ESERCIZI SU SEGNATURA

Titolo nota 26/04/2016

$$\begin{pmatrix} \boxed{1} & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad 1, 1, -3, -1$$

$$\quad\; , \;\;\; , \;\;\; ,$$
$$\quad\; p \;\;\; c \;\;\; p$$

LEMMA 2.3.9 p41
2 permanenze
1 cambiamento

segnatura è $(2,1,0)$

$$\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = (-1)\cdot\det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = -1$$

$$S: \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & 8 & -4 \\ 0 & -4 & 2 \end{pmatrix} \qquad 1, 1, 8, 0$$

$i_0 > 0$ perché $\det S = 0$

$$W = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

$S|_W$ è definito positivo $\qquad i_+ \geq 2$

$i_0 + i_+ + i_- = 3$ $\qquad$ l'unica possibilità è $(2,0,1)$

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & -1 & \\ & & & & & & 0 \\ & & & & & & & 0 \end{pmatrix}$$

$$\det({}^tM\,S\,M) =$$
$$= \det(S)\cdot\det({}^tM)\cdot\det(M)$$
$$= \det(S)\cdot\underbrace{(\det(M))^2}_{>0}$$

$\det(S)$ e $\det({}^tMSM)$ hanno lo stesso segno

$$S = \begin{pmatrix} \boxed{1} & 2 & 1 \\ 2 & 0 & \alpha \\ 1 & \alpha & \alpha^2 \end{pmatrix} \qquad 1, 1, -4, -5\alpha^2+4\alpha$$

$$\quad\; , \;\;\; , \;\;\; ,$$
$$\quad\; p \;\;\; c \;\;\; ?$$

$$\det\begin{pmatrix}1&2&1\\2&0&\alpha\\1&\alpha&\alpha^2\end{pmatrix} = -2\det\begin{pmatrix}2&1\\\alpha&\alpha^2\end{pmatrix} - \alpha\det\begin{pmatrix}1&2\\1&\alpha\end{pmatrix} =$$

$$-2(2\alpha^2-\alpha)-\alpha(\alpha-2) = -4\alpha^2+2\alpha-\alpha^2+2\alpha$$

$$= -5\alpha^2+4\alpha$$

$-5\alpha^2+4\alpha>0$ se $0<\alpha<\frac{4}{5}$

$-5\alpha^2+4\alpha<0$ se $\alpha<0$ o $\alpha>\frac{4}{5}$

Se $0<\alpha<\frac{4}{5}$ $\quad (1,2,0)$

se $\alpha<0$ o $\alpha>\frac{4}{5}$ $\quad (2,1,0)$

I casi $\alpha=0$ $\quad \alpha=4/5$ vanno studiati a parte

$$S_0=\begin{pmatrix}\boxed{1}&2&1\\2&0&0\\1&0&0\end{pmatrix} \qquad (1,1,1) \quad \text{per } \alpha=0$$

Cerchiamo una base che diagonalizza S_0

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}0\\1\\-2\end{pmatrix} (x\ y\ z)\begin{pmatrix}1&2&1\\2&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=0$$

$$(x\ y\ z)\begin{pmatrix}1\\2\\1\end{pmatrix}=0 \qquad \boxed{x+2y+z=0}$$

scelgo $\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$$(x\ y\ z)\begin{pmatrix}1&2&1\\2&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}=0 \qquad (x\ y\ z)\begin{pmatrix}0\\2\\1\end{pmatrix}=0 \quad \boxed{2y+z=0}$$

$z=-2y \quad x=0 \quad y=1 \quad z=-2$

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} \qquad {}^tM = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}$$

$${}^tM S_0 M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix}} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = -1$$

$\updownarrow \perp$

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

Consideriamo il sottospazio $\operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} 0 & \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} =$$

$$=(0\ \alpha\ \beta)\begin{pmatrix}2\alpha+\beta\\0\\0\end{pmatrix}=0 \quad \forall\ \alpha,\beta$$

Caso $\alpha = 4/5$

$$S_{4/5}=\left(\begin{array}{cc|c}1&2&1\\2&0&4/5\\\hline 1&4/5&\frac{16}{25}\end{array}\right)$$

$d_1 \quad d_2 \quad d_3$

$1,\ 1,\ -4,\ 0$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}2\\-1\\0\end{pmatrix}\begin{pmatrix}-4\\-3\\10\end{pmatrix} \qquad (x\ y\ z)\begin{pmatrix} & & \\ & S_{4/5} & \\ & & \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=0$$

$$x+2y+z=0$$

$$(x\ y\ z)\begin{pmatrix} & & \\ & S_{4/5} & \\ & & \end{pmatrix}\begin{pmatrix}2\\-1\\0\end{pmatrix}=0$$

$$(x\ y\ z)\begin{pmatrix}0\\4\\6/5\end{pmatrix}=0 \qquad 4y+\frac{6}{5}z=0$$

$$10y+3z=0$$

$$\begin{cases}x+2y+z=0\\10y+3z=0\end{cases} \qquad \begin{aligned}z&=10\\y&=-3\\x&=-4\end{aligned}$$

$$M=\begin{pmatrix}1&2&-4\\0&-1&-3\\0&0&10\end{pmatrix} \qquad {}^tM=\begin{pmatrix}1&0&0\\2&-1&0\\-4&-3&10\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\+2&-1&0\\-4&-3&10\end{pmatrix}\begin{pmatrix}1&2&1\\2&0&4/5\\1&4/5&16/25\end{pmatrix}\begin{pmatrix}1&2&-4\\0&-1&-3\\0&0&10\end{pmatrix}=$$

$$\begin{pmatrix} 1 & 0 & 0 \\ +2 & -1 & 0 \\ -4 & -3 & 10 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 6/5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{-4 - \frac{12}{5} + \frac{32}{5} = 0}$$

segnatura $(1, 1, 1)$ anche per $\alpha = 4/5$

$$S = \begin{pmatrix} \alpha & \alpha+1 & \alpha+2 \\ \alpha+1 & \alpha+2 & \alpha+1 \\ \alpha+2 & \alpha+1 & \alpha \end{pmatrix} \qquad 1 \quad \alpha \quad -1$$

$$\begin{array}{c} 2C_3 \\ \begin{pmatrix} 2\alpha+4 \\ 2\alpha+2 \\ 2\alpha \end{pmatrix} \end{array} \quad \begin{array}{c} C_1+C_2 \\ \begin{pmatrix} 2\alpha+1 \\ 2\alpha+3 \\ 2\alpha+3 \end{pmatrix} \end{array}$$

$$\det \begin{pmatrix} \alpha & \alpha+1 \\ \alpha+1 & \alpha+2 \end{pmatrix} = \alpha^2 + 2\alpha - \alpha^2 - 2\alpha - 1 = -1 \quad \forall \alpha$$

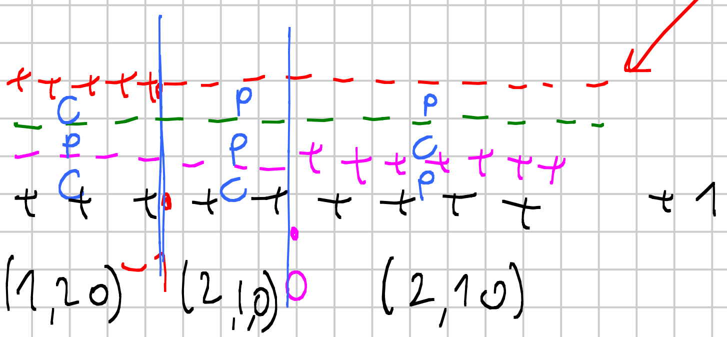
$$\det S = \det \begin{pmatrix} \alpha & \alpha+1 & 3 \\ \alpha+1 & \alpha+2 & -1 \\ \alpha+2 & \alpha+1 & -3 \end{pmatrix} = \det \begin{pmatrix} 2\alpha+2 & 2\alpha+2 & 0 \\ \alpha+1 & \alpha+2 & -1 \\ \alpha+2 & \alpha+1 & -3 \end{pmatrix} =$$

$$= \det \begin{pmatrix} 2\alpha+2 & 0 & 0 \\ \alpha+1 & 1 & -1 \\ \alpha+2 & -1 & -3 \end{pmatrix} = (2\alpha+2)(-3-1)$$

$$-8(\alpha+1)$$

$$1, \boxed{\alpha}, \boxed{-1}, \boxed{-1-\alpha}$$

Successione dei segni dei d_i



$\alpha = -1$

$\alpha = 0$ vanno studiati a parte

$\alpha = 0$

$$S_0 = \begin{pmatrix} \boxed{0} & 1 & 2 \\ 1 & \boxed{2} & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$i_0 = 0$

$i_+ \geq 1$

i_- ?

$(3\ 0\ 0)$?
$(2\ 1\ 0)$
$(1\ 2\ 0)$?

$\det(S_0) \neq 0$

certamente $i_0 = 0$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ può stare in base

$(x\ y\ z) \begin{pmatrix} & & \\ & S_0 & \\ & & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow \boxed{x + 2y + z = 0} \rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

secondo vettore di base

$(x\ y\ z) \begin{pmatrix} & & \\ & S_0 & \\ & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \quad (x\ y\ z) \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = 0 \quad \boxed{-2x + 2z = 0}$

$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ terzo vettore di base

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad {}^tM = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$${}^tM S_0 M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$i_+ = 2$
$i_- = 1$
$i_0 = 0$

$\mathrm{prd} = 0$

Caso $\alpha = -1$

$$S_{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix}1\\0\\1\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \begin{pmatrix}1\\0\\-1\end{pmatrix} \quad (x\ y\ z)\begin{pmatrix} & & \\ & S_{-1} & \\ & & \end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = 0$$

sempre

$$(x\ y\ z)\begin{pmatrix}-1 & 0 & 1\\0 & 1 & 0\\1 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = 0 \qquad y = 0$$

$$M = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & -1\end{pmatrix} \qquad {}^tM = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & -1\end{pmatrix}$$

$${}^tM\, S_{-1}\, M = \begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & -4\end{pmatrix}$$

segnatura $(1\ \ 1\ \ 1)$ per $\alpha = -1$

2.13 $\quad \langle A, B\rangle = tr(AB)$

$$A = \begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix} \qquad B = \begin{pmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{pmatrix}$$

$$AB = \begin{pmatrix}a_{11}b_{11} + a_{12}b_{21} & *\\ * & a_{22}b_{22} + a_{21}b_{12}\end{pmatrix}$$

$$tr(AB) = a_{11}b_{11} + a_{22}b_{22} + a_{12}b_{21} + a_{21}b_{12}$$

$$A = a_{11}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + a_{12}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + a_{21}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + a_{22}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix} b_{11} & b_{12} & b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}$$

$\det S < 0$

i_- è dispari

$i_0 = 0$

$(1\ 3\ 0)$

$(3\ 1\ 0)$ 2 casi possibili

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

${}^tMSM =$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & +1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$(3, 1, 0)$ è la segnatura

2.1.14

$$S = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & -8 \end{pmatrix}$$

$\det S = 0$

successione dei det

$1, 1, -1, 0$

Segnatura $(1, 1, 1)$

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2x+y=z \right\}$$

base di W è, ad esempio $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$W^{\perp}$

$$\begin{cases} (x\ y\ z) \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 0 \\ (x\ y\ z) \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & -8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \end{cases}$$

$$\begin{cases} (x\ y\ z) \begin{pmatrix} -1 \\ 4 \\ -17 \end{pmatrix} = 0 \\ (x\ y\ z) \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix} = 0 \end{cases}$$

$$\begin{cases} -x+4y-17z=0 \\ x+4y-7z=0 \end{cases}$$

$\Downarrow$

$2x+10z=0$ $\quad x=-5z$

$y=3z$

$$\begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$$

Una base di $W^{\perp}$ è $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} \right\}$

2.16 $\quad p(x) = p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0$

$q(x) = q_4x^4 + q_3x^3 + q_2x^2 + q_1x + q_0$

$$\langle p, q \rangle = p(1)\,q(-1) + p(-1)\,q(1)$$

$$p(1) = (p_4 + p_2 + p_0) + (p_3 + p_1)$$

$$p(-1) = (p_4 + p_2 + p_0) - (p_3 + p_1)$$

$$q(1) = (q_4 + q_2 + q_0) + (q_3 + q_1)$$

$$q(-1) = (q_4 + q_2 + q_0) - (q_3 + q_1)$$

$$\langle p, q \rangle = 2\left[(p_4 + p_2 + p_0)(q_4 + q_2 + q_0) - (p_3 + p_1)(q_3 + q_1)\right]$$

$$\begin{array}{c|c}
 & \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \end{array} \\
\begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} & \begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & -2 & 0 & -2 & 0 \\ 2 & 0 & 2 & 0 & 2 \\ 0 & -2 & 0 & -2 & 0 \\ 2 & 0 & 2 & 0 & 2 \end{pmatrix}
\end{array}$$

vettori $\perp$ a qualunque vettore di V

$$\begin{cases} q_4 + q_2 + q_0 = 0 \\ q_3 + q_1 = 0 \end{cases}$$

2 eq e 5 incognite

∞^3 vettori!

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

base per $\mathcal{N}_0$

sono una base per il radicale