PRODOTTI SCALARI - SEGNATURA

Note Title 11/04/2017

3.13 $\quad S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \overset{p}{1} \; \overset{p}{1} \quad (2,0,0)$

$\langle \;\; \rangle : \mathbb{R}^2 \to \mathbb{R}$

$\{e_1, ?\}$

$\uparrow$

$e_1 - e_2$

$\langle \alpha e_1 + \beta e_2, e_1 \rangle = 0$

$\alpha \langle e_1, e_1 \rangle + \beta \langle e_2, e_1 \rangle = 0$

$\alpha \cdot 1 + \beta \cdot 1 = 0$

$\beta = -\alpha$ ad esempio

$\beta = -1 \quad \alpha = 1$

$S' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \langle e_1 - e_2, e_1 - e_2 \rangle = 1 + 2 - 1 - 1 = 1$

Controlliamo che $S' = {}^tMSM \quad M = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$S \begin{pmatrix} \boxed{0} & -1 \\ -1 & 0 \end{pmatrix}$ $\quad \det(S) = -1 \neq 0 \implies i_0 = 0$

$(2 \; 0 \; 0)$

$(0 \; 2 \; 0)$?

$(1 \; 1 \; 0) \leftarrow$

$\langle e_1 - e_2, e_1 - e_2 \rangle = 0 - (-1) - (-1) + 0 = +2$

$\langle \alpha e_1 + \beta e_2, e_1 - e_2 \rangle = 0 \implies \alpha - \beta = 0$

$\alpha = \beta = 1 \longrightarrow e_1 + e_2 \qquad \{e_1 - e_2, e_1 + e_2\}$

$S' = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $\quad \langle e_1 + e_2, e_1 + e_2 \rangle = 0 - 1 - 1 + 0 = 2$

$S' = {}^tM S M \qquad M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ segnatura $(1, 1, 0)$ OK!

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$\langle \quad \rangle : \mathbb{R}^3 \to \mathbb{R} \qquad S \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$\{e_1 - e_3,\ e_2,\ e_1 + e_3\}$

$S' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

segnatura $= (2, 1, 0)$

3.14 $\quad \langle \quad \rangle : \mathbb{R}^3 \to \mathbb{R} \qquad S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & t \end{pmatrix}$

$i_0 \geq 1$ perché $\det(S) = 0$

$i_0 < 3$ perché $\langle e_1, e_1 \rangle = 1 > 0$

$i_+ \geq 1$

$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 2 & 0 & 1 \end{pmatrix}$?
$\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$

$\{e_1,\ e_1 - e_2,\ e_1 - e_3\}$?

$\alpha e_1 + \beta e_2 + \gamma e_3\}$

$\langle e_1, \alpha e_1 + \beta e_2 + \gamma e_3 \rangle = 0$

$\boxed{\alpha + \beta + \gamma = 0}$

$\alpha = 1 \quad \beta = -1 \quad \gamma = 0$

$\langle \alpha' e_1 + \beta' e_2 + \gamma' e_3, e_1 \rangle = 0 \to \alpha' + \beta' + \gamma' = 0$

$\langle \boxed{\alpha' e_1 + \beta' e_2 + \gamma' e_3}, e_1 - e_2 \rangle = 0 \to 0 = 0$ INUTILE!

$\alpha' = 1 \quad \beta' = 0 \quad \gamma' = -1$

$S' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & ? \end{pmatrix}$

$\langle e_1 - e_2, e_1 - e_2 \rangle = 1 - 1 - 1 + 1 = 0$

$\langle e_1 - e_3, e_1 - e_3 \rangle = 1 - 1 - 1 + t = t - 1$

$$S' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t-1 \end{pmatrix}$$

$t = 1 \quad (1\ 0\ 2)$

$t > 1 \quad (2\ 0\ 1)$

$t < 1 \quad (1\ 1\ 1)$

$\langle \quad \rangle : \mathbb{R}^4 \to \mathbb{R}$

$$S = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\mathbb{R}^4 = V \oplus W$

$V = \text{span}\{e_1, e_2\} \qquad W = \text{span}\{e_3, e_4\}$

$V \perp W \qquad \forall v \in V, w \in W$ si ha $\langle v, w \rangle = 0$

$\{e_1 + e_2, e_1 - e_2, e_3 + e_4, e_3 - e_4\}$

$$S' \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

segnatura è $(2, 2, 0)$

3.16

$V = \mathbb{R}_3[x]$

$\dim(V) = 4 \qquad \{1, x, x^2, x^3\}$ è una base di $\mathbb{R}^3[x]$

$p = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

$q = q_0 + q_1 x + q_2 x^2 + q_3 x^3$

$p(1) = p_0 + p_1 + p_2 + p_3$

$p(-1) = p_0 - p_1 + p_2 - p_3$

$q(1) = q_0 + q_1 + q_2 + q_3$

$q(-1) = q_0 - q_1 + q_2 - q_3$

$\langle p, q \rangle = p(1)\, q(-1) + p(-1)\, q(1) =$

$= (p_0 + p_1 + p_2 + p_3)(q_0 - q_1 + q_2 - q_3) + (p_0 - p_1 + p_2 - p_3) \cdot$

$(q_0 + q_1 + q_2 + q_3) = 2 p_0 q_0 + 2 p_0 q_2 + 2 p_2 q_0$

$+ 2 p_2 q_2 - 2 p_1 q_1 - 2 p_3 q_3 - 2 p_1 q_3$

$- 2 p_3 q_1$

$$S = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & -2 & 0 & -2 \\ 2 & 0 & 2 & 0 \\ 0 & -2 & 0 & -2 \end{pmatrix}$$

$\{1, x^2, x, x^3\}$

$$S' = \left(\begin{array}{cc|cc} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \hline 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

$(1\ 0\ 1)$ $\qquad$ $(0\ +1,\ 1)$

signatura $= (1 \quad 1 \quad 2)$

$$\left(\begin{array}{c|c} + \; 0 & 0 \\ \hline 0 & - \; 0 \end{array}\right)$$

$\langle \alpha + \beta x^2, \ \gamma x + \delta x^3 \rangle$

$(\alpha + \beta)(-\gamma - \delta) + (\alpha + \beta)(\gamma + \delta) = 0$

$\alpha + \beta x^2 \perp \gamma x + \delta x^3 \quad \forall \ \alpha \beta \gamma \delta$

$p_0 + p_1 x + p_2 x^2 + p_3 x^3 \ \perp q \quad \forall q \in \mathbb{R}_3[x]$ per determinare il radicale.

$$\langle p_0+p_1x+p_2x^2+p_3x^3,\ 1\rangle = 0 \Rightarrow 2p_0+2p_2=0$$
$$\langle \quad '' \quad ,\ x\rangle = 0 \Rightarrow -2p_1-2p_3=0$$
$$\langle \quad '' \quad ,\ x^2\rangle = 0 \Rightarrow 2p_0+2p_2=0$$
$$\langle \quad '' \quad ,\ x^3\rangle = 0 \Rightarrow -2p_1-2p_3=0$$

$$\begin{cases} p_0+p_2=0 \\ p_1+p_3=0 \end{cases} \Rightarrow \begin{cases} p_2=-p_0 \\ p_3=-p_1 \end{cases} \qquad p_0(1-x^2)+p_3(x-x^3)$$

$$V^\perp = \text{Span}\{1-x^2,\ x-x^3\}$$

$$W = \text{span}\{x+x^2\}$$

$W^\perp$ è lo stesso di prima!

$$\langle p_0+p_1x+p_2x^2+p_3x^3,\ \alpha(x+x^2)\rangle = 0$$

$$-2\alpha(p_1+p_3)+2\alpha(p_0+p_2)=0$$

4 "incognite"
1 equazione

$$2\alpha\underbrace{(p_0+p_2-p_1-p_3)}=0$$

— ricavo p_3 in funzione di p_0, p_1, p_2

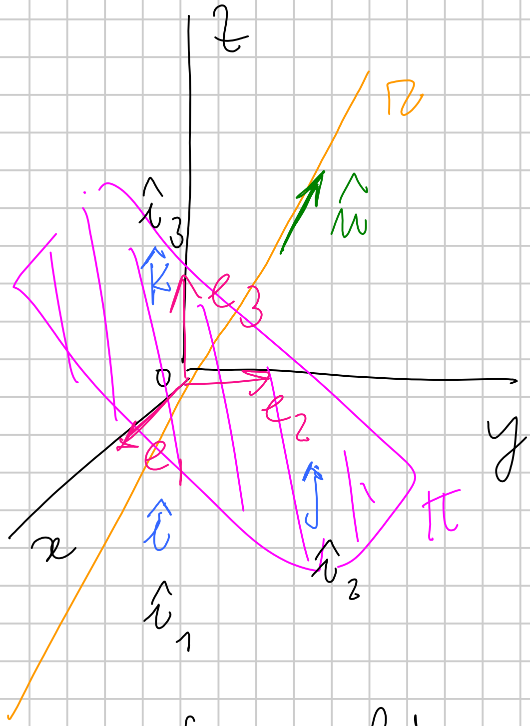
$$p_3 = p_0-p_1+p_2$$

$$p(x) = \overset{\text{libero}}{\boxed{p_0}} + \overset{\text{libero}}{\boxed{p_1}}x + \overset{\text{libero}}{\boxed{p_2}}x^2 + \overset{\text{dipende dagli altri 3}}{(p_0-p_1+p_2)}x^3$$

$$p_0(1+x^3)+p_1(x-x^3)+p_2(x^2+x^3)$$

$$\text{Span}\{1+x^3,\ x-x^3,\ x^2+x^3\}$$

Parte più geometrica: prodotto scalare è euclideo. V è $\mathbb{R}^2$ o $\mathbb{R}^3$ $\rightarrow$



$v \neq 0, \quad \|v\| = \sqrt{v \cdot v} > 0$

$$\hat{v} = \frac{v}{\|v\|}$$

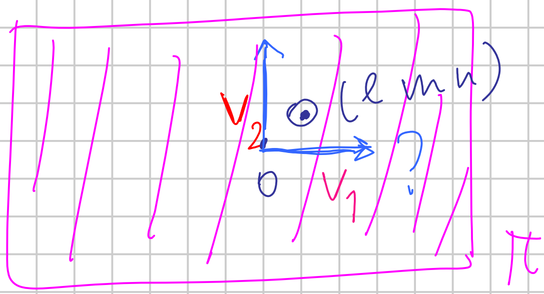
$$\hat{u} = \frac{l\hat{\imath} + m\hat{\jmath} + n\hat{k}}{\sqrt{l^2+m^2+n^2}}$$

Supponiamo, da ora in poi, $l^2+m^2+n^2=1$

$$r: \begin{cases} x = lt \\ y = mt \\ z = nt \end{cases} \quad t \in \mathbb{R} \text{ qualsiasi}$$

$\pi: \; lx + my + nz = 0 \qquad r \perp \pi$

base su π



$\hat{\imath}\cdot\hat{\imath} = 1$
$\hat{\jmath}\cdot\hat{\jmath} = 1$
$\hat{k}\cdot\hat{k} = 1$
$\hat{\imath}\cdot\hat{\jmath} = 0 \quad \hat{\imath}\cdot\hat{k} = 0$
$\hat{\jmath}\cdot k = 0$

$$(l\hat{\imath} + m\hat{\jmath} + n\hat{k}) \cdot (\alpha\hat{\imath} + \beta\hat{\jmath} + \gamma\hat{k}) = 0$$

$\Rightarrow \alpha l + \beta m + \gamma n = 0$

Se l ed m non sono entrambi nulli $\quad \alpha = m$
$\beta = -l$
$\gamma = 0$

$V_1 = m\hat{\imath} - l\hat{\jmath}$

$V_2 = nl\hat{\imath} + nm\hat{\jmath} + (n^2-1)\hat{k}$

$$\begin{cases} \alpha' l + \beta' m + \gamma' n = 0 \\ \alpha' m - \beta' l = 0 \end{cases}$$

$$\alpha' = n\ell \qquad \beta' = nm \qquad n \neq 0$$

$$\cancel{n}\ell^2 + \cancel{n}m^2 + \gamma'\cancel{n} = 0$$

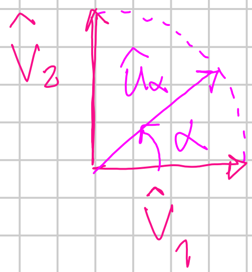
$$\ell^2 + m^2 + n^2 = 1 \qquad \gamma' = n^2 - 1$$

$$\|v_1\|^2 = m^2 + \ell^2 = 1 - n^2$$

$$\|v_2\|^2 = n^2\ell^2 + n^2m^2 + (n^2-1)^2 = n^2(1-n^2) + (1-n^2)^2$$
$$= (1-n^2)(n^2 + 1 - n^2) = 1 - n^2$$

$$\|v_1\| = \|v_2\|$$

$$\hat{v}_1 = \frac{1}{\sqrt{1-n^2}}\left(m\hat{\imath} - \ell\hat{\jmath}\right)$$



$$\hat{v}_2 = \frac{1}{\sqrt{1-n^2}}\left(n\ell\hat{\imath} + nm\hat{\jmath} + (n^2-1)\hat{k}\right)$$

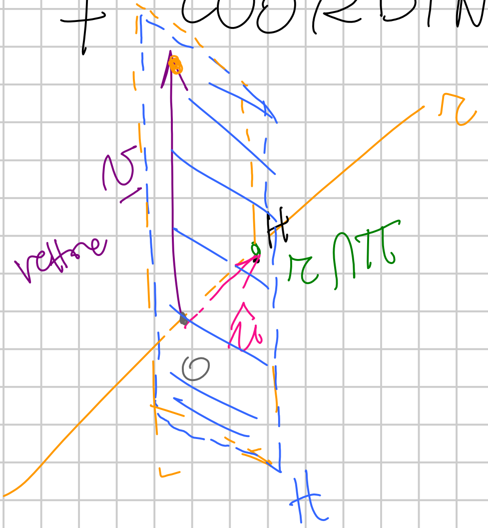
$v = a_1\hat{v}_1 + a_2\hat{v}_2$ è certamente $\perp$ π

$\cos\alpha\,\hat{v}_1 + \operatorname{sen}\alpha\,\hat{v}_2$ è un generico versore che è $\perp$ ad π con $\alpha \in [0, 2\pi)$ li trova tutti

$$\hat{u}_\alpha = \frac{1}{\sqrt{1-n^2}}\left[\cos\alpha\left(m\hat{\imath} - \ell\hat{\jmath}\right) + \operatorname{sen}\alpha\left(n\ell\hat{\imath} + nm\hat{\jmath} + (n^2-1)\hat{k}\right)\right]$$

$$= \frac{1}{\sqrt{1-n^2}}\left[(n\ell\operatorname{sen}\alpha + m\cos\alpha)\hat{\imath}\,(mn\operatorname{sen}\alpha - \ell\cos\alpha)\hat{\jmath} + (n^2-1)\operatorname{sen}\alpha\,\hat{k}\right]$$

COMPONENTE (ORTOGONALE) $\neq$ COORDINATA



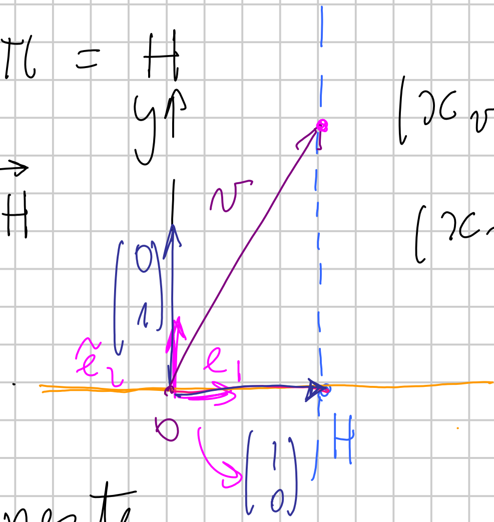
estremità libera

$\pi \perp r$

π passa per la "punta" di $\underline{v}$

$r \cap \pi = H$

$\underline{v}_{\parallel} = \overrightarrow{OH}$



(x_v, y_v)

$(x_v, y_v) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\hat{x} = x_v$

$\| \overrightarrow{OH} \| = | \underline{v} \cdot \hat{u} |$

$\underline{v} \cdot \hat{u} = v_r$ la componente (ortogonale) di v lungo r

$\underline{v} = 4\hat{\imath} + 8\hat{\jmath} + \hat{k}$

$r_1: \begin{cases} x = -2t \\ y = 2t \\ z = t \end{cases}$ $\quad r_2: \begin{cases} x = s \\ y = s \\ z = 0 \end{cases}$

$r_3: \begin{cases} x = -u \\ y = u \\ z = -4u \end{cases}$

$v_{r_1} = \underline{v} \cdot \hat{u}_1$

$$\hat{u}_1 = \frac{-2\hat{\imath} + 2\hat{\jmath} + \hat{k}}{\sqrt{(-2)^2 + (+2)^2 + 1^2}} = -\frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}$$

SCALARE

$$v_{r_1} = 4 \cdot \left(-\frac{2}{3}\right) + 8 \cdot \left(\frac{2}{3}\right) + 1 \cdot \left(\frac{1}{3}\right) = -\frac{8}{3} + \frac{16}{3} + \frac{1}{3} = 3$$

$$\hat{u}_2 = \frac{\hat{\imath} + \hat{\jmath}}{\sqrt{2}} \qquad v_{r_2} = (4\hat{\imath} + 8\hat{\jmath} + \hat{k}) \cdot \frac{\hat{\imath} + \hat{\jmath}}{\sqrt{2}} = 6\sqrt{2}$$

$$\hat{u}_3 = \frac{-\hat{\imath} + \hat{\jmath} - 4\hat{k}}{\sqrt{1 + 1 + 16}} = \frac{-\hat{\imath} + \hat{\jmath} - 4\hat{k}}{3\sqrt{2}}$$

$$v_{r_3} = \underline{v} \cdot \hat{u}_3 = (4\hat{i} + 8\hat{j} + \hat{k}) \cdot \frac{(-\hat{i} + \hat{j} - 4\hat{k})}{3\sqrt{2}} = 0$$

$v_{r_1}\hat{u}_1 + v_{r_2}\hat{u}_2 + v_{r_3}\hat{u}_3 = ?$ ottengo di nuovo $\underline{v}$ oppure no?

Dipende da chi sono $\hat{u}_1$ $\hat{u}_2$ $\hat{u}_3$!

Se sono una base ortogonale di $\mathbb{R}^3$, sì!

$$3 \cdot \left(-\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right) + 6\sqrt{2}\left(\frac{\hat{i} + \hat{j}}{\sqrt{2}}\right) + 0\left(\frac{-\hat{i} + \hat{j} - 4\hat{k}}{3\sqrt{2}}\right)$$

$$= -2\hat{i} + 2\hat{j} + \hat{k} + 6\hat{i} + 6\hat{j} = 4\hat{i} + 8\hat{j} + \hat{k} \text{ OK}$$

COME MAI? $\hat{u}_1 \cdot \hat{u}_2 = \left(-\frac{2}{3\sqrt{2}} + \frac{2}{3\sqrt{2}}\right) \boxed{= 0}$

$$\hat{u}_1 \cdot \hat{u}_3 = \left(+\frac{2}{9\sqrt{2}} + \frac{2}{9\sqrt{2}} - \frac{4}{9\sqrt{2}}\right) \boxed{= 0}$$

$$\hat{u}_2 \cdot \hat{u}_3 = -\frac{1}{6} + \frac{1}{6} \boxed{= 0}$$

$\hat{u}_1, \hat{u}_2, \hat{u}_3$ è una base ortonormale di $\mathbb{R}^3$

in questo caso, ricomponendo le componenti ortogonali si ottiene il vettore!

$\underline{V} = 6\hat{i} + 5\hat{j} + 5\hat{k}$

$$r_1: \begin{cases} x = -2t \\ y = 2t \\ z = t \end{cases} \qquad r_2: \begin{cases} x = -2s \\ y = s \\ z = 2s \end{cases} \qquad r_3: \begin{cases} x = u \\ y = 2u \\ z = -2u \end{cases}$$

$v_{r_1} = 1$ $\qquad$ $v_{r_3} = 2$

$v_{r_2} = 1$

$\hat{i}$ $\qquad$ $\hat{j}$ $\qquad$ $\hat{k}$

$$\hat{i} + \hat{j} + 2\hat{k} \neq V$$

Questo accade perché le 3 rette non sono a due a 2 ortogonali

BISOGNA TROVARE LE COORDINATE!

α, β, γ tali che

$$\underline{v} = \alpha \hat{u}_1 + \beta \hat{u}_2 + \gamma \hat{u}_3$$

$$\hat{u}_1 = \frac{-2\hat{i} + 2\hat{j} + \hat{k}}{3} \qquad \hat{u}_2 = \frac{-2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

$$\hat{u}_3 = \frac{\hat{i} + 2\hat{j} - 2\hat{k}}{3}$$

(ortogonali)

$\alpha = -71$ $\beta = 81$ $\gamma = 38$ provare!

COORDINATE, NON COMPONENTI !!!