ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 17 – 3.12.2024

Teo (Regolarità lipschitz)
GIA' VISTO

Teo (Regolarità C^1) $u \in Lip([a,b])$, $L = L(x,y,z)$, $L \in C^0$

$\exists \frac{\partial L}{\partial z} \in C^0$ $\quad z \mapsto \frac{\partial L}{\partial z}(x,y,z)$ iniettiva $\forall (x,y)$

$\frac{\partial L}{\partial z}(x, u(x), u'(x)) \in W^{1,1}$.

Allora $u \in C^1([a,b])$

dim Rademacher: $u \in Lip \Rightarrow u'(x)$ esiste per q.o. x

$$\frac{\partial L}{\partial z}(x, u(x), u'(x)) = g(x) \quad \tilde{\forall} x \text{ con } g \in C^0([a,b])$$

Sia $E = \left\{ x \in [a,b] : \exists u'(x), \frac{\partial L}{\partial z}(x, u(x), u'(x)) = g(x) \right\}$

$|[a,b] \setminus E| = 0 \Rightarrow E$ è denso

CLAIM 1

$\forall x \in [a,b]$ se $\begin{cases} x_k \in E, & x_k \to x, & u'(x_k) \to v \\ x_k' \in E, & x_k' \to x, & u'(x_k') \to w \end{cases}$

allora $v = w$ e $\frac{\partial L}{\partial z}(x, u(x), v) = g(x)$.

$$\frac{\partial L}{\partial z}\Big(x_k, u(x_k), u'(x_k)\Big) \overset{(x_k \in E)}{=} g(x_k)$$

$$\downarrow \quad x_k \to x, \quad u(x_k) \to u(x), \quad u'(x_k) \to v$$

$$\frac{\partial L}{\partial z}\Big(x, u(x), v\Big) = g(x)$$

ma anche $\frac{\partial L}{\partial z}\Big(x, u(x), w\Big) = g(x)$

ma $\frac{\partial L}{\partial z}$ è iniettiva $\Rightarrow v = w$ $\boxed{1}$

Posso definire $v(x)$. Presa $x_k \in E$, $x_k \to x$

$u'(x_k)$ è limitata ($u \in \mathrm{Lip}$)

$\exists k_j$ $\quad u'(x_{k_j}) \to v$ $\quad$ v non dipende dalla successione scelta.

(dunque $v(x) = \lim\limits_{\substack{t \to x \\ t \in E}} u'(t)$)

(ovviamente $v(x) = u'(x)$ se $x \in E$)

$v(x) = u'(x)$ $\forall x$.

e $\forall x \in [a,b]$ $\quad \frac{\partial L}{\partial z}\Big(x, u(x), v(x)\Big) = g(x)$ $\circledast$

<u>CLAIM 2</u> $\quad v \in C^0$

Fissato $x \in [a,b]$, $\forall x_k \in [a,b]$ $\quad x_k \to x$

devo mostrare che $v(x_k) \longrightarrow v(x)$

v limitata $\quad \exists k_j \ \exists w : v(x_{k_j}) \to w$

$$\frac{\partial L}{\partial z}\left(x_{k_j}, u(x_{k_j}), v(x_{k_j})\right) \overset{(*)}{=} g(x_{k_j})$$

$$x_{k_j} \to x, \quad u(x_{k_j}) \to u(x), \quad v(x_{k_j}) \to w, \quad g(x_{k_j}) \to g(x)$$

$$\frac{\partial L}{\partial z}\left(x, u(x), w\right) = g(x)$$

ma anche $\frac{\partial L}{\partial z}\left(x, u(x), v(x)\right) \underset{(*)}{=} g(x)$

$\frac{\partial L}{\partial z}$ iniettiva $\Rightarrow w = v(x)$.

$\Rightarrow v \in C^0 \qquad \boxed{2}$

da ogni successione trovo una estratta che converge a $v(x)$
$\Downarrow$
la funzione converge a $v(x)$

CONCLUSIONE

$$u'(x) = v(x) \quad \forall x \in [a,b]$$

$$v \in C^0 \Rightarrow$$

$$\tilde{u}(x) = \int_a^x v \qquad \tilde{u} \in C^1$$

$u \in C^0 \ \forall x$

$(\tilde{\forall} x)$

$$\tilde{u}'(x) = v(x) = u'(x) \ \tilde{\forall} x \Rightarrow u(x) = \tilde{u}(x) + c$$

C^1

$$\Rightarrow u \in C^1. \qquad \square$$

Teo (Regolarità C^∞) Sia $u \in C^1([a,b])$, $L \in C^k$, $2 \leq k \leq +\infty$, u soddisfa E-L

$$\boxed{\frac{d}{dx} \underbrace{\frac{\partial L}{\partial z}}_{C^{k-1}} (x, u, \underbrace{u'}_{C^0}) = \frac{\partial L}{\partial y} (x, u, u')} \quad \text{in } W^{1,1}.$$

e inoltre $\frac{\partial^2 L}{\partial z^2}(x, u(x), u'(x)) > 0 \quad \forall x \in [a,b]$.

Allora $u \in C^k$.

dim ($k=2$)

$$\underset{C^1}{H(x,z)} = \underset{C^1}{\frac{\partial L}{\partial z}}(x, \underset{C^1}{u(x)}, z) - \underbrace{\int_a^x \frac{\partial L}{\partial y}(t, u(t), u'(t))\, dt}_{C^1}$$

$$\frac{d}{dx} \underbrace{H(x, u'(x))}_{C^0} = \frac{d}{dx} \frac{\partial L}{\partial z}(x, u(x), u'(x)) - \frac{\partial L}{\partial y}(x, u(x), u'(x)) \overset{E\text{-}L}{=} 0$$

derivata debole

$H(x, u'(x))$ è costante q.o. ma è continua $\Rightarrow$ è costante.

$(x, u'(x))$ è contenuto in un insieme di livello di H.

Posso applicare il teorema del Dini?

$H \in C^1$

$$\begin{cases} \frac{\partial H}{\partial x}(x,z) = \frac{\partial^2 L}{\partial z \partial x}(x, u(x), z) + \frac{\partial^2 L}{\partial z \partial y}(x, u(x), z) u'(x) - \frac{\partial L}{\partial y}(x, u(x), u'(x)) \\ \frac{\partial H}{\partial z}(x,z) = \frac{\partial^2 L}{\partial z^2}(x, u(x), z) \end{cases}$$

$$\frac{\partial H}{\partial z}\big(x, u'(x)\big) = \frac{\partial^2 L}{\partial z^2}\big(x, u(x), u'(x)\big) > 0$$
$$\neq 0$$

Posso applicare Dini!

Fissato $x_0 \in (a,b)$ c'è un intorno di x_0 in cui la curva $(x, u'(x))$ è un grafico: $\boxed{u'(x) = z(x)}$ con $z \in C^1$.

$\exists!\, z$

$$u'(x) \in C^1 \qquad H(x, u'(x)) \equiv \text{costante} \qquad u \in C^2$$

$$\Big\downarrow \frac{d}{dx}$$

$$\frac{\partial H}{\partial x}\big(x, u'(x)\big) + \frac{\partial H}{\partial z}\big(x, u'(x)\big)\, u''(x) = 0$$

$$u''(x) = -\frac{\frac{\partial H}{\partial x}\big(x, u'(x)\big)}{\frac{\partial H}{\partial z}\big(x, u'(x)\big)} = -\frac{\overset{C^{k-1}}{\frac{\partial^2 L}{\partial z \partial x}}\overset{C^{k-1}}{(x,u,u')} + \overset{C^{k-1}}{\frac{\partial^2 L}{\partial z \partial y}}\overset{C^{k-1}}{(x,u,u')}\,\overset{C^{k}}{u'} - \frac{\partial L}{\partial y}\overset{C^{k-1}}{(x,u,u')}}{\underset{C^{k-1}}{\frac{\partial^2 L}{\partial z^2}}\underset{C^{k-1}}{(x,u,u')}}$$

$$\Uparrow$$

$$u \in C^k, \; L \in C^{k+1}$$

BOOTSTRAP!

$$C^{k-1} \Rightarrow u \in C^{k+1}$$

□