ANALISI MATEMATICA B
LEZIONE 70 – 4.4.2024

Sistemi di equazioni: I ordine, 2×2 lineari, coefficienti costanti, omogenei

Oss Un sistema non lineare può essere "linearizzato".
(Es. già visto: Volterra-Lotka, pendolo)

$\underline{u}'(x) = A\,\underline{u}(x)$ $\qquad$ A matrice $n\times n$

$n=2$ $\qquad \underline{u}(x) \rightsquigarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

le soluzioni rappresentano curve nel piano xy.

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{cases} x'(t) = a\,x(t) + b\,y(t) \\ y'(t) = c\,x(t) + d\,y(t) \end{cases}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Se cambiamo variabili $\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} X \\ Y \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = M \begin{pmatrix} X' \\ Y' \end{pmatrix}$$

M costante

L'equazione $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ diventa:

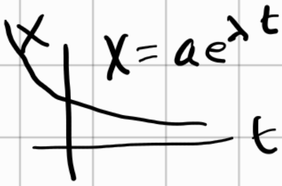
$$M\begin{pmatrix}X'\\Y'\end{pmatrix} = A \cdot M\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}X'\\Y'\end{pmatrix} = \underbrace{M^{-1}AM}\begin{pmatrix}X\\Y\end{pmatrix}$$

1 Caso Se A è diagonalizzabile esiste M tale che $M^{-1}AM = \begin{pmatrix}\lambda & 0\\0 & \mu\end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$

$$\begin{cases} X' = \lambda X \\ Y' = \mu Y \end{cases} \qquad \begin{cases} X(t) = a\, e^{\lambda t} \\ Y(t) = b\, e^{\mu t} \end{cases}$$

λ, μ autovalori di A.

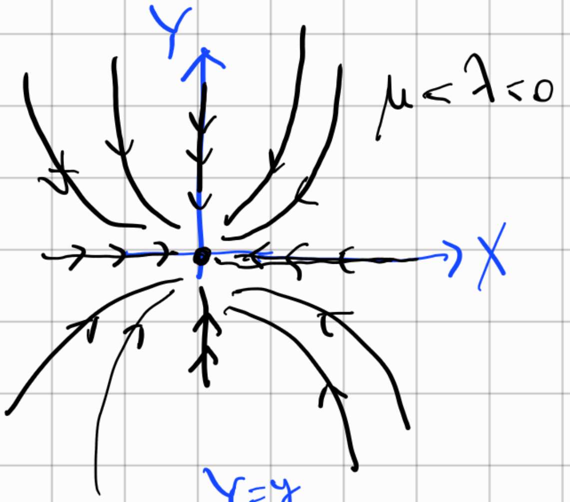


$\lambda < 0$

$\mu < \lambda < 0$

NODO

asintoticamente (STABILE)



$$t = \frac{1}{\mu} \ln \frac{Y}{b}$$

$$X = a\, e^{\frac{\lambda}{\mu} \ln \frac{Y}{b}}$$

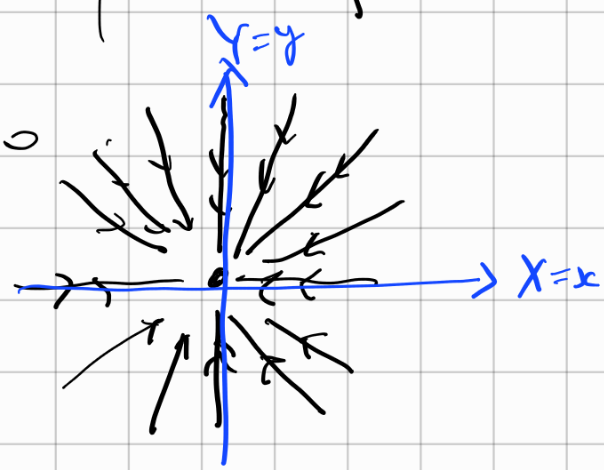
$$= a\left(\frac{Y}{b}\right)^{\frac{\lambda}{\mu}} = c\, Y^{\frac{\lambda}{\mu}}$$

$$\boxed{Y = \tilde{c}\, X^{\frac{\mu}{\lambda}}}$$

$\lambda = \mu < 0$

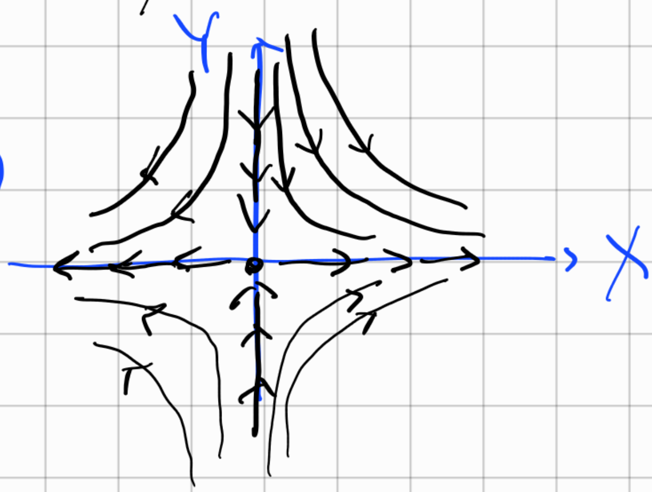
STELLA

asintoticamente (STABILE)



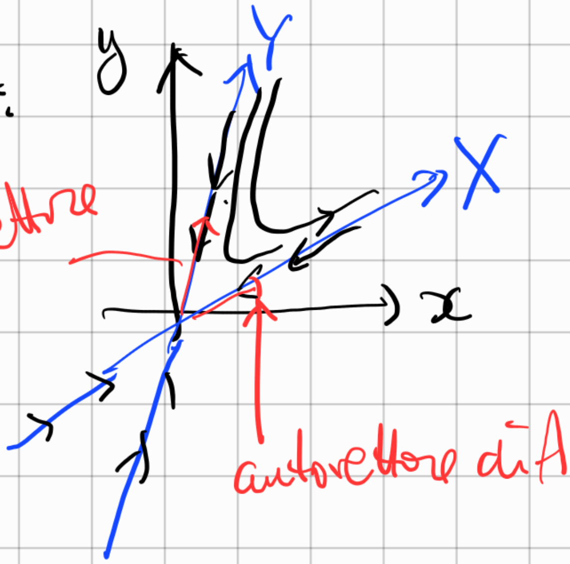
Se $\mu < 0 < \lambda$

SELLA (instabile)



$$\underline{Y = c\, X^{\frac{\mu}{\lambda}}}$$

Nelle coordinate originarie:



Se $P(\lambda) = \det(A - \lambda I)$

Se λ, μ radici reali di P

se $\lambda \neq \mu$ A è diagonalizzabile. $(n=2)$

Caso 2: $\lambda = \mu \in \mathbb{R}$ ma A non è diag.bile.

Allora A è simile a $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ [decomposizione a blocchi di Jordan]

esistono un autovettore:

$\underline{u}$ t.c. $A\underline{u} = \lambda \underline{u}$

$(A - \lambda I)\underline{u} = 0$

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_1 \end{bmatrix} \leftarrow$$

è uno pseudo-autovettore:

$\underline{v}$ t.c. $\quad A\underline{v} = \lambda \underline{v} + \underline{u}$

$$\begin{cases} X' = \lambda X + Y \\ Y' = \lambda Y \end{cases} \qquad Y = b e^{\lambda t}$$

$$X' = \lambda X + b e^{\lambda t}$$

$$X' - \lambda X = b e^{\lambda t}$$

$X' - \lambda X = 0$ ha soluzione $X = a e^{\lambda t}$

Una sol. particolare è $c t e^{\lambda t}$

$$X = (a + ct) e^{\lambda t} \qquad t = \frac{1}{\lambda} \ln \frac{Y}{b}$$

$$X = \left(a + \frac{c}{\lambda} \ln \frac{Y}{b} \right) \frac{Y}{b}$$

$\mu = \lambda < 0$

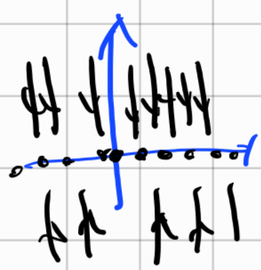
(non diag.bile)

NODO IMPROPRIO

(stabile)

CASI degeneri

$\lambda = 0 \quad \mu \neq 0$



$\lambda = 0 \quad \mu = 0$

Caso 3 $\qquad \lambda, \mu$ complessi coniugati

($\text{radici di } \det(A - tI) = 0$)

$\lambda = \alpha + i\beta \qquad \alpha, \beta \in \mathbb{R}$ {FORMA di JORDAN REALE}

$\mu = \alpha - i\beta.$

(A è diag.bile in $\mathbb{C}$ ma non in $\mathbb{R}$).

Sia $\underline{v} + i\underline{w}$ un autovettore relativo all'autovalore $\alpha - i\beta$.

(Scrivo A nella base $\underline{v}$ $\underline{w}$.)

$$A(\underline{v} + i\underline{w}) = (\alpha - i\beta)\cdot(\underline{v} + i\underline{w})$$

$$= \alpha\underline{v} + \beta\underline{w} + i(-\beta\underline{v} + \alpha\underline{w})$$

$$A\underline{v} + iA\underline{w} \qquad \begin{cases} A\underline{v} = \alpha\underline{v} + \beta\underline{w} \\ A\underline{w} = -\beta\underline{v} + \alpha\underline{w} \end{cases}$$

$$A \sim \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$i \cong J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$1 \cong I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbb{C} \cong \operatorname{Span}_{\mathbb{R}^{2\times 2}}\{I, J\}$$

$$\begin{cases} X' = \alpha X - \beta Y \\ Y' = \beta X + \alpha Y \end{cases} \qquad \beta Y = \alpha X - X'$$

$$X'' = \alpha X' - \beta Y' = \alpha X' - \beta(\beta X + \alpha Y)$$

$$= \alpha X' - \beta\left(\beta X + \frac{\alpha}{\beta}(\alpha X - X')\right)$$

$$X'' = \alpha X' - \beta^2 X - \alpha^2 X + \alpha X'$$

$$X'' - 2\alpha X' + (\alpha^2 + \beta^2) X = 0$$

$$P(\lambda) = \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2$$

$$\lambda_{12} = \alpha \pm \sqrt{\alpha^2 - (\alpha^2 + \beta^2)}$$

$$= \alpha \pm \sqrt{-\beta^2} = \alpha \pm i\beta$$

$$\begin{cases} X(t) = a e^{\alpha t} \cos\beta t + b e^{\alpha t} \sin\beta t \\ Y(t) = \dfrac{\alpha X - X'}{\beta} = \dots = a e^{\alpha t} \sin\beta t - b e^{\alpha t} \cos\beta t \end{cases}$$

$\alpha = 0$

$$\begin{cases} X(t) = \rho e^{\alpha t}\cos(\beta t+\theta) \\ Y(t) = \rho e^{\alpha t}\sin(\beta t+\theta)\end{cases}$$

← definisco θ

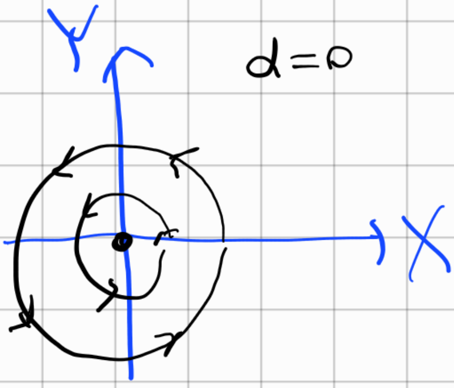
← VERIFICARE!

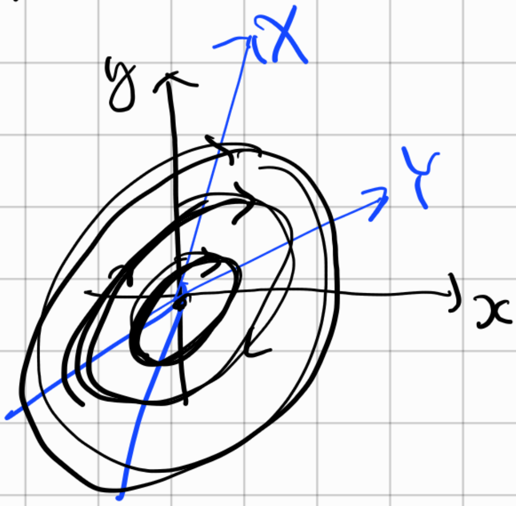
$\alpha = 0$

CENTRO

(stabile)

(ma non asintoticamente)





$\alpha < 0$

FUOCO

(asintoticamente stabile)