$$
$$= \frac{f_1 - f_3}{\sqrt{2}}$$

3.30 in 2 modi diversi $\qquad U = \{x + y + z = 0\}$

P_U proiezione ortogonale su U _ Scrivere la matrice rispetto alla b.c. di $\mathbb{R}^3$

$e_1 \; e_2 \; e_3$ $\qquad f_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \quad f_2 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix} \quad f_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$\{f_1 \; f_2\}$ base di U $\qquad f_3 \perp f_1 \quad f_3 \perp f_2$

$\vec{OP} = x_1 e_1 + x_2 e_2 + x_3 e_3 \qquad P_U(\vec{OP}) = ?$

$= x_1' f_1 + x_2' f_2 + x_3' f_3 \qquad P_U(\vec{OP}) = x_1' f_1 + x_2' f_2$

$$f_1 = \frac{e_1 - e_2}{\sqrt{2}} \qquad f_2 = \frac{e_1 + e_2 - 2e_3}{\sqrt{6}} \qquad f_3 = \frac{e_1 + e_2 + e_3}{\sqrt{3}}$$

$$e_1 = \frac{f_1}{\sqrt{2}} + \frac{f_2}{\sqrt{6}} + \frac{f_3}{\sqrt{3}} \qquad e_2 = -\frac{f_1}{\sqrt{2}} + \frac{f_2}{\sqrt{6}} + \frac{f_3}{\sqrt{3}}$$

$$e_3 = -\frac{2f_2}{\sqrt{6}} + \frac{f_3}{\sqrt{3}}$$

$$\vec{OP} = x_1 e_1 + x_2 e_2 + x_3 e_3 = x_1\left(\frac{f_1}{\sqrt{2}} + \frac{f_2}{\sqrt{6}} + \frac{f_3}{\sqrt{3}}\right) +$$

$$+ x_2\left(-\frac{f_1}{\sqrt{2}} + \frac{f_2}{\sqrt{6}} + \frac{f_3}{\sqrt{3}}\right) + x_3\left(-\frac{2f_2}{\sqrt{6}} + \frac{f_3}{\sqrt{3}}\right) =$$

$$= \left(\frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}\right) f_1 + \left(\frac{x_1}{\sqrt{6}} + \frac{x_2}{\sqrt{6}} - \frac{2x_3}{\sqrt{6}}\right) f_2 + \left(\frac{x_1 + x_2 + x_3}{\sqrt{3}}\right) f_3$$

$$P_U(\vec{OP}) = \left(\frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}\right) f_1 + \left(\frac{x_1 + x_2 - 2x_3}{\sqrt{6}}\right) f_2 =$$

$$\frac{1}{\sqrt{2}}(x_1 - x_2)\frac{(e_1 - e_2)}{\sqrt{2}} + \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3)\frac{e_1 + e_2 - 2e_3}{\sqrt{6}} =$$

$$\frac{1}{2}(x_1 e_1 - x_1 e_2 - x_2 e_1 + x_2 e_2) + \frac{1}{6}(x_1 e_1 + x_1 e_2 - 2x_1 e_3)$$

$$+ \frac{1}{6}(x_2 e_1 + x_2 e_2 - 2x_2 e_3) + \frac{1}{6}(-2x_3 e_1 - 2x_3 e_2 + 4x_3 e_3)$$

$$e_1\left(\frac{x_1}{2} - \frac{x_2}{2} + \frac{x_1}{6} + \frac{x_2}{6} - \frac{x_3}{3}\right) + e_2\left(-\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{6}x_1 + \frac{1}{6}x_2 - \frac{1}{3}x_3\right)$$

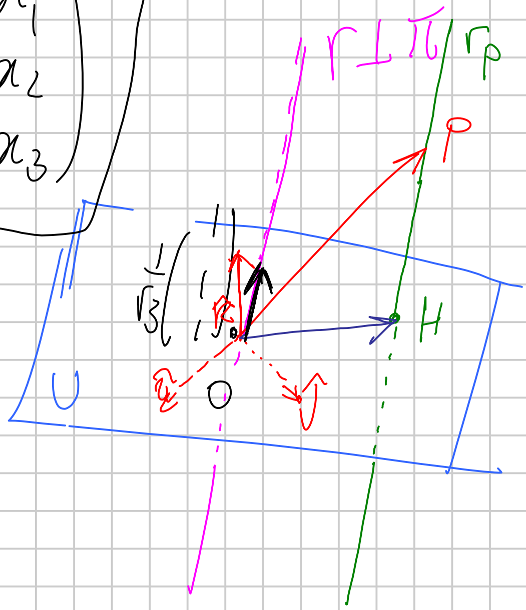
$$+ e_3\left(-\frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{2}{3}x_3\right) =$$

$$\frac{e_1}{3}\left(2x_1 - x_2 - x_3\right) + \frac{e_2}{3}\left(-x_1 + 2x_2 - x_3\right) +$$

$$+ \frac{e_3}{3}\left(-x_1 - x_2 + 2x_3\right)$$

$$\left[\frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\right]^2 = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$P_U\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



$$\overrightarrow{OP} = x_1 e_1 + x_2 e_2 + x_3 e_3$$
$$= x_p \hat{\imath} + y_p \hat{\jmath} + z_p \hat{k}$$

$$\overrightarrow{OH} = P_U\left(\overrightarrow{OP}\right)$$

$$r: \begin{cases} x = 1/\sqrt{3}\, t \\ y = 1/\sqrt{3}\, t \\ z = 1/\sqrt{3}\, t \end{cases} \qquad r_p: \begin{cases} x = \frac{1}{\sqrt{3}} t + x_p \\ y = \frac{1}{\sqrt{3}} t + y_p \\ z = \frac{1}{\sqrt{3}} t + z_p \end{cases}$$

$t \in \mathbb{R}$

$$H = r_p \cap U \qquad U = \{x + y + z = 0\}$$

$$\left(\frac{1}{\sqrt{3}}\bar{t} + x_p\right) + \left(\frac{1}{\sqrt{3}}\bar{t} + y_p\right) + \left(\frac{1}{\sqrt{3}}\bar{t} + z_p\right) = 0$$

$$\sqrt{3}\,\bar{t} + x_p + y_p + z_p = 0$$

$$\bar{t} = -\frac{x_p + y_p + z_p}{\sqrt{3}}$$

$$\begin{cases} x_H = \frac{1}{\sqrt{3}}\bar{t} + x_p \\ y_H = \frac{1}{\sqrt{3}}\bar{t} + y_p \\ z_H = \frac{1}{\sqrt{3}}\bar{t} + z_p \end{cases}$$

$$\begin{cases} x_H = -\frac{x_p + y_p + z_p}{3} + x_p \\ y_p = -\frac{x_p + y_p + z_p}{3} + y_p \\ z_p = -\frac{x_p + y_p + z_p}{3} + z_p \end{cases}$$

$$\begin{cases} x_H = \frac{2x_p - y_p - z_p}{3} \\ y_H = \frac{-x_p + 2y_p - z_p}{3} \\ z_H = \frac{-x_p - y_p + 2z_p}{3} \end{cases}$$

Stesso risultato di prima!

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

1 è autovalore (reale)

l'autovettore corrisp. a +1 dà la direz. dell'asse di rotaz.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\begin{cases} \beta = \alpha \\ -\gamma = \beta \\ -\alpha = \gamma \end{cases} \quad \alpha^2 + \beta^2 + \gamma^2 = 1$$

$\alpha = \beta = \frac{1}{\sqrt{3}}$ $\quad \gamma = -\frac{1}{\sqrt{3}}$ ad esempio

direz. dell'asse di rotaz. è $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\quad \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$$A' = \begin{pmatrix} \boxed{\begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

$$M^{-1} A M = A'$$

ortogonali

$$\mathrm{tr}(A) = \mathrm{tr}(A') = 1 + 2\cos\theta \leftarrow \text{angolo di rotazione intorno a } \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\cos\theta = \frac{\mathrm{tr}(A) - 1}{2} = \frac{0-1}{2} = -\frac{1}{2}$$

$$\cos\theta = -\frac{1}{2} \qquad \theta = \pm\frac{2\pi}{3}$$

3.35 rotaz. R di $\frac{2\pi}{3}$ intorno a $\begin{pmatrix}1\\1\\1\end{pmatrix}$

$R^3 = I \qquad R^3 - I = 0 \qquad$ provate ....