ESERCIZI GEOM

Titolo nota 27/05/2016

Cambiamento di base $\boxed{\text{affine}}$ → SI SPOSTA L'ORIGINE

p 64

$x \in \mathbb{R}^n$ $(x_1 \dots x_n)$ rispetto ad una fissata base

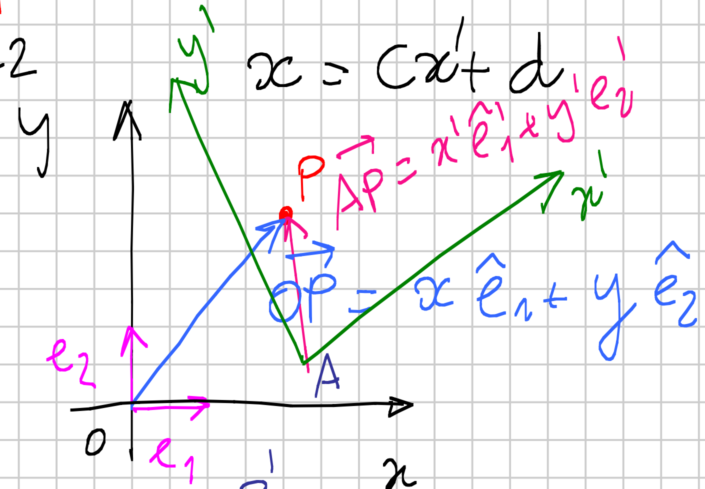
$\downarrow$

$x' \in \mathbb{R}^n$ $(x'_1 \dots x'_n)$ rispetto ad un'altra fissata base

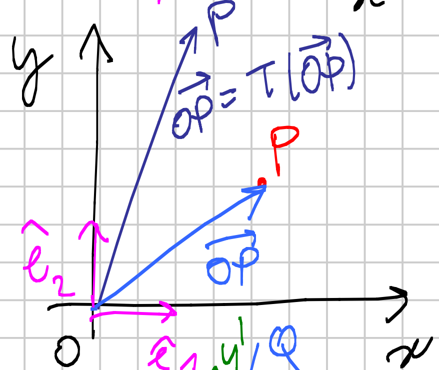
stesso vettore espresso in due modi diversi

$n=2$

$x = Cx' + d$

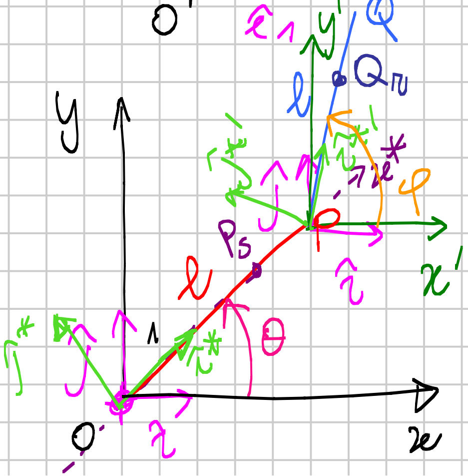


CAMBIO DI COORDINATE



$$\vec{OP'} = \underset{\substack{\uparrow \\ \text{matrice}}}{C}(\vec{OP}) + \underset{\substack{\uparrow \\ \text{vettore}}}{d}$$

TRASFORMAZIONE DEL PIANO IN SE'



$\overline{OP} = l > 0$ $\overline{PQ} = l > 0$

$$\vec{OP} = x_P \underset{\substack{\uparrow \\ \hat{\imath}}}{\hat{e}_1} + y_P \underset{\substack{\uparrow \\ \hat{\jmath}}}{\hat{e}_2} \qquad \vec{OQ} = x_Q\,\hat{\imath} + y_Q\,\hat{\jmath}$$

$$\vec{PQ} = x'_Q\,\hat{\imath} + y'_Q\,\hat{\jmath}$$

$$\vec{OQ} = \vec{OP} + \vec{PQ}$$

$$x_Q = x_P + x'_Q$$

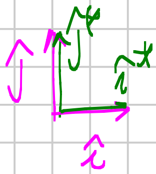
$$y_Q = y_P + y'_Q$$

$$x_P = l\cos\theta \qquad \theta \in [0, 2\pi)$$

$$y_P = l\,\mathrm{sen}\,\theta$$

$$\hat{\imath}^* = \cos\theta\,\hat{\imath} + \mathrm{sen}\,\theta\,\hat{\jmath}$$

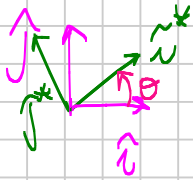
$$\hat{\jmath}^* = -\mathrm{sen}\,\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$



sovrapposti

$\theta = 0$

rotazione $\rightsquigarrow$ di un angolo θ intorno ad O



$$\begin{pmatrix} \hat{\imath}^* \\ \hat{\jmath}^* \end{pmatrix} = \begin{pmatrix} \cos\theta & \mathrm{sen}\,\theta \\ -\mathrm{sen}\,\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \end{pmatrix}$$

$$\begin{pmatrix} x_P(\theta) \\ y_P(\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\mathrm{sen}\,\theta \\ +\mathrm{sen}\,\theta & \cos\theta \end{pmatrix} \begin{pmatrix} l \\ 0 \end{pmatrix}$$

$$x_P(\theta) = l\cos\theta$$

$$y_P(\theta) = l\,\mathrm{sen}\,\theta$$

$$\vec{OP} = l\,\hat{\imath}^* \qquad \hat{\imath}^* = \cos\theta\,\hat{\imath} + \mathrm{sen}\,\theta\,\hat{\jmath}$$

$$\vec{OP} = l(\cos\theta\,\hat{\imath} + \mathrm{sen}\,\theta\,\hat{\jmath}) = \underbrace{l\cos\theta}_{x_P(\theta)}\,\hat{\imath} + \underbrace{l\,\mathrm{sen}\,\theta}_{y_P(\theta)}\,\hat{\jmath}$$

$$\vec{OP_s} = s\,\hat{\imath}^* \qquad 0 \le s \le l$$

$$\vec{OP_s} = s\cos\theta\,\hat{\imath} + s\,\mathrm{sen}\,\theta\,\hat{\jmath}$$

$$\vec{PQ} = l\cos\varphi\,\hat{\imath} + l\,\mathrm{sen}\,\varphi\,\hat{\jmath}$$

adesso cambio sistema di riferimento

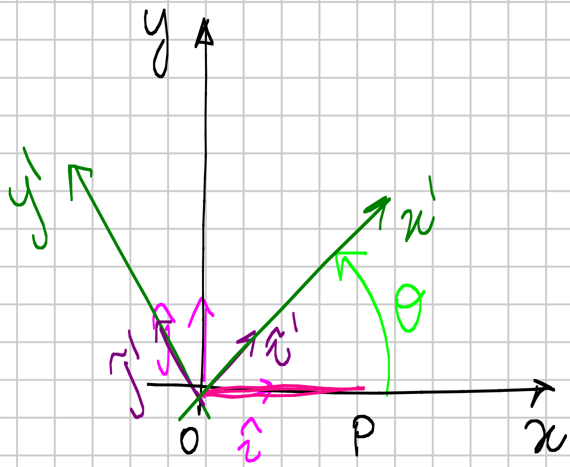
$$\vec{OQ} = \vec{OP} + \vec{PQ} = l(\cos\theta + \cos\varphi)\,\hat{\imath} + l(\mathrm{sen}\,\theta + \mathrm{sen}\,\varphi)\,\hat{\jmath}$$

$$\vec{PQ_r} = r\cos\varphi\,\hat{\imath} + r\,\mathrm{sen}\,\varphi\,\hat{\jmath} \qquad 0 \le r \le l \qquad \varphi \in [0, 2\pi)$$

$$\vec{OQ_r} = (l\cos\theta + r\cos\varphi)\,\hat{\imath} + (l\,\mathrm{sen}\,\theta + r\,\mathrm{sen}\,\varphi)\,\hat{\jmath}$$

$\theta \to \theta(t)$ velocità . . . . . .

$\varphi \to \varphi(t)$



$$\hat{i}' = \cos\theta\,\hat{i} + \operatorname{sen}\theta\,\hat{j}$$

$$\hat{j}' = -\operatorname{sen}\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\begin{pmatrix} \hat{i}' \\ \hat{j}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \end{pmatrix}$$

$$\begin{pmatrix} \hat{i} \\ \hat{j} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{i}' \\ \hat{j}' \end{pmatrix}$$

$$\vec{OP} = \ell\,\hat{i}$$

$$\vec{OP} = \ell\left(\cos\theta\,\hat{i}' - \operatorname{sen}\theta\,\hat{j}'\right)$$

in Oxy $\quad P \equiv (\ell, 0)$

in $Ox'y'$ $\quad P \equiv (\ell\cos\theta, -\ell\operatorname{sen}\theta)$

$$\vec{OP} = x_p\,\hat{i} + y_p\,\hat{j}$$

lo stesso vettore sarà espresso nella forma

$$\vec{OP} = x_p\left(\cos\theta\,\hat{i}' - \operatorname{sen}\theta\,\hat{j}'\right) + y_p\left(\operatorname{sen}\theta\,\hat{i}' + \cos\theta\,\hat{j}'\right) =$$

$$\underbrace{\left(x_p\cos\theta + y_p\operatorname{sen}\theta\right)}_{x'_p}\hat{i}' + \left(-x_p\operatorname{sen}\theta + y_p\cos\theta\right)\hat{j}'$$

le coordinate di P in Oxy sono (x_p, y_p)

le coordinate di P in $Ox'y'$ sono $(x'_p\ y'_p)$

$$x'_p = x_p\cos\theta + y_p\operatorname{sen}\theta$$

$$y'_p = -x_p\operatorname{sen}\theta + y_p\cos\theta$$

C^{-1} p64

$d=0$

$$\begin{pmatrix} x'_p \\ y'_p \end{pmatrix} = \boxed{\begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

CAMBIO DI COORDINATE

$x = Cx' + d \qquad x \rightarrow \begin{pmatrix} x_p \\ y_p \end{pmatrix} \qquad x' \rightarrow \begin{pmatrix} x'_p \\ y'_p \end{pmatrix}$

$x' = C^{-1}x - (C^{-1}d)$

l'origine delle due "terne" è la stessa $\Rightarrow d = 0$

$P \equiv (\ell, 0)$ in Oxy

$P \equiv (\ell\cos\theta, -\ell \operatorname{sen}\theta)$ in $Ox'y'$ Supponiamo che θ vari nel tempo

Un osservatore in Oxy vede P fermo

Un osservatore in $Ox'y'$ vede P muoversi lungo un arco di circonferenza di centro O e raggio ℓ

$$\begin{cases} x'_p = \ell\cos\theta \\ y'_p = -\ell\operatorname{sen}\theta \end{cases} \qquad \begin{cases} \cos\theta = \dfrac{x'}{\ell} \\ \operatorname{sen}\theta = -\dfrac{y'}{\ell} \end{cases} \qquad \begin{aligned} &\cos^2\theta + \operatorname{sen}^2\theta = 1 \\ &\frac{x'^2}{\ell^2} + \frac{y'^2}{\ell^2} = 1 \end{aligned}$$

$x'^2 + y'^2 = \ell^2$ Circonferenza di centro O e raggio ℓ

$Kx^2 + 4xy + y^2 - 4x - 2y + 5 = 0$

$$A = \begin{vmatrix} K & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 5 \end{vmatrix}$$

se $K = 4$ la 1^a colonna è il doppio della 2^a

$car(A) \geq 2$ $\det(A)$ è di 1° grado in K

$$A_2 = \begin{vmatrix} K & 2 \\ 2 & 1 \end{vmatrix} = K - 4$$ per $K = 4$ si annulla anche $\det(A_2)$

Coppia di rette parallele non reali

$$4x^2+4xy+y^2=0 \qquad (2x+y)^2=0$$

$$(2x+y+c_1)(2x+y+c_2) = 4x^2+4xy+y^2-4x-2y+5=0$$

$$\begin{cases} c_1+c_2=-2 \\ 2c_1+2c_2=-4 \\ c_1c_2=5 \end{cases} \qquad t^2+2t+5=0 \quad \Delta<0$$

Se $k-4<0$, cioè $k<4$, abbiamo iperboli

$$kx^2+4xy+y^2=0 \qquad \frac{x}{y}=t$$

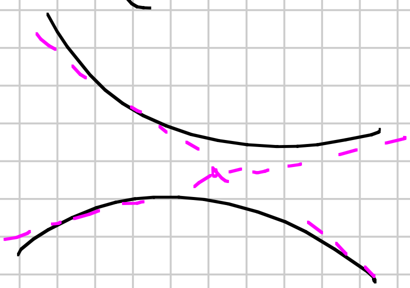
$$kt^2+4t+1=0 \qquad t=-2\pm\sqrt{4-k}$$

>0 se $k<4$

$$\frac{x}{y}=-2\pm\sqrt{4-k} \qquad x=\left(-2\pm\sqrt{4-k}\right)y$$

$$x+\left(2-\sqrt{4-k}\right)y+c_1=0$$

$$x+\left(2+\sqrt{4-k}\right)y+c_2=0$$

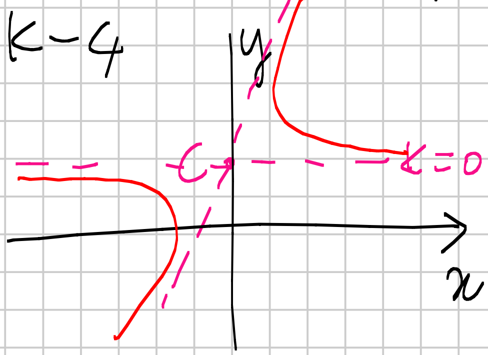


Cerchiamo il centro

$$k(x'+x_c)^2+4(x'+x_c)(y'+y_c)+(y'+y_c)^2-4(x'+x_c)-2(y'+y_c)+5=0$$

$$\begin{cases} 2kx_c+4y_c-4=0 \\ 4x_c+2y_c-2=0 \end{cases} \qquad \begin{cases} kx_c+2y_c=2 \\ 2x_c+y_c=1 \end{cases}$$

$$x_c=\frac{\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix}}{k-4}=0 \quad \forall k\neq 4 \qquad y_c=\frac{k-4}{k-4}=1 \quad \forall k\neq 4$$



Caso $k=0$

$$\begin{cases} 4xy+y^2-4x-2y+5=0 \\ x=0 \end{cases}$$

$$y^2-2y+5=0 \quad \Delta<0$$

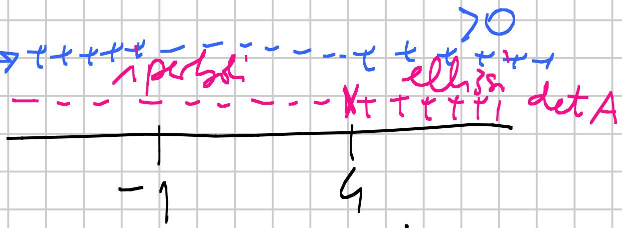
Ellissi per $K > 4$ sono reali o immaginarie?

$$\begin{vmatrix} K & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 5 \end{vmatrix} \qquad K(5-1) - 2(10-2) - 2(0) =$$

$$= 4K - 16 = 4(K-4)$$

$tr(A_2) = K+1$

$tr(A_2)\, det(A) = \boxed{4(K-4)(K+1)}$



Ellisse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

$$\begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{oppure} \quad \begin{pmatrix} -1/a^2 & 0 & 0 \\ 0 & -1/b^2 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

$det(A) = -\frac{1}{a^2 b^2}$ | $det(A) = +\frac{1}{a^2b^2}$

$tr(A_2) = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)$ | $tr(A_2) = -\frac{1}{a^2} - \frac{1}{b^2}$

$\boxed{det(A) \cdot tr(A_2) < 0}$ | $\boxed{det(A) \cdot tr(A_2) < 0}$

3.2 V g_V definito positivo $U \subset V$ s.s.v di V

P_U è un endomorfismo simmetrico di V

$$g(P_U(v), w) = g(v, P_U(w)) \quad \text{simmetrico}$$

$$\boxed{\begin{array}{l} g(x,y)=0 \\ x \in U \\ y \in U_\perp \end{array}}$$

$U \subset V$ e p.s. $\qquad V = U \oplus U_\perp$ in modo unico

$\forall v \in V \qquad v = v_U + v_{U^\perp}$ in modo unico

$w = w_U + w_{U^\perp}$ in modo unico

$P_U(v) = v_U$

$P_U(w) = w_U$

$$g(P_U(v), w) = g(v_U, w_U + w_{U^\perp}) =$$

$$= g(v_U, w_U) + \cancel{g(v_U, w_{U^\perp})} = \boxed{g(v_U, w_U)}$$

$$g(v, P_U(w)) = g(v_U + v_{U^\perp}, w_U) = g(v_U, w_U) + \cancel{g(v_{U^\perp}, w_U)} =$$

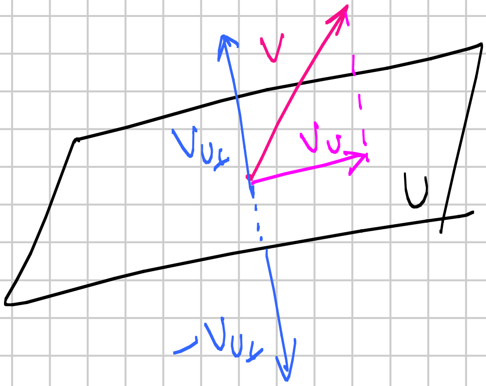
$$= \boxed{g(v_U, w_U)}$$

$$g(P_U(v), w) = g(v, g_U(w))$$

P_U è un operatore simmetrico

$Rif_U(v) = v_U - v_{U^\perp}$

$Rif_U(w) = w_U - w_{U^\perp}$



$$g(Rif_U(v), w) = g(v_U - v_{U^\perp}, w_U + w_{U^\perp}) =$$

$$= \boxed{g(v_U, w_U) - g(v_{U^\perp}, w_{U^\perp})}$$

$$g(v, Rif_U(w)) = g(v_U + v_{U^\perp}, w_U - w_{U^\perp}) =$$

$$= \boxed{g(v_U, w_U) - g(v_{U^\perp}, w_{U^\perp})}$$

$$g(Rif_U(v), w) = g(v, Rif_U(w))$$

$Rif_U(v)$ è un operatore simmetrico

3,5 A matrice quadrata diagonalizzabile con $\lambda_i > 0$

Mostra che $\exists B$ quadrata tale che $A = B^2$

c'è una base di V $\{e_1 \dots e_n\}$ tale che

$$A' = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad A' = M^{-1} A M$$

$$A = M A' M^{-1}$$

$$A' = \text{"}\sqrt{A'}\text{"} \cdot \text{"}\sqrt{A'}\text{"} \qquad \text{"}\sqrt{A'}\text{"} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix} = A''$$

$$A = M A'' \cdot A'' \cdot M^{-1}$$

$$= M A'' I A'' M^{-1} \qquad I = M^{-1} M$$

$$= M A'' M^{-1} M A'' M^{-1}$$

$$= \underbrace{(M A'' M^{-1})}_{B} \underbrace{(M A'' M^{-1})}_{B} = B^2$$

$A = M A'' A'' {}^tM$

...

Posso trovare M tale che

$${}^tM A M = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

rinormalizzando i vettori di base

${}^tM A M = I$ $\qquad A =$

$$({}^tM)^{-1} {}^tM A M M^{-1} = ({}^tM)^{-1} I M^{-1} = ({}^tM^{-1}) M^{-1}$$

$$A = ({}^tM^{-1})(M^{-1})$$