EXISTENCE OF MINIMAL CLUSTERS

Let us consider the *cluster isoperimetric problem.* That is, we are given $m \in \mathbb{N}$ and a vector $V \in (\mathbb{R}^+)^M$. A *cluster of volume* V is a family $\mathcal{E} = \{E_1, E_2, \dots, E_m\}$ where $E_i \subseteq \mathbb{R}^N$ are pairwise disjoint Borel sets, with volumes $|E_i| = V_i$ for each $1 \leq i \leq m$. We will write for brevity $|\mathcal{E}| = V$. We say that $\mathcal{E}$ is a *cluster of finite perimeter* if each E_i is a set of finite perimeter, and we call *perimeter of the cluster* $\mathcal{E}$ the quantity

$$P(\mathcal{E}) = \mathscr{H}^{N-1}\Big(\bigcup_{i=1}^m \partial^* E_i\Big).$$

We define then

$$\mathfrak{J}(V) = \inf\big\{P(\mathcal{E}) :\ |\mathcal{E}| = V\big\}.$$

The isoperimetric problem for cluster consists then in looking for minimizers of the functional $\mathfrak{J}$. Each cluster $\mathcal{E}$ such that $P(\mathcal{E}) = \mathfrak{J}(|\mathcal{E}|)$ is called a *minimal cluster.* Our goal is to show the following result.

Theorem 1. *For every* $V \in (\mathbb{R}^+)^M$ *there exist minimal clusters of volume* V.

To prove the result, we start with some simple observations.

Lemma 2. *For every* $V, V', V'' \in (\mathbb{R}^+)^m$ *one has*

$$\mathfrak{J}(V' + V'') \leq \mathfrak{J}(V') + \mathfrak{J}(V''), \tag{1}$$

$$\mathfrak{J}(V) \geq \mathfrak{J}(|V|, 0, \dots, 0) = N\omega_N^{1/N} |V|^{\frac{N-1}{N}}, \tag{2}$$

where we write $|V| = V_1 + V_2 + \dots + V_m$ *to denote the* L^1 *norm of the vector* $V \in \mathbb{R}^m$.

Proof. To prove the first inequality, for every $\varepsilon > 0$ we take two bounded clusters $\mathcal{E}'$ and $\mathcal{E}''$ with volumes $|\mathcal{E}'| = V'$ and $|\mathcal{E}''| = V''$ and such that $P(\mathcal{E}') < \mathfrak{J}(V') + \varepsilon$ and $P(\mathcal{E}'') < \mathfrak{J}(V'') + \varepsilon$. Up to a translation we can assume that the union of the sets E_i' does not intersect the union of the sets E_j''. Hence, define $\mathcal{E}$ the cluster defined by $E_i = E_i' \cup E_i''$ for every $1 \leq i \leq m$. By construction, we have that $|\mathcal{E}| = V' + V''$, thus

$$\mathfrak{J}(V' + V'') \leq P(\mathcal{E}) \leq P(\mathcal{E}') + P(\mathcal{E}'') < \mathfrak{J}(V') + \mathfrak{J}(V'') + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, (1) is proved.

To obtain (2), for any cluster $\mathcal{E}$ with volume $|\mathcal{E}| = V$ we just define F the set $F = \cup_{i=1}^m E_i$. The set F has volume $|V|$, and by construction $\partial^* F \subseteq \cup_{i=1}^m \partial^* E_i$, so

$$P(\mathcal{E}) \geq P(F) \geq \mathfrak{J}(|V|, 0, \dots, 0).$$

Taking the infimum over all possible clusters $\mathcal{E}$, we obtain (2). Notice that $N\omega_N^{1/N}|V|^{\frac{N-1}{N}}$ is the perimeter of a ball of volume $|V|$, which coincides with $\mathfrak{J}(|V|, 0, \dots, 0)$ by the isoperimetric inequality. $\square$

It is reasonable to guess that inequality (1) is strict unless either m' or m'' are the zero vector. We will discusse this issue in the end in Remark 12. We give then the following definition.

Definition 3. *Let $V \in (\mathbb{R}^+)^M$. We say that V is* irreducible *if for every V', V'' such that $V' + V'' = V$ and $\min\{|V'|, |V''|\} > 0$ one has*

$$\mathfrak{J}(V) < \mathfrak{J}(V') + \mathfrak{J}(V'') .$$

We are going to prove Theorem 1 in some steps. First of all, we will show that there exist minimal clusters for every irreducible volume; then, we will use this to deduce the existence of minimal cluster for every volume.

Proposition 4. *Let $V \in (\mathbb{R}^+)^M$ be an irreducible volume. Then, there exist minimal clusters of volume V.*

Proof. Let $\{\mathcal{E}^n\}$, with $n \in \mathbb{N}$, be an optimal sequence of clusters of volume V, that is,

$$|\mathcal{E}^n| = V \quad \forall\, n \in \mathbb{N} , \qquad\qquad P(\mathcal{E}^n) \xrightarrow[n\to\infty]{} \mathfrak{J}(V) .$$

Since V is irreducible, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\mathfrak{J}(V) < \mathfrak{J}(V') + \mathfrak{J}(V'') - \delta \quad \forall\, V',\, V'' \,:\, V' + V'' = V , \quad \min\{|V'|,\, |V''|\} > \varepsilon , \tag{3}$$

as one readily obtains thanks to the observation that $\mathfrak{J} : (\mathbb{R}^+)^M \to \mathbb{R}$ is continuous. Let us then fix $\varepsilon > 0$, and let $n \in \mathbb{N}$ be any number such that

$$P(\mathcal{E}^n) < \mathfrak{J}(V) + \frac{\delta}{3} . \tag{4}$$

Calling for brevity $F^n = \cup_{i=1}^m E_i^n$, up to a translation we can assume that

$$\bigl|F^n \cap x_i > 0\bigr| = \frac{|V|}{2} \quad \forall\, 1 \leq i \leq N . \tag{5}$$

Let then $t \in \mathbb{R}$ be such that

$$\mathscr{H}^{N-1}\bigl(F^n \cap \{x_1 = t\}\bigr) < \frac{\delta}{3} . \tag{6}$$

Call now $\mathcal{E}'$ and $\mathcal{E}''$ the two clusters obtained by intersecting $\mathcal{E}^n$ with the two half-spaces $\{x_1 < t\}$ and $\{x_1 > t\}$, that is, for every $1 \leq j \leq m$ we set

$$E_j' = E_j \cap \{x_1 < t\} , \qquad\qquad E_j'' = E_j \cap \{x_1 > t\} .$$

We have then by (6) and (4)

$$\mathfrak{J}(|\mathcal{E}'|) + \mathfrak{J}(|\mathcal{E}''|) \leq P(\mathcal{E}') + P(\mathcal{E}'') \leq P(\mathcal{E}^n) + 2\mathscr{H}^{N-1}(F^n \cap \{x_1 = t\}) \leq P(\mathcal{E}^n) + \frac{2}{3}\,\delta < \mathfrak{J}(V) + \delta ,$$

and by (3) we deduce

$$\min\{|\mathcal{E}'|,\, |\mathcal{E}''|\} \leq \varepsilon . \tag{7}$$

By Fubini Theorem, we can find $t_1 < 0 < t_2$ such that (6) is satisfied both with $t = t_1$ and $t = t_2$, and such that

$$\max\{|t_1|,\, |t_2|\} < 2\,\frac{|V|}{\delta} .$$

As a consequence, keeping in mind (5), (7) ensures that

$$\big|F^n\cap\{x_1>2|V|/\delta\}\big|\leq\big|F^n\cap\{x_1>t_2\}\big|\leq\varepsilon\,,$$

and similarly

$$\big|F^n\cap\{x_1<-2|V|/\delta\}\big|\leq\big|F^n\cap\{x_1<t_1\}\big|\leq\varepsilon\,,$$

Repeating the same argument in the other $N-1$ directions, we finally deduce that

$$\left|F^n\cap\left[-2\,\frac{|V|}{\delta},2\,\frac{|V|}{\delta}\right]^N\right|\geq|V|-2N\varepsilon \tag{8}$$

for every n for which (4) is true.

Notice now that, for every $1\leq j\leq m$, the characteristic functions $\chi_{E_j^n}$ are bounded in $BV(\mathbb{R}^N)$. Hence, up to a subsequence, we can assume that for every $1\leq j\leq m$ one has $\chi_{E_j^n}\xrightarrow[BV_{\rm loc}]{*}f_j$ when $n\to\infty$. Since the convergence is locally strong in L^1, each function f_j is in fact a characteristic function, say the characteristic function of a set $\overline{E}_j$. We immediately deduce that the sets $\overline{E}_j$ are essentially disjoint, hence $\overline{\mathcal{E}}=\{\overline{E}_1,\overline{E}_2,\,\dots\,,\overline{E}_m\}$ is a cluster. By lower semicontinuity of the perimeter, together with the simple observation that for any cluster $\mathcal{E}$ one has

$$P(\mathcal{E})=\frac{1}{2}\,\left(\sum_{j=1}^m P(E_i)+P\big(\mathbb{R}^N\setminus\cup_{j=1}^m E_j\big)\right),$$

we deduce that

$$P(\overline{\mathcal{E}})\leq\liminf P(\mathcal{E}^n)=\mathfrak{J}(V)\,.$$

Of course, the proof will be concluded as soon as we check that $|\overline{\mathcal{E}}|=V$. In fact, by lower semicontinuity it is clear that $|\overline{\mathcal{E}}|\leq V$ (that is, for every $1\leq j\leq m$ one has $|\overline{E}_j|\leq V_j$), hence it suffices to check that $|F|=|V|$, being $F=\cup_{j=1}^m\overline{E}_j$. Since the convergence of the characteristic functions is strong in $L^1(D)$, being $D=[-2|V|/\delta,2|V|/\delta]^N$, thanks to (8) we know that

$$|F|\geq|V|-2N\varepsilon\,.$$

And finally, since $\varepsilon>0$ was arbitrary, the proof is concluded. □

We can also show that every minimal cluster is bounded.

Proposition 5. *Let $\mathcal{E}$ be a minimal cluster. Then $\mathcal{E}$ is bounded.*

In order to proof this proposition, the following technical result is needed.

Lemma 6. *Let $\mathcal{E}$ be any cluster. For any constant $c>0$, there exist a ball B and a constant $\bar\varepsilon>0$ such that the following holds. Let some constants $-\bar\varepsilon<\varepsilon_j<\bar\varepsilon$ for $1\leq j\leq m$ be given. Then, there exists another cluster $\mathcal{E}'$ such that $E_j\setminus B=E'_j\setminus B$, that is, the two clusters differ only inside the ball B, and moreover*

$$|E'_j|=|E_j|+\varepsilon_j\quad\forall\,1\leq j\leq m\,,\qquad\qquad P(\mathcal{E}')\leq P(\mathcal{E})+c\bigg(\sum_{i=1}^m|\varepsilon_i|\bigg)^{\frac{N-1}{N}}\,. \tag{9}$$

This result is true in a wide generality, and in fact it can be proved in the much stronger version where the constant c is given, but the exponent $\frac{N-1}{N}$ in (9) is replaced by 1. However, this claim is simpler to prove and suffices for our purposes here. The proof of Lemma 6 can be found in the Appendix.

Proof (of Proposition 5). Let us call again $F = \cup_{i=1}^m E_i$, and let B and $\bar\varepsilon > 0$ be given by Lemma 6 with the choice

$$c = \frac{N\omega_N^{1/N}}{2}\,.$$

There exists a large constant R such that the ball B is contained in the ball B_R centered at the origin and with radius R. For every $t > R$, let us call

$$\varphi(t) = \big|F \setminus B_t\big|\,.$$

For every $1 \leq j \leq m$, let us call $E_j^{\rm ext} = E_j \setminus B_t$, and $\varepsilon_j = |E_j^{\rm ext}|$, so that the cluster $\mathcal{E}^{\rm ext}$ satisfies $|\mathcal{E}^{\rm ext}| = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$. By construction, $\sum \varepsilon_j = \varphi(t)$. If t is big enough, then $\varphi(t) < \bar\varepsilon$. As a consequence, applying Lemma 6 we can find a cluster $\mathcal{E}'$ such that (9) holds. For every $1 \leq j \leq m$, let us then call $E_j'' = E_j' \cap B_t$. By construction, $\mathcal{E}''$ is a cluster satisfying $|\mathcal{E}''| = |\mathcal{E}|$. Being $\mathcal{E}$ a minimal cluster, also by (9) we get then

$$P(\mathcal{E}) \leq P(\mathcal{E}'') \leq P(\mathcal{E}) + c\varphi(t)^{\frac{N-1}{N}} - P(\mathcal{E}^{\rm ext}) + \mathscr{H}^{N-1}(F \cap \partial B_t)\,. \tag{10}$$

Let us now notice that $\mathscr{H}^{N-1}(F \cap \partial B_t) = -\varphi'(t)$. Moreover, by (2) we have

$$P(\mathcal{E}^{\rm ext}) \geq J(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_M) \geq J(\varphi(t), 0, \dots, 0) = N\omega_N^{1/N}\varphi(t)^{\frac{N-1}{N}}\,.$$

From the estimate (10) we deduce then

$$-\varphi'(t) \geq \frac{N\omega_N^{1/N}}{2}\,\varphi(t)^{\frac{N-1}{N}}\,,$$

and since $\frac{N-1}{N} < 1$ we further obtain that φ vanishes in finite time, that is, there exists some $\bar t > 0$ such that $\varphi(\bar t) = 0$. This precisely means that $\mathcal{E}$ is bounded. $\square$

Let us now concentrate ourselves on the question whether some given $V \in (\mathbb{R}^+)^M$ is irreducible or not. In positive case, Proposition 4 ensures that there is a minimal cluster of volume V, hence Theorem 1 is already proved. In negative case, instead, there are some nonzero vectors V', V'' such that $V' + V'' = V$ and $\mathfrak{J}(V) = \mathfrak{J}(V') + \mathfrak{J}(V'')$. If both V' and V'' are irreducible, then there are a minimal cluster of volume V' and another minimal cluster of volume V''. Since we can assume the two clusters to have empty intersection, because they are both bounded by Proposition 5, their union is a minimal cluster of volume V, hence Theorem 1 is again proved. As a consequence, in some sense we only have to avoid that this argument has to be repeated infinitely many times. Let us be more precise.

Definition 7. *Let $V^1, V^2, \dots, V^K$ be vectors in $(\mathbb{R}^+)^m$ such that $|V^1| \geq |V^2| \geq \dots \geq |V^K|$, and let $V = \sum_{\ell=1}^K V^\ell$. We say that $(V^1, V^2, \dots, V^K)$ is a* reduction of V *if*

$$\mathfrak{J}(V) = \sum_{\ell=1}^K \mathfrak{J}(V^\ell)\,.$$

Moreover, we say that the reduction $(V^1, V^2, \dots, V^K)$ *is* balanced *if* $K = 1$ *or*

$$|V^{K-1}| \geq \frac{|V^1|}{2}\,.$$

Lemma 8. *Let* $V \in (\mathbb{R}^+)^M$ *be a vector. If* $(V^1, V^2, \dots, V^K)$ *is a reduction of* V*, then there exists also a balanced reduction* $(\widetilde{V}^1, \widetilde{V}^2, \dots, \widetilde{V}^H)$ *of* V *with* $\widetilde{V}^1 = V^1$*.*

Proof. If $K < 3$ or $K \geq 3$ and $|V^{K-1}| \geq |V^1|/2$, then the original reduction is already balanced. Otherwise, let us define a reduction $(\widehat{V}^1, \widehat{V}^2, \dots, \widehat{V}^{K-1})$ as follows. The $K-1$ vectors $\widehat{V}^\ell$ coincide with the $K-2$ vectors V^ℓ for $1 \leq \ell \leq K-2$ together with the vector $V^{K-1} + V^K$, and they are numbered so that $|\widehat{V}^1| \geq |\widehat{V}^2| \geq \dots \geq |\widehat{V}^{K-1}|$. We have that $\widehat{V}^1 = V^1$ since by construction

$$|V^{K-1}| + |V^K| \leq 2|V^{K-1}| < |V^1|\,.$$

Moreover, by (1) we have

$$\mathfrak{J}(V) = \sum_{\ell=1}^{K} \mathfrak{J}(V^\ell) \geq \sum_{\ell=1}^{K-2} \mathfrak{J}(V^\ell) + \mathfrak{J}(V^{K-1} + V^K) = \sum_{\ell=1}^{K-1} \mathfrak{J}(\widehat{V}^\ell) \geq \mathfrak{J}(V)\,,$$

so $(\widehat{V}^1, \widehat{V}^2, \dots, \widehat{V}^{K-1})$ is actually a reduction. With an obvious recursion, after at most $K-2$ steps we obtain a balanced reduction. □

Lemma 9. *For every reduction* $(V^1, V^2, \dots, V^K)$ *of any vector* $V \in (\mathbb{R}^+)^M$ *one has*

$$|V^1| \geq \frac{1}{m}\,|V|\,.$$

Proof. A possible cluster of volume V is given by m disjoint balls of volumes $V_1, V_2, \dots, V_M$. Hence,

$$\mathfrak{J}(V) \leq N\omega_N^{1/N}\Big(V_1^{\frac{N-1}{N}} + V_2^{\frac{N-1}{N}} + \dots + V_m^{\frac{N-1}{N}}\Big)\,.$$

On the other hand, for every reduction $(V^1, V^2, \dots, V^K)$ of V by (2) we have

$$\mathfrak{J}(V) = \sum_{\ell=1}^{K} \mathfrak{J}(V^\ell) \geq N\omega_N^{1/N} \sum_{\ell=1}^{K} |V^\ell|^{\frac{N-1}{N}} \geq \frac{N\omega_N^{1/N}}{|V^1|^{1/N}} \sum_{\ell=1}^{K} |V^\ell| = \frac{N\omega_N^{1/N}|V|}{|V^1|^{1/N}}\,.$$

Putting together the last two estimates we get

$$|V^1| \geq |V|\,\frac{|V|^{N-1}}{\Big(V_1^{\frac{N-1}{N}} + V_2^{\frac{N-1}{N}} + \dots + V_m^{\frac{N-1}{N}}\Big)^N}\,,$$

which concludes the proof since by concavity we have

$$\frac{\Big(V_1^{\frac{N-1}{N}} + V_2^{\frac{N-1}{N}} + \dots + V_m^{\frac{N-1}{N}}\Big)^N}{|V|^{N-1}} \leq m\,.$$

□

Corollary 10. *A balanced reduction* $(V^1, V^2, \dots, V^K)$ *of any vector* $V \in (\mathbb{R}^+)^M$ *has at most length* $K = 2m$*.*

Lemma 11. *Let $V \in (\mathbb{R}^+)^M$ be a given vector. Then, there is a reduction $(V^1, V^2, \dots, V^K)$ of V such that V^1 is irreducible.*

Proof. Let us define

$$\eta = \inf\Big\{|V^1| :\ (V^1, V^2, \dots, V^K) \text{ is a reduction of } V\Big\},$$

and notice that $\eta \geq |V|/m$ by Lemma 9. We aim to prove that η is in fact a minimum.

To do so, by Lemma 8 we can take a sequence of balanced reductions for which $|V^1|$ converges to η. Keeping in mind Corollary 10, and calling $K = 2m$, this means that there are reductions $(V^{1,n}, V^{2,n}, \dots, V^{K,n})$ of V with $n \in \mathbb{N}$ such that $|V^{1,n}| \to \eta$ for $n \to \infty$. By compactness and by the continuity of $\mathfrak{J}$, we immediately deduce that there is a reduction $(\overline{V}^1, \overline{V}^2, \dots, \overline{V}^K)$ of V for which $|\overline{V}^1| = \eta$.

To complete the proof, we have to show that $\overline{V}^1$ is necessarily irreducible. More precisely, since the order or vectors with the same norm is not fixed, we aim to show that there is some irreducible $\overline{V}^\ell$ with $|\overline{V}^\ell| = \eta$. In fact, if none of the vectors $\overline{V}^\ell$ with $|\overline{V}^\ell| = \eta$ is irreducible, substituting each of them with a non-trivial reduction provides another reduction of V having all vectors of norm strictly smaller than η, against the definition of η. □

We are finally in position to prove our main result.

Proof (of Theorem 1). Let us fix the vector $V \in (\mathbb{R}^+)^M$. By Lemma 11 and Lemma 9, we can find a sequence of irreducible vectors $V^n \in (\mathbb{R}^+)^m$ such that

$$V = \sum_{n\in\mathbb{N}} V^n\,, \qquad\qquad \mathfrak{J}(V) = \sum_{n\in\mathbb{N}} \mathfrak{J}(V^n)\,. \tag{11}$$

By Proposition 4, we find a minimal cluster for each volume V^n, that is, a cluster $\mathcal{E}^n$ with $|\mathcal{E}^n| = V^n$ and $P(\mathcal{E}^n) = \mathfrak{J}(V^n)$. Since all the clusters $\mathcal{E}^n$ are bounded by Proposition 5, up to translations we can assume them to be pairwise disjoint. And finally, calling $\mathcal{E}$ the cluster obtained by the union of the clusters $\mathcal{E}^n$, by (11) we have that $\mathcal{E}$ is a cluster of volume m and with perimeter $\mathfrak{J}(V)$, hence a minimal cluster for mass V. The proof is then concluded. □

Remark 12. *Once Theorem 1 has been proved, it is actually not hard to guess that every volume is irreducible. In fact, otherwise the proof provided us with two different "connected components" of a minimal cluster. And in turn, at least if the cluster is regular enough, one can bring two different connected components very close to each other, so that there are two small pieces of the boundaries close to each other and more or less parallel. And then, it is enough to "glue these two pieces together", strictly lowering the total perimeter. A formal proof for the general case of regions of finite perimeter is less obvious.*

Appendix A. Proof of Lemma 6

This appendix is devoted to present a simple proof of Lemma 6. This will be done in three steps. We start with the case of a single set.

Lemma 13. *The result of Lemma 6 is true for the special case $m = 1$, so for a single set E.*

Proof. Let us take a point $x \in \partial^* E$. Up to a translation and a rotation, and just for simplicity of notation, let us suppose that $x = O \in \mathbb{R}^N$ and that $\nu_E(x) = e_N$. Let $\rho \ll 1$ be a small parameter, depending on c and to be specified later. By the blow-up properties of sets of finite perimeter, there exists a small $\overline{r}$ such that, for every $r < \overline{r}$,

$$\frac{\Big|\Big(E \cap \big[-r, r\big]^N\Big)\Delta\Big(\big[-r,r\big]^{N-1} \times [-r, 0]\Big)\Big|}{r^N} < \rho^2\,, \qquad \frac{\Big|P\big(E;(-r,r)^N\big) - r^{N-1}\Big|}{r^{N-1}} < \rho\,. \tag{12}$$

Let us call then $\bar\varepsilon = \rho\overline{r}^N$. For every $|\varepsilon| < \bar\varepsilon$, let us take

$$r = \sqrt[N]{\frac{|\varepsilon|}{\rho}} < \overline{r}\,, \tag{13}$$

and call for brevity

$$D = \Big(E \cap \big[-r, r\big]^N\Big)\Delta\Big(\big[-r,r\big]^{N-1} \times [-r, 0]\Big)\,.$$

By Fubini Theorem and (12), there is some $r/2 < r' < r$ such that, setting $\Gamma = [-r', r']^N$, one has

$$\mathscr{H}^{N-1}\Big(D \cap \partial\Gamma\Big) \leq 2\rho^2 r^{N-1}\,. \tag{14}$$

We make now a first modification to F by setting

$$F = \big(E \setminus \Gamma\big) \cup \big(\Gamma \cap \{x_N < 0\}\big)\,.$$

By (12) we clearly have that

$$\big|E\Delta F\big| < \rho^2 r^N\,. \tag{15}$$

Let us now estimate $P(F) - P(E)$. First of all, let us consider a point $x \in \partial\Gamma$ which belongs to $\partial^* F \setminus \partial^* E$. Up to remove a $\mathscr{H}^{N-1}$-negligible set, such a point has density $1/2$ for F, and either 0 or 1 for E. By definition of F, if $x_N < 0$ then this density has to be 0 (indeed, if it is 1 for E, then *a fortiori* it is 1 for F), and similarly if $x_N > 0$ it has to be 1. Therefore, by (14)

$$\mathscr{H}^{N-1}\big((\partial^* F \setminus \partial^* E) \cap \partial\Gamma\big) \leq \mathscr{H}^{N-1}\big(D \cap \partial\Gamma\big) \leq 2\rho^2 r^{N-1}\,.$$

Moreover, by construction and applying (12) with r', we have

$$P\big(F;(-r',r')^N\big) - P\big(E;(-r',r')^N\big) = r'^{N-1} - (1-\rho)r'^{N-1} = \rho r'^{N-1} \leq \rho r^{N-1}\,.$$

Putting together the last two estimates, and keeping in mind that $\rho \ll 1$, we have then

$$P(F) \leq P(E) + 2\rho r^{N-1}\,.$$

Let us now set ℓ the constant such that

$$r'^{N-1}\ell = \varepsilon - |F| + |E|\,,$$

and notice that by (13) and (15)

$$|\ell| < 2^N \rho r < r'\,,$$

where the last inequality is clearly true as soon as ρ is small enough. We finally define E' as

$$E' = \big(E \setminus \Gamma\big) \cup \big(\Gamma \cap \{x_N < \ell\}\big)\,.$$

By construction we have that $|E'| = |E| + \varepsilon$, and moreover

$$\begin{aligned} P(E') &\leq P(F) + 2(N-1){r'}^{N-2}|\ell| \leq P(E) + 2\rho r^{N-1} + 2^{N+1}(N-1)\rho r^{N-1} \\ &= P(E) + \big(2 + 2^{N+1}(N-1)\big)\rho^{1/N}|\varepsilon|^{\frac{N-1}{N}} . \end{aligned}$$

The thesis is then obtained up to choose ρ sufficiently small, depending on c. □

We can now pass to the case of clusters, modifing the sets around a point which belongs to two different boundaries. More precisely, calling for brevity $E_0 = \mathbb{R}^N \setminus \cup_{i=1}^m E_i$ for any cluster $\mathcal{E} = (E_1, \dots, E_m)$, we have the following result.

Lemma 14. *Let $\mathcal{E}$ be a cluster, and let $x \in \partial^* E_i \cap \partial^* E_j$, where $0 \leq i < j \leq m$. Then, for every constant $c > 0$, there exist two constants $\bar\varepsilon, \bar r > 0$ such that, for every $|\varepsilon| < \bar\varepsilon$, there exists another cluster $\mathcal{E}'$, coinciding with $\mathcal{E}$ outside of a ball B of radius $\sqrt{N}\bar r$ centered at x, and such that*

$$|E'_h \cap B| = \begin{cases} |E_i \cap B| + \varepsilon & \text{if } h = i\,, \\ |E_j \cap B| - \varepsilon & \text{if } h = j\,, \\ |E_h \cap B| & \text{if } h \notin \{i,\, j\}\,, \end{cases} \qquad\qquad P(\mathcal{E}') \leq P(\mathcal{E}) + c|\varepsilon|^{\frac{N-1}{N}} . \tag{16}$$

Proof. The proof is basically the same as that of the previous lemma. Up to a translation and rotation we can assume that $x = 0$ and that $\nu_{E_i}(x) = e_N = -\nu_{E_j}(x)$. Then, for a suitably small ρ to be specified later, we define $\bar r$, $\bar\varepsilon$, r, r' and Γ as in the previous lemma (keeping in mind that x has density $1/2$ for E_i and E_j, and 0 for every other E_h), and this time we define the cluster $\mathcal{F}$ by putting $F_h = E_h$ in $\mathbb{R}^N \setminus \Gamma$ for every $0 \leq h \leq N$, and

$$E'_i \cap \Gamma = \Gamma \cap \{x_N < 0\}\,, \qquad E'_j \cap \Gamma = \Gamma \cap \{x_N > 0\}\,, \qquad E'_h \cap \Gamma = \emptyset \quad \forall\, h \notin \{i,\, j\}\,.$$

The estimate (15) is still valid, in the sense that $|E_i \Delta F_i| \leq \rho^2 r^N$ for every $0 \leq i \leq m$, and exactly as before we readily get

$$P(\mathcal{F}) \leq P(\mathcal{E}) + 2\rho r^{N-1}\,.$$

Again exactly as before, we can define $-2^N\rho r < \ell < 2^N \rho r$ so that, setting $\mathcal{E}'' = \mathcal{E}$ outside of Γ, and

$$E''_i \cap \Gamma = \Gamma \cap \{x_N < \ell\}\,, \qquad E''_j \cap \Gamma = \Gamma \cap \{x_N > \ell\}\,, \qquad E''_h \cap \Gamma = \emptyset \quad \forall\, h \notin \{i,\, j\}\,,$$

we have $|E''_i \cap B| = |E_i \cap B| + \varepsilon$. As in the previous lemma we get

$$P(\mathcal{E}'') \leq P(\mathcal{E}) + \big(3 + 2^{N+1}(N-1)\big)\rho^{1/N}|\varepsilon|^{\frac{N-1}{N}} . \tag{17}$$

The first equality in (16) with $\mathcal{E}''$ in place of $\mathcal{E}'$ is not necessarily true, but by construction it holds with the index i and

$$\big||E''_j \cap B| - |E_j \cap B| + \varepsilon\big| < \rho^2 r^N\,, \qquad \big||E''_h \cap B| - |E_h \cap B|\big| < \rho^2 r^N \quad \forall\, h \notin \{i,\, j\}\,.$$

We can then define $\mathcal{E}'$ with an obvious modification of $\mathcal{E}''$ in $B \setminus [-r, r]^N$ so to adjust the volumes –that is, $\mathcal{E}'$ satisfies the first inequality in (16)– and in such a way that

$$P(\mathcal{E}') \leq P(\mathcal{E}'') + mN\omega_N^{1/N}\rho^{2\frac{N-1}{N}}\, r^{N-1} = P(\mathcal{E}'') + mN\omega_N^{1/N}\rho^{\frac{N-1}{N}}\,|\varepsilon|^{\frac{N-1}{N}} .$$

Putting this estimate together with (17) we get then the full validity of (16), up to have chosen a suitably small ρ. □

We are then finally in position to prove Lemma 6.

Proof (of Lemma 6). It is of course sufficient to show the result in the special case when $\varepsilon_i \neq 0$ for a single $1 \leq i \leq m$. This corresponds to adding a quantity ε_i to the volume of the set E_i without modifying any of the other volumes. Lemma 14 already gives us the result if there is some point $x \in \partial^* E_i \cap \partial^* E_0$, which is of course true if $\mathscr{H}^{N-1}(\partial^* E_i \cap \partial^* E_0) > 0$. Applying twice Lemma 14 gives the result also if there is some $1 \leq j \leq m$ such that $\mathscr{H}^{N-1}(\partial^* E_i \cap \partial^* E_j) > 0$ and $\mathscr{H}^{N-1}(\partial^* E_j \cap \partial^* E_0) > 0$, since the first time we add ε_i to $|E_i|$ and remove ε_i to $|E_j|$, and the second time we adjust the volume of E_j "against E_0". More in general, we can find a "chain" of indices $i_0 = i$, i_1, i_2, $i_n = 0$ such that $\mathscr{H}^{N-1}(\partial^* E_{i_h} \cap \partial^* E_{i_{h+1}}) > 0$ for every $0 \leq h < n$, of course with $n \leq m$, and this will clearly conclude the thesis.

To show the existence of such a "chain" it is enough to consider, in the set $\{0, 1, 2, \ldots, m\}$, the equivalence relation generated by the property that $h \sim k$ if $\mathscr{H}^{N-1}(\partial^* E_h \cap \partial^* E_k) > 0$, and observing that the union of the sets in an equivalence class must necessarily be the whole $\mathbb{R}^N$, which means that all the indices are equivalent, thus the searched chain exists. □