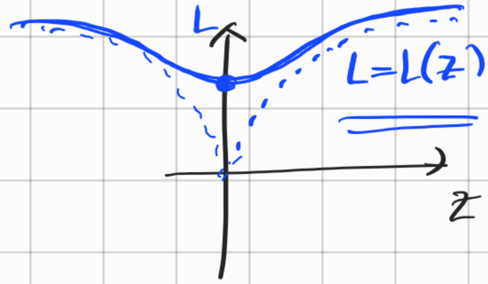
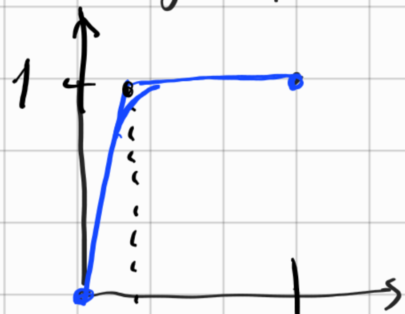
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 7

Es

$$\begin{cases} \mathcal{L}(u) = \int_0^1 \sqrt[3]{1+(u'(x))^2} \\ u(0)=0 \\ u(1)=1 \end{cases} \qquad L(x,y,z) = \sqrt[3]{1+z^2}$$

$$\mathcal{L}(u) \to \min$$





$$\int_0^1 |u'(x)|\,dx = \underbrace{u(1)-u(0)}$$

$L(z)=z$

$\hookrightarrow$ qualunque funzione crescente paga uguale.

$$u_k(x) = \begin{cases} kx & \text{se } x \le \frac{1}{k} \\ 1 & \text{se } x \geq \frac{1}{k} \end{cases} \qquad u_k \in C^1_p([0,1])$$

$$\mathcal{L}(u_k) = \int_0^{1/k} \sqrt[3]{1+k^2} + \int_{\frac{1}{k}}^1 \sqrt[3]{1+0^2}$$

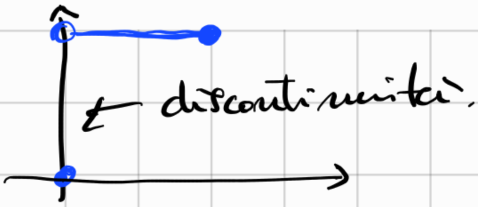
$$= \underbrace{\cancel{\frac{\sqrt[3]{1+k^2}}{k}}}_{0} + \Big(1 - \underbrace{\cancel{\frac{1}{k}}}_{0}\Big) \xrightarrow[k\to\infty]{} 1$$

$$\mathcal{L}(u) = \int_0^1 \sqrt[3]{1+(u')^2} \geq \int_0^1 1 = 1$$

lemma di allisciamento

$$1 = \inf_{\begin{cases} C^1_p \\ u(0)=0 \\ u(1)=1 \end{cases}} \mathcal{L} \overset{\checkmark}{=} \inf_{\begin{cases} C^\infty \\ u(0)=0 \\ u(1)=1 \end{cases}} \mathcal{L}$$

$$\mathcal{L}(u) = 1 \Rightarrow (u')^2 = 0$$
$$\Downarrow$$
$$u' = 0$$
$$u = \text{costante}$$

Moralmente il minimo è 

Esempio (Weierstraß)

$$\begin{cases} \mathcal{I}(u) = \dfrac{1}{2}\displaystyle\int_0^1 x \cdot |u'(x)|^2\, dx \longrightarrow \min \\ u(0) = 1 \\ u(1) = 0 \end{cases}$$

$$L(x,y,z) = x \cdot z^2$$

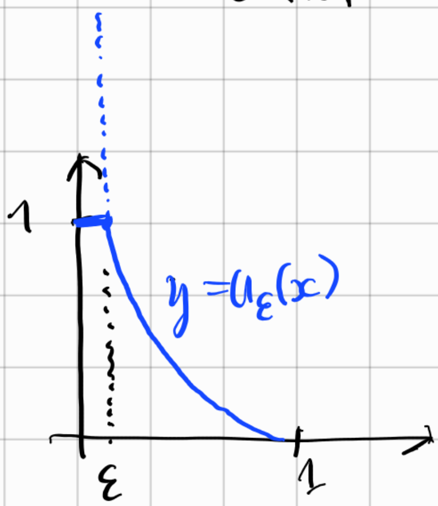
è convessa in z $\forall x$ ma non uniformemente.

$$\frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial z} = 2xz$$

E-L $\qquad 0 = \dfrac{d}{dx}\left(2x u'(x)\right)$

$$x u'(x) = c \qquad (\text{costante})$$

$$u'(x) = \frac{c}{x} \qquad \begin{cases} u(x) = c \ln x + d \\ u(1) = 0 \qquad d = 0 \\ u(0) \to \pm\infty \neq 1 \end{cases}$$



$$u_\varepsilon(x) = \begin{cases} 1 & \text{se } x \leq \varepsilon \\ \dfrac{\ln x}{\ln \varepsilon} & \text{se } x \geq \varepsilon \end{cases}$$

$$1 = u_\varepsilon(\varepsilon) = c \ln \varepsilon \qquad c = \frac{1}{\ln \varepsilon}$$

$$\mathcal{I}(u_\varepsilon) = \int_0^\varepsilon \cancel{x \cdot 0}\, dx + \int_\varepsilon^1 x \cdot \left(\frac{1}{x \ln \varepsilon}\right)^2 dx$$

$$= \frac{1}{\ln^2\varepsilon}\int_\varepsilon^1 \frac{1}{x}\,dx = \frac{1}{\ln^2\varepsilon}\left[\ln x\right]_\varepsilon^1$$

$$= \frac{-\ln\varepsilon}{\ln^2\varepsilon} = -\frac{1}{\ln\varepsilon} \to 0^+$$

$\mathcal{L}(u) \geq 0 \quad \forall u.$ $\qquad \inf_{C^1} \mathcal{L} = 0$

(ma anche C^∞)

$\left.\begin{array}{l} u(0)=1 \\ u(1)=0 \end{array}\right\}$

ma se fosse $\mathcal{L}(u)=0 \Rightarrow u$ costante impossibile.

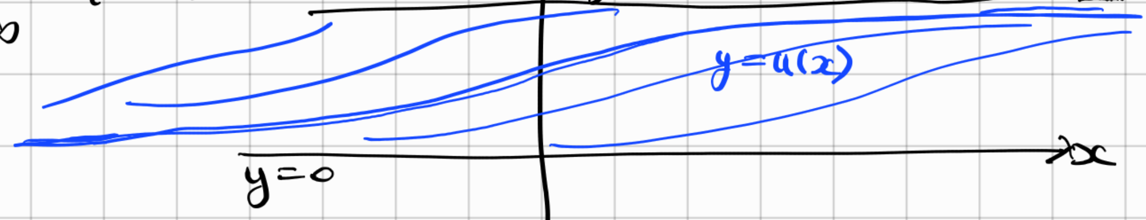
Un esempio in cui $L(y,z)$ è convessa in z $(\forall y)$.

ma non nella coppia (y,z)

Esempio (transizione di fase)

$$\begin{cases} \mathcal{L}(u) = \displaystyle\int_{-\infty}^{+\infty}\left[W(u(x)) + \frac{1}{2}|u'(x)|^2\right]dx \\ \lim\limits_{x\to-\infty} u(x) = 0 \\ \lim\limits_{x\to+\infty} u(x) = 1 \end{cases}$$

$$W(y) = \frac{1}{2}\, y^2(1-y)^2$$



$$L(x,y,z) = W(y) + \frac{1}{2}z^2 \qquad \frac{\partial L}{\partial z} = z$$

<u>Eq. Beltrami</u> $\quad L - u' \frac{\partial L}{\partial z} = \text{cost.} \qquad [L = L(y,z)]$

$$W(u(x)) + \frac{1}{2}(u')^2 - u' \cdot u' = c$$

$$\frac{1}{2}u^2(1-u)^2 - \frac{1}{2}(u')^2 = c$$

$$(u')^2 = u^2(1-u)^2 - 2c$$

$u \to 0$ se $x \to -\infty$

$\Downarrow$

$u^2(1-u)^2 \to 0$

$(u')^2 \to -2c$

$\Downarrow$

$u' \to 0 \Rightarrow c = 0$

$$u' = \pm u(1-u)$$

- $0 \le u \le 1$
- $u' \geq 0$

$$u' = u(1-u) \qquad \frac{u'}{u(1-u)} = 1$$

$$\int \frac{1}{u(1-u)}\,du = x + c \qquad \frac{1}{u(1-u)} = \frac{1}{u} + \frac{1}{1-u}$$

$$\int \frac{1}{u(1-u)}\,du = \int \frac{1}{u} + \int \frac{1}{1-u} = \ln u - \ln(1-u)$$

$$\ln u - \ln(1-u) = x + c$$

$$\ln \frac{u}{1-u} = x + c \qquad \frac{u}{1-u} = e^{x+c}$$

$$u = (1-u)e^{x+c} \qquad u = e^{x+c} - u\,e^{x+c}$$

$$(1+e^{x+c})u = e^{x+c} \qquad u = \frac{e^{x+c}}{1+e^{x+c}}$$

c mi dà una traslazione lungo l'asse delle x.

scegliamo $c = 0$

$$u_0(x) = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}$$

è crescente

$$\left.\begin{array}{l} u_0(x) \to 0 \quad \text{per } x \to -\infty \\ u_0(x) \to 1 \quad \text{per } x \to +\infty \\ u_0(x) \text{ è strett. crescente} \end{array}\right\} \quad \text{ok!}$$

u_0 è minimo di $\mathcal{I}(u)$?

$$\mathcal{I}(u) = \frac{1}{2}\int_{-\infty}^{+\infty} \left[u^2(1-u)^2 + (u')^2 \right] \overset{?}{\geq} \mathcal{I}(u_0)$$

$$a^2 + b^2 \underset{\text{Young}}{\geq} 2ab \qquad \frac{a^2+b^2}{2} \geq ab$$

$$0 \leq (a-b)^2 = a^2 + b^2 - 2ab$$

$$\frac{1}{2}\int_{-\infty}^{+\infty} \left[\left(u(1-u)\right)^2 + (u')^2 \right] \geq \int_{-\infty}^{+\infty} u(1-u) \, \underbrace{u'(x)\,dx}$$

$u = u(x)$

$du = u'(x)\,dx$

$$= \int_{u(-\infty)}^{u(+\infty)} (u - u^2)\,du = \left[\frac{u^2}{2} - \frac{u^3}{3} \right]_0^1$$

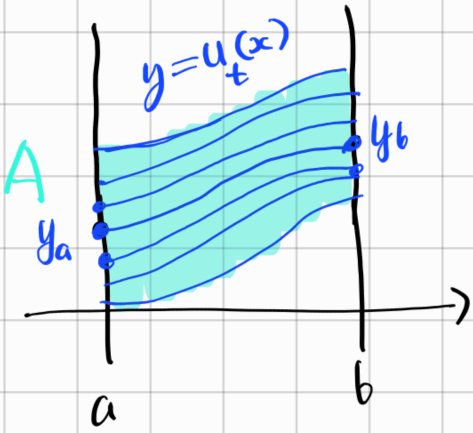
$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\mathcal{I}(u) = \frac{1}{6} \iff a^2 + b^2 = 2ab$$

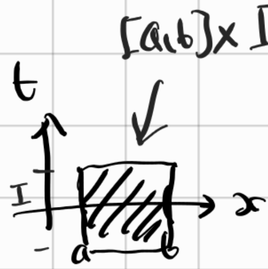
$$\Downarrow$$

$a = b$ cioè $u(1-u) = u' \implies \underline{\underline{u = u_0}}$ $\square$

CALIBRAZIONI

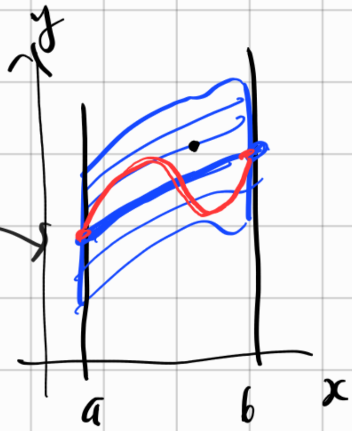


$$\begin{cases} \mathcal{L}(u)=\int_a^b L(x,u(x),u'(x))\,dx \\ u(a)=y_a \\ u(b)=y_b \end{cases}$$



al variare di $t\in I$ intervallo intorno di 0
trovo delle soluzioni di E–L: $u_t(x)$
che "fibrano" una regione A cioè

$$f(x,t)=(x,u_t(x))$$



$R=[a,b]\times I$
$(x\ ,\ t)$

IPOTESI di CONVESSITÀ

$\forall\, x,y$
$z \mapsto L(x,y,z)$
è convessa

$f: R \to A$ $\quad$ f bigettiva. (in realtà sarà un diffeomorfismo)

Teorema/Metodo Allora $u_t(x)$ realizza

il minimo di $\mathcal{L}(u)$ sulla classe di

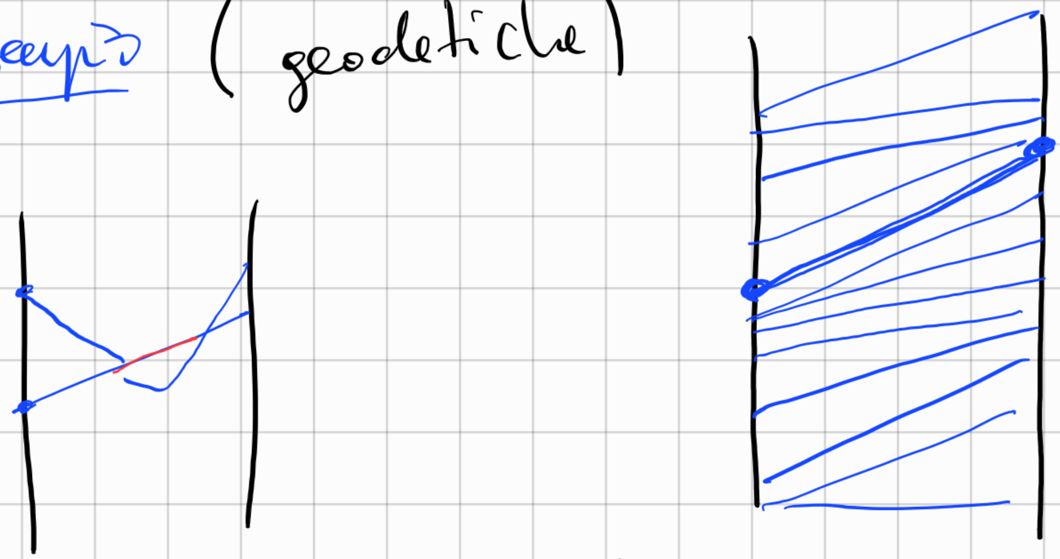
tutte le $u:[a,b]\to\mathbb{R}$ tali che:

$u(a)=u_t(a)$, $u(b)=u_t(b)$

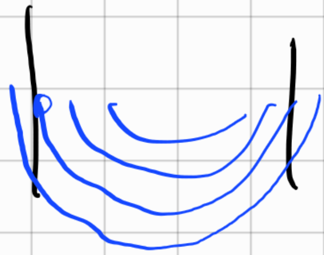
e $(x,u(x))\in A$ $\forall x$

+ ipotesi di regolarità

Esempio (geodetiche)



Esempio (brachistocrona)



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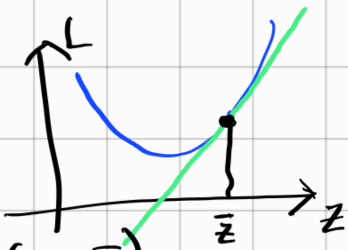
dim (esecuzione del metodo)

Passo 1 Usare la convessità per rendere il funzionale "lineare" in z



Fissati (x,y) e scelto $\bar z = \bar z(x,y)$

$$\underbrace{L(x,y,z)}_{\text{Grafico di } L} \geq \underbrace{L(x,y,\bar z(x,y)) + \frac{\partial L}{\partial z}(x,y,\bar z(x,y))\cdot(z-\bar z)}_{\text{retta tangente in } z=\bar z}$$

Scelgo:

$$\bar z(x,y) = \bar z(x,u_t(x)) = u_t'(x)$$

$(x,y)\in A \Rightarrow (x,y) = (x,u_t(x))$ $\exists!\, t$

$$y = u_t(x)$$
$$\bar z = u_t'(x)$$

$$M(x,y,z) = L(x,y,\bar z(x,y)) + \frac{\partial L}{\partial z}(x,y,\bar z(x,y))\cdot(z-\bar z(x,y))$$

$$\mathcal{L}(u) = \int_a^b L(x,\underset{y}{u(x)},\underset{z}{u'(x)})\,dx \geq \int_a^b M(x,u(x),u'(x))\,dx = \mathcal{M}(u)$$

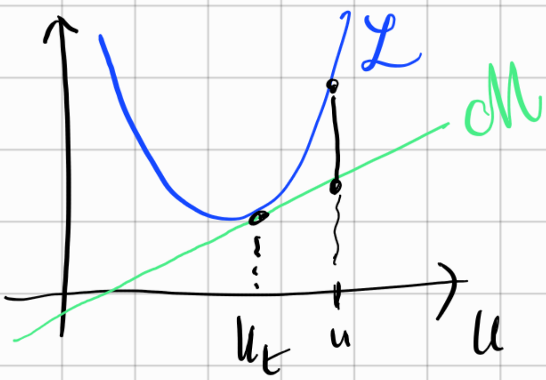
$$\mathcal{L}(u_t) \overset{?}{=} \mathcal{M}(u_t)$$

$$\mathcal{L}(u_t) = \int_a^b L(x, u_t(x), u_t'(x))\,dx \qquad u_t'(x) = \Xi_t(x, u(x))$$

$$= \int_a^b L(x, u_t(x), \Xi_t(x, u_t(x)))\,dx$$

$$= \int_a^b M(x, u_t(x), u_t'(x))\,dx = \mathcal{M}(u_t)$$

<u>lemma</u> banale



Se $\mathcal{L}(u) \geq \mathcal{M}(u)$ $\quad \forall u$ t.c. $(x, u(x)) \in A$.

e se $\mathcal{L}(u_t) = \mathcal{M}(u_t)$

Se u_t è minimo per $\mathcal{M}$

allora u_t è minimo per $\mathcal{L}$.

Basta mostrare che u_t è minimo per $\mathcal{M}$.

(PROSSIMA LEZIONE)