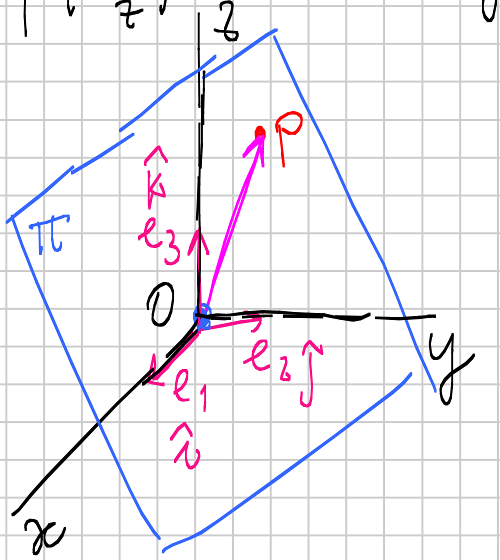
VETTORI ⊥ → PROIETTORI

Titolo nota 12/04/2016

Esercizio 2,7

$\pi : x+y+z=0$ visione "geometrica"

$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x+y+z=0 \right\} \subset \mathbb{R}^3$ "definizione astratta"



$P \equiv (x_P, y_P, z_P)$

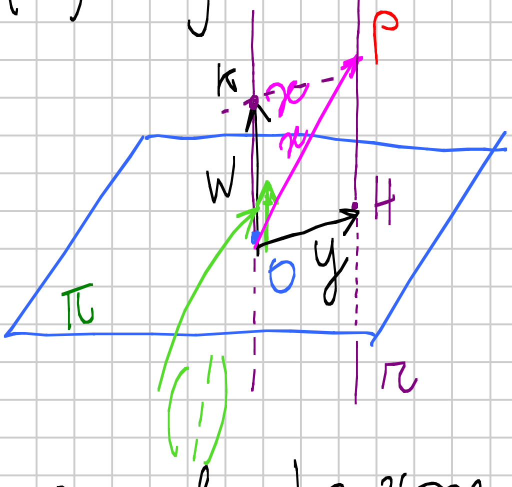
$P \in \pi \iff x_P + y_P + z_P = 0$

$\vec{OP} = x_P e_1 + y_P e_2 + z_P e_3 \in U$

$p_U(x) = y$ tale che $y \in U$

$x = y + w$

$w \perp \pi$



$\vec{OP}$ lo chiamo x

$\vec{OP} = \vec{OH} + \vec{OK}$

OHPK è un rettangolo

Conosco la direzione di $\underset{w}{\vec{OK}}$

$y = x - w$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = g$

$w = \dfrac{\langle \vec{x}, g \rangle}{(\langle g, g \rangle)^2} g$

$\vec{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$

$w = \dfrac{x_1 + x_2 + x_3}{3} (e_1 + e_2 + e_3)$

$\vec{g} = e_1 + e_2 + e_3$

$$x = y + w \qquad y = x - w = x_1 e_1 + x_2 e_2 + x_3 e_3$$

$$- \frac{1}{3}(x_1 + x_2 + x_3)(e_1 + e_2 + e_3) =$$

$$= \left[x_1 - \frac{1}{3}(x_1 + x_2 + x_3)\right] e_1 + \left[x_2 - \frac{1}{3}(x_1 + x_2 + x_3)\right] e_2 +$$

$$\left[x_3 - \frac{1}{3}(x_1 + x_2 + x_3)\right] e_3 =$$

$$= \left(\frac{2}{3} x_1 - \frac{1}{3} x_2 - \frac{1}{3} x_3\right) e_1 + \left(-\frac{1}{3} x_1 + \frac{2}{3} x_2 - \frac{1}{3} x_3\right) e_2$$

$$+ \left(-\frac{1}{3} x_1 - \frac{1}{3} x_2 + \frac{2}{3} x_3\right) e_3 =$$

$$= \frac{1}{3}\left[(2x_1 - x_2 - x_3) e_1 + (-x_1 + 2x_2 - x_3) e_2 + (-x_1 - x_2 + (x_3) e_3\right]$$

$$P_u = \left(p_U(e_1) \,\vdots\, p_U(e_2) \,\vdots\, p_U(e_3) \right)$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} x_1 = 1 \\ x_2 = x_3 = 0 \end{matrix} \qquad p_U(e_1) = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{matrix} x_2 = 1 \\ x_1 = x_3 = 0 \end{matrix} \qquad p_U(e_2) = \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \end{pmatrix} \Rightarrow P_U = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$e_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{matrix} x_3 = 1 \\ x_1 = x_2 = 0 \end{matrix} \qquad p_U(e_3) = \begin{pmatrix} -1/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

$P_U^2 = P_U$ Verifichiamo!

$$\frac{1}{9}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad \text{OK!}$$

$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = f_1$ vettori della nuova base

$p_u(f_1) = 0_{\mathbb{R}_3}$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = f_2$ $\qquad p_u(f_2) = f_2$ $\qquad$ nella nuova base

coordinate di un vettore della nuova base rispetto alla vecchia $\quad p_u(f_3) = f_3$ $\qquad P_u' \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$(x_1' \;\; x_2' \;\; x_3')$ tale che

$\langle f_1, f_3 \rangle_E = 0 \rightarrow x_1' + x_2' + x_3' = 0$

$\langle f_2, f_3 \rangle_E = 0 \qquad x_1' - x_3' = 0 \rightarrow x_1' = x_3'$

$\langle f_3, f_3 \rangle_E = 1 \qquad x_1'^2 + x_2'^2 + x_3'^2 = 1$

$x_2' + 2x_3' = 0 \qquad x_2' = -2x_3'$

$(x_3')^2 + 4(x_3')^2 + (x_3')^2 = 1$

$6(x_3')^2 = 1 \qquad x_3' = \pm \frac{1}{\sqrt{6}}$ scelgo ad esempio quella col +

$x_2' = -\frac{2}{\sqrt{6}} \qquad x_1' = \frac{1}{\sqrt{6}}$

$$M = \begin{pmatrix} \frac{1}{\sqrt{3}} & 1/\sqrt{2} & 1/\sqrt{6} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\det(M) = -\frac{1}{\sqrt{2}} \det\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} + \frac{1}{\sqrt{2}} \det\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix} =$$

$$= -\frac{1}{\sqrt{2}}\left(\frac{1}{3\sqrt{2}} + \frac{2}{3\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(-\frac{2}{3\sqrt{2}} - \frac{1}{3\sqrt{2}}\right) = -1$$

Matrice ortogonale

$$M^{-1} = {}^tM \qquad \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\frac{1}{3}\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

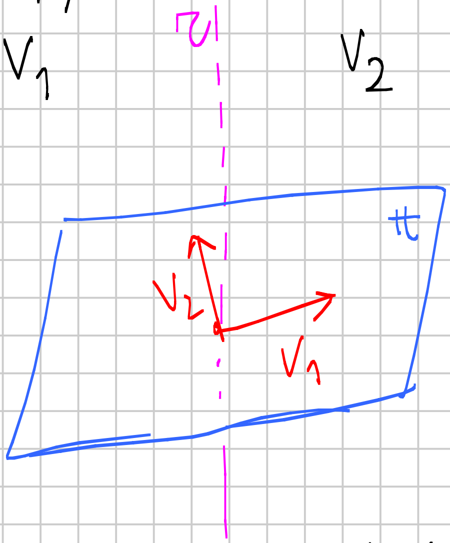
$$= \frac{1}{3}\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} 0 & 3/\sqrt{2} & 3/\sqrt{6} \\ 0 & 0 & -6/\sqrt{6} \\ 0 & -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{6}} \end{pmatrix} =$$

$$= \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \boxed{1} & \boxed{0} \\ 0 & \boxed{0} & \boxed{1} \end{pmatrix}$$

$\mathbb{R}^3$, base canonica

$$\underset{v_1}{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}} \xrightarrow{R} \underset{v_2}{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}}$$

tutti i vettori che appartengono a $\mathrm{Span}(v_1, v_2)$ vengano "trasformati" da R in vettori $\in$ allo stesso sottospazio



r retta $\perp$ a π

$\hat{v}_1 = \frac{1}{3} v_1 \quad \hat{v}_2 = \frac{1}{3} v_2 \quad \langle v_1, v_2 \rangle_E = 0$

$v_1 \perp v_2$

vettore che dà la diret. di r ?

$(x_1 \; x_2 \; x_3)$

v_3

$$\begin{cases} 2x_1 + 2x_2 + x_3 = 0 \\ x_1 - 2x_2 + 2x_3 = 0 \\ x_1^2 + x_2^2 + x_3^2 = 1 \end{cases} \to 3x_1 + 3x_3 = 0$$

$x_3 = -x_1$

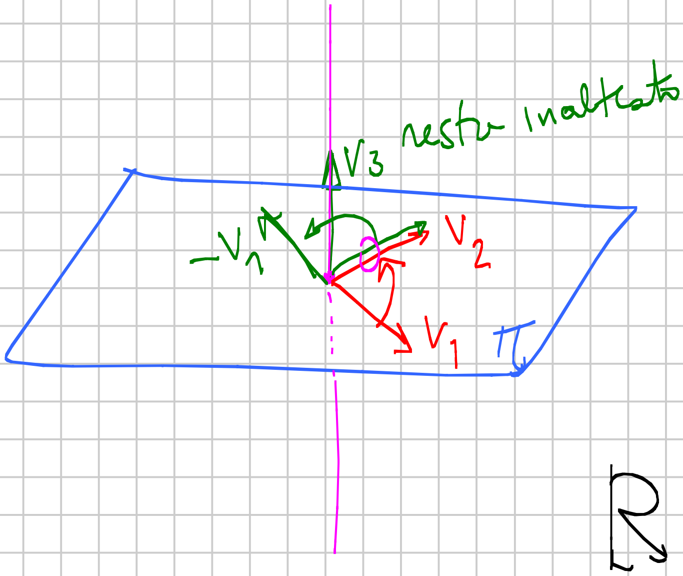
$2x_2 + x_1 = 0$

$x_1 = -2x_2$ $\qquad 4x_2^2 + x_2^2 + 4x_2^2 = 1 \qquad x_2 = \frac{-1}{3}$

$x_3 = +2x_2$ $\qquad x_1 = \frac{2}{3} \qquad x_3 = -\frac{2}{3}$

I vettori $\in$ allo Span$[V_3]$ non devono essere "modificati" da R

I vettori $\in$ allo Span$[V_1, V_2]$ devono "subire" una rotazione intorno alla retta r ($\perp \pi$)



$V_1 \xrightarrow{R} V_2$

$V_2 \rightarrow -V_1$

$V_3 \rightarrow V_3$

$$R = \begin{pmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{1} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta & 0 \\ \operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{V_1}{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}} \xrightarrow{R} \underset{V_2}{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}} \qquad \underset{V_2}{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}} \xrightarrow{R} \underset{-V_1}{\begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}}$$

$$\underset{V_3}{\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}} \xrightarrow{R} \underset{V_3}{\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}}$$

come è fatta R_c^c

(rispetto alla base canonica)

$$R_c^c = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

BRUTALE!

$$\begin{cases} 2a + 2b + c = 1 \\ a - 2b + 2c = -2 \\ 2a - b - 2c = 2 \end{cases} \rightsquigarrow \quad a = \frac{4}{9} \quad b = \frac{4}{9} \quad c = -\frac{7}{9}$$

$$\begin{cases} 2d + 2e + f = -2 \\ d - 2e + 2f = -2 \\ 2d - e - 2f = -1 \end{cases} \qquad d = -\frac{8}{9} \quad e = \frac{1}{9} \quad f = -\frac{4}{9}$$

$$\begin{cases} 2g + 2h + i = 2 \\ g - 2h + 2i = -1 \\ 2g - h - 2i = -2 \end{cases} \qquad g = -\frac{1}{9} \quad h = \frac{8}{9} \quad i = \frac{4}{9}$$

$$R_c^c = \frac{1}{9} \begin{pmatrix} 4 & 4 & -7 \\ -8 & 1 & -4 \\ -1 & 8 & 4 \end{pmatrix}$$

è una matrice ortogonale

$$\mathrm{Tr}(R_c^c) = 1 = 1 + 2\cos\theta$$

angolo di rotaz.

$$\to \cos\theta = 0 \to \theta = \pi/2 \quad \text{oppure } \frac{3\pi}{2}$$

$$\frac{4^2 + (-8)^2 + (-1)^2}{81} = \frac{16+64+1}{81} = \frac{81}{81} = 1$$

$$\frac{(-7)^2 + (-4)^2 + (+4)^2}{81} = \frac{49+16+16}{81} = \frac{81}{81} = 1$$

$$M = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ 1 & 2 & -2 \end{pmatrix} \qquad M^{-1} = {}^tM = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{pmatrix}$$

andate a caccia dei 7

$$M^{-1}R_c^c M = \frac{1}{81}\begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} 4 & 4 & -7 \\ -8 & 1 & -4 \\ -1 & 8 & 4 \end{pmatrix}\begin{pmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ 1 & 2 & -2 \end{pmatrix}$$

$$= \frac{1}{81}\begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} 9 & -18 & 18 \\ -18 & -18 & -9 \\ +18 & -9 & -18 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & -1 \\ +2 & -1 & -2 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & -9 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

OK = a quella scritta nella "miglior base possibile"

$$\lambda_1 = 1 \qquad \lambda_2 = i \qquad \lambda_3 = -i$$

$\lambda_1 =$ corrisponde al sottospazio dei vettori diretti come r, che non sono affetti dalla rotazione

Una base nel sottospazio $\perp$ ad r (vettori $\in$ al piano π) che non è una base di autovettori, perché autovettori non ce ne sono!