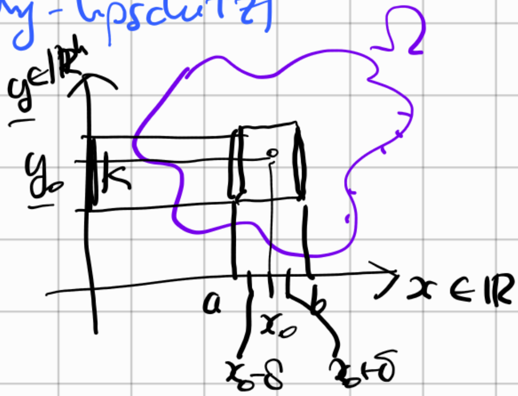
ANALISI MATEMATICA B
LEZIONE 66

Teorema di esistenza e unicità locale (Cauchy-Lipschitz)

$$\begin{cases} \underline{u}'(x) = \underline{f}(x, \underline{u}(x)) \\ \underline{u}(x_0) = \underline{y}_0 \end{cases} \qquad \underline{f} : \Omega \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

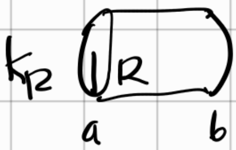
$[a,b] \subseteq \mathbb{R}$

$x_0 \in [a,b]$



$$K_R = \{ \underline{y} \in \mathbb{R}^n : \|\underline{y} - \underline{y}_0\| \le R \}$$

(1) f continua in $[a,b] \times K_R$



(2) $\exists L > 0$ t.c. $\forall x \in [a,b]$ $\forall \underline{y}_1, \underline{y}_2 \in K_R$

$$\|\underline{f}(x, y_1) - \underline{f}(x, y_2)\| \le L \, \|y_1 - y_2\|$$

ovvero $$\frac{\|f(x, y_1) - f(x, y_2)\|}{\|y_1 - y_2\|} \le L$$

Ipotesi di C-L.

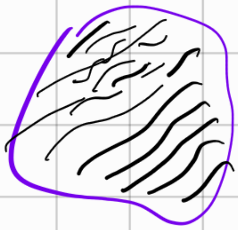
Diremo che $\underline{f} : \Omega \to \mathbb{R}^n$ soddisfa le ipotesi di C-L localmente se:

$$\forall (x_0, \underline{y}_0) \in \Omega \quad \exists \; a < x_0 < b \quad \exists R > 0$$

t.c. $[a,b] \times K_R \subseteq \Omega$

e valgono le ipotesi (1) e (2).

Oss Sotto queste ipotesi nell'intorno di ogni punto di Ω c'è una soluzione dell'equazione $u' = f(x,u)$ passante per quel punto.



Teorema

x definizione $\qquad$ x definizione

$$f \in C^\infty(\Omega) \Rightarrow f \in C^1(\Omega) \Rightarrow \begin{cases} f \in C^0(\Omega) \\ \dfrac{\partial f}{\partial y_1}, \dots, \dfrac{\partial f}{\partial y_n} \in C^0(\Omega) \end{cases} \overset{?}{\Rightarrow} \boxed{f \text{ soddisfa C-L localmente}}$$

[def $f \in C^0$ significa f continua

$f \in C^n$ significa di esistere $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y_1}, \dots, \dfrac{\partial f}{\partial y_n} \in C^{n-1}$]

Es

$$u' = \frac{\sin\left(e^{x \ln(u(x)+2)}\right)}{\cos u(x)}$$

$$f(x,y) = \frac{\sin\left(e^{x \cdot \ln(y+2)}\right)}{\cos y} \qquad f \in C^\infty.$$

ma anche $\quad u' = \sqrt[3]{x} \cdot e^{x u(x)} \qquad f(x,y) = \sqrt[3]{x}\, e^{x \cdot y}$

$f \in C^0 \qquad \dfrac{\partial f}{\partial x}$ non esiste in 0

ma $\dfrac{\partial f}{\partial y} \in C^0$

dim Supponiamo $f \in C^0(\Omega)$, $\frac{\partial f}{\partial y_k} \in C^0(\Omega)$ $k = 1 \dots n$.

se $y \in \mathbb{R}$ $(n=1)$

$$\frac{f(x,y) - f(x,z)}{y-z} = \frac{\partial f}{\partial y}(x, w) \quad \left(\exists\, w \in (y,z)\right)$$

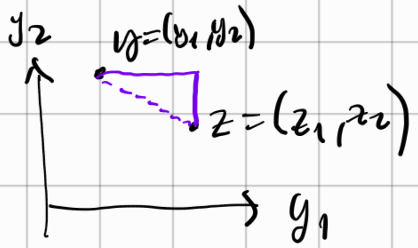
Teorema di Lagrange.

Se $\frac{\partial f}{\partial y}$ è continua per Weierstraß $\left|\frac{\partial f}{\partial y}\right| \le L$

su ogni cilindro $[a,b] \times K_R$ $\exists L$

(2) $$|f(x,y) - f(x,z)| \le L\,|y - z|. \qquad \square$$

Se $y \in \mathbb{R}^n$



$$|f(x, y_1, \dots, y_n) - f(x, z_1, \dots, z_n)| \le |f(x, y_1, \dots, y_n) - f(x, z_1, y_2, \dots, y_n)| +$$
$$+ |f(x, z_1, y_2, \dots, y_n) - f(x, z_1, z_2, y_3, \dots, y_n)| + \dots +$$
$$+ |f(x, z_1, \dots, z_{n-1}, y_n) - f(x, z_1, \dots, z_n)|.$$

Lagrange

$$= \left|\frac{\partial f}{\partial y_1}(x, w_1, y_2, \dots, y_n) \cdot \underline{(y_1 - z_1)}\right| + \left|\frac{\partial f}{\partial y_2}(x, z_1, w_2, y_3, \dots, y_n)(y_2 - z_2)\right|$$
$$+ \dots + \left|\frac{\partial f}{\partial y_n}(x, z_1, \dots, z_{n-1}, w_n) \cdot (y_n - z_n)\right|$$

$$= \left|\frac{\partial f}{\partial y_1}(\dots)\right| \cdot |y_1 - z_1| + \dots + \left|\frac{\partial f}{\partial y_n}(\dots)\right| \cdot |y_n - z_n|$$

$$\le L_1 \|y - z\| + \dots + L_n \|y - z\| \le L \cdot \|y - z\|$$

$L = n \cdot \max\{L_1 \dots L_n\}$

$\Rightarrow$ vale la condizione (2). $\square$

Sistema di equazioni:

$$\begin{cases} u' = f(x, u(x), v(x)) \\ v' = g(x, u(x), v(x)) \\ u(x_0) = y_0 \\ v(x_0) = z_0 \end{cases} \qquad \underline{w}(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}$$

$$\underline{w}'(x) = \begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} \underline{w}'(x) = \underline{h}(x, \underline{w}(x)) \\ \underline{w}(x_0) = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \end{cases} \qquad \underline{h}(x, y, z) = \begin{pmatrix} f(x, y, z) \\ g(x, y, z) \end{pmatrix}$$

Teorema esistenza e unicità locale per equazioni di ordine n

$$(*) \begin{cases} u^{(n)}(x) = g(x, u(x), u'(x), \dots, u^{(n-1)}(x)) \\ u(x_0) = \bar{y}_1 \\ u'(x_0) = \bar{y}_1 \\ \vdots \\ u^{(n-1)}(x_0) = \bar{y}_n \end{cases}$$

Si riconduce ad un sistema del primo ordine:

$$\boxed{\underline{v}(x) = J_u(x) = \begin{pmatrix} u(x) \\ u'(x) \\ \vdots \\ u^{(n-1)}(x) \end{pmatrix}} \leftarrow$$

$$\underline{v}'(x) = \begin{pmatrix} u'(x) \\ u''(x) \\ \vdots \\ u^{(n)}(x) \end{pmatrix} = \begin{pmatrix} v_2(x) \\ v_3(x) \\ \vdots \\ v_n(x) \\ g(x, v_1(x), \dots, v_n(x)) \end{pmatrix}$$

$$\begin{cases} \underline{v}'(x) = \underline{f}(x, \underline{v}(x)) \\ \underline{v}(x_0) = \begin{pmatrix}\bar{y}_1 \\ \vdots \\ \bar{y}_n\end{pmatrix} = \bar{\underline{y}} \end{cases} \qquad \underline{f}(x, y_1 \dots y_n) = \begin{pmatrix} y_2 \\ \vdots \\ y_n \\ g(x, y_1, \dots y_n) \end{pmatrix}$$

Se g soddisfa le condizioni di C-L.

$$g = g(x, \underline{y}) \qquad g : \Omega \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

allora anche $\underline{f}$ soddisfa le condizioni di C-L

$\Rightarrow$ $\exists!$ soluzione $\underline{v}(x)$ del sistema del I ordine

$\Rightarrow$ $\exists!$ $u(x) = v_1(x)$ soluzione dell'equazione di ordine n. $\square$

Es [SPAZIO delle FASI, esempio del pendolo]

$F = ma$ $\qquad u'' = -\sin u$

$u = u(x)$ $\qquad x =$ tempo $\qquad u =$ angolo

$v = u'(x)$

$g(x, y, z) = \sin y$

$$J_u(x) = \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

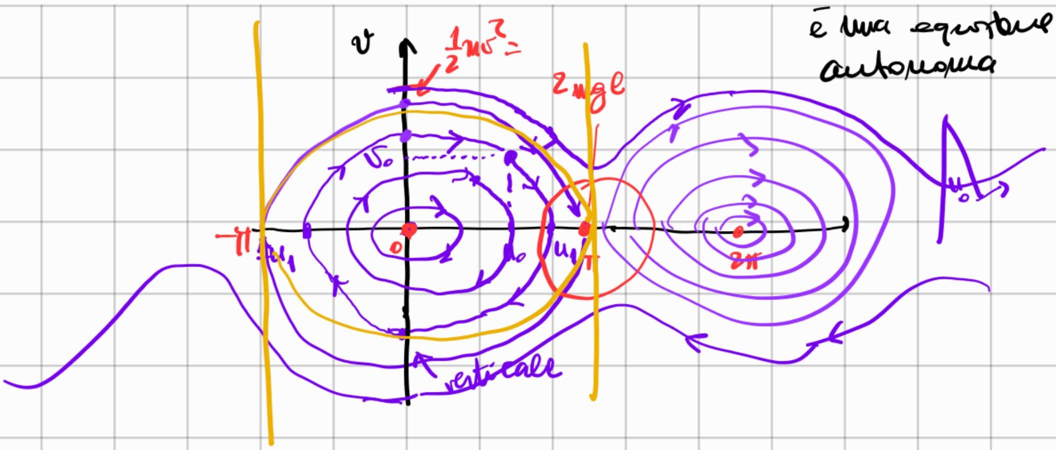
$$\boxed{\underline{f}(x, y, z) = \begin{pmatrix} z \\ -\sin y \end{pmatrix}}$$

$$u'' = -\sin u \longleftrightarrow \begin{cases} v'(x) = -\sin u \\ v(x) = u'(x) \end{cases}$$

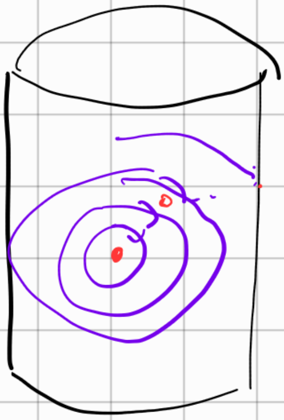
$$\updownarrow$$

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \underline{f}(x, u(x), v(x)) \longleftrightarrow \begin{cases} u'(x) = v(x) \\ v'(x) = -\sin u \end{cases}$$

é una equazione autonoma



ma $u = 2\pi$ é uguale a $u = 0$ $\qquad 2\pi = 0$



vero spazio delle fasi