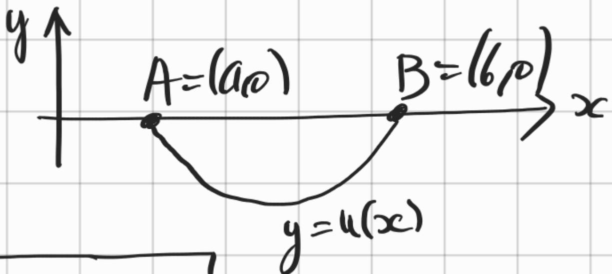
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 3 - 3.10.2024

Catenaria



$$\mathcal{F}(u) = \int_a^b u(x) \cdot \sqrt{1 + (u'(x))^2}\, dx$$

$$\mathcal{G}(u) = \int_a^b \sqrt{1+(u'(x))^2}\, dx$$

$$\begin{cases} \mathcal{F}(u) \longrightarrow \min \\ \mathcal{G}(u) = L \\ u(a) = u(b) = 0 \end{cases} \qquad \text{"} \nabla \mathcal{F}(u) = \lambda \nabla \mathcal{G}(u) \text{"}$$

$$\mathcal{L}(u) = \mathcal{F}(u) - \lambda \mathcal{G}(u) \qquad \text{(moltiplicatori di Lagrange)}$$

$$= \int_a^b (u-\lambda)\sqrt{1+(u')^2}\, dx$$

$$L(x,y,z) = L(y,z) = (y-\lambda)\cdot\sqrt{1+z^2}$$

Posso usare l'eq. di Beltrami: $L - u' \dfrac{\partial L}{\partial z} = \text{cost.}$

$$\frac{\partial L}{\partial z} = (y-\lambda)\frac{z}{\sqrt{1+z^2}}$$

$$(u-\lambda)\sqrt{1+(u')^2} - u' \cdot (u-\lambda)\frac{u'}{\sqrt{1+(u')^2}} = c$$

$$(u-\lambda)\left(1+\cancel{(u')^2}-\cancel{(u')^2}\right) = c\sqrt{1+(u')^2}$$

$$(u-\lambda)^2 = c^2\left(1+(u')^2\right)$$

$$(u')^2 = \frac{(u-\lambda)^2-c^2}{c^2} = \left(\frac{u-\lambda}{c}\right)^2 - 1$$

$$u' = \pm\sqrt{\left(\frac{u-\lambda}{c}\right)^2-1}$$

$$\frac{u'}{\sqrt{\left(\frac{u-\lambda}{c}\right)^2-1}} = \pm 1 \qquad \int \frac{du}{\sqrt{\left(\frac{u-\lambda}{c}\right)^2-1}} = \pm(x-x_0)$$

$$\int \frac{1}{\sqrt{s^2-1}}\,ds = \operatorname{settcosh} s$$

$$\left\{\begin{aligned} \cosh s &= \frac{e^s+e^{-s}}{2} \\ \sinh s &= \frac{e^s-e^{-s}}{2} \end{aligned}\right. \qquad \boxed{\cosh^2 s - \sinh^2 s = 1}$$

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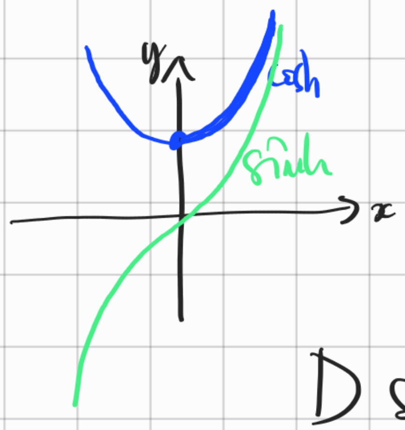


$$\operatorname{settsinh} = \sinh^{-1}$$

$$D\operatorname{settsinh} s = \frac{1}{\cosh(\operatorname{settsinh} s)} = \frac{1}{\sqrt{s^2+1}}$$

$$D\operatorname{settcosh} s = \frac{1}{\sinh(\operatorname{settcosh} s)} = \frac{1}{\sqrt{s^2-1}}$$

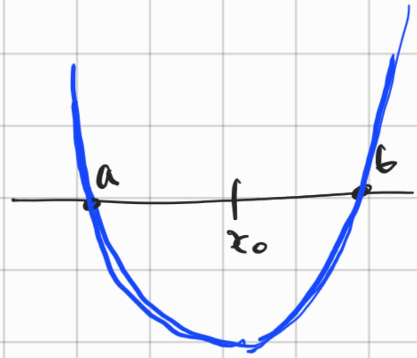
$$\int \frac{du}{\sqrt{\left(\frac{u-\lambda}{c}\right)^2 - 1}} = c \ \text{sett}\cosh\left(\frac{u-\lambda}{c}\right) = \pm (x - x_0)$$

$$\underbrace{\text{sett}\cosh\left(\frac{u-\lambda}{c}\right)}_{} = \pm \frac{x - x_0}{c}$$

$\left\{ \right.$ CONDIZIONI VARIE.

$$\frac{u-\lambda}{c} = \cosh\left(\frac{x-x_0}{c}\right)$$

$$u(x) = c \cosh\left(\frac{x-x_0}{c}\right) + \lambda$$



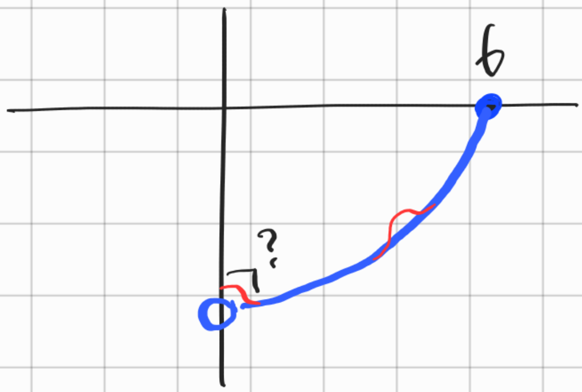
$$u(a) = u(b) = 0 \qquad \underbrace{x_0 = \frac{a+b}{2}}_{}$$

$$0 = u(a) = c \cosh\left(\frac{x - x_0}{c}\right) + \lambda$$

$$\lambda = - c \cosh\left(\frac{x - x_0}{c}\right) \quad \leftarrow \quad \text{trovo } c$$

$$G[u] = \int_a^b \sqrt{1 + (u')^2} = L \quad \leftarrow \quad \text{e } \lambda$$

ESTREMI LIBERI



$$\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx$$

$$\begin{cases} \mathcal{L}(u) \to \min \\ u(b) = y_b \\ \text{ma } u(a) \text{ è libero} \end{cases}$$

VARIAZIONI:

$$0 = \frac{d}{d\varepsilon} \mathcal{L}(u + \varepsilon \varphi) \Big|_{\varepsilon = 0} = \int_a^b \left[\frac{\partial L}{\partial y} \cdot \varphi + \frac{\partial L}{\partial z} \varphi' \right] =$$

$$0 \overset{\forall\varphi}{=} \underbrace{\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial z}\right]\cdot\varphi(x)\,dx}_{1} + \underbrace{\left[\frac{\partial L}{\partial z}\cdot\varphi\right]_a^{\cancel{b}}}$$ $\boxed{\varphi(b)=0}$

Se scelgo φ con anche $\varphi(a)=0$

ottengo la solita E-L : $\boxed{\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial z}}$.

$\Rightarrow$ 1 è sempre nullo!

dunque: $$0 = \underbrace{\frac{\partial L}{\partial z}\big(a, u(a), u'(a)\big)}\cdot\varphi(a)$$

Se scelgo φ con $\varphi(a)\neq 0$ trovo

$$\boxed{\frac{\partial L}{\partial z}\big(a, u(a), u'(a)\big) = 0}$$

Se avessi fissato $u(a)$ e lasciato libero $u(b)$ avrei avuto

$$\boxed{\left.\frac{\partial L}{\partial z}\right|_{x=b} = 0}$$

Se lasciavi liberi sia $u(a)$ che $u(b)$

otterrei $$\begin{cases} \left.\frac{\partial L}{\partial z}\right|_{x=a} = 0 \\ \left.\frac{\partial L}{\partial z}\right|_{x=b} = 0 \end{cases}$$

Nell'esempio della catenaria
EL è la stessa equazione. In più ottengo:

$$\frac{\partial L}{\partial z}\big(a, u(a), u'(a)\big) = 0 \qquad\qquad L = (y-\lambda)\sqrt{1+z^2}$$

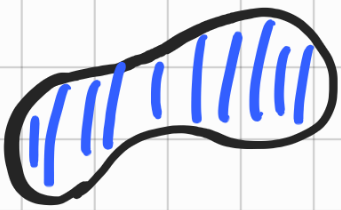
$$\frac{\partial L}{\partial z} = (y-\lambda)\frac{z}{\sqrt{1+z^2}}$$

$$\downarrow$$

$$(u(a)-\lambda)\frac{u'(a)}{\sqrt{1+(u'(a))^2}} = 0 \quad < \begin{array}{l} u(a)=\lambda \quad ✗ \\ \boxed{u'(a)=0} \quad ✓ \end{array}$$

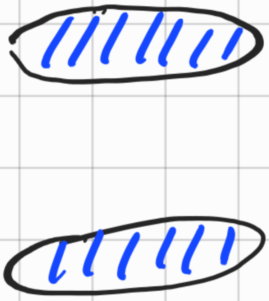
$$\left[\, u(x) = c\cdot\cosh\left(\frac{x-x_0}{c}\right) + \lambda \neq \lambda \qquad \text{se } c \neq 0, \quad \cosh \geq 1 \,\right]$$

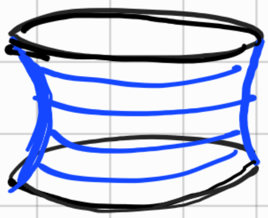
CATENOIDE



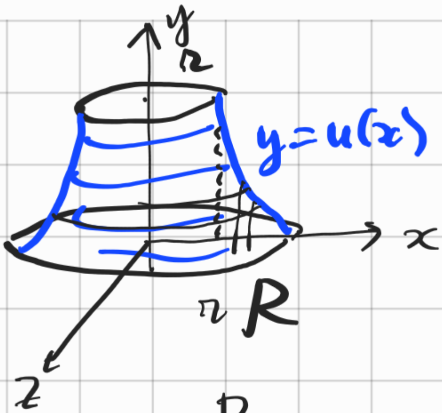
Problema di Plateau:

Fissata una curva nello spazio
trovo la superficie di area minima
che "borda" la curva data.





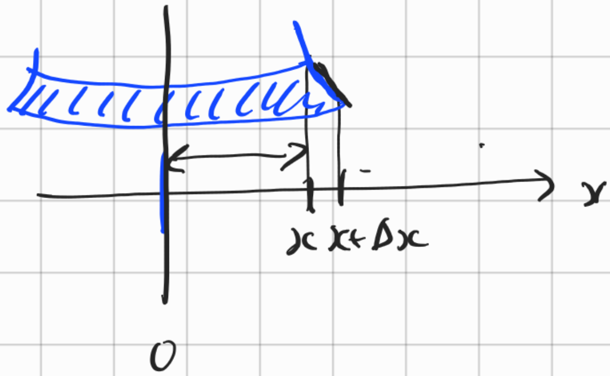
Supponiamo che la superficie sia una superficie
di rotazione attorno all'asse comune dei due cerchi



$u:[r,R]\to\mathbb{R}$

$u(R)=0$

$u(r)=h$



$$\mathcal{L}(u)=2\pi\int_r^R x\cdot\sqrt{1+(u'(x))^2}\,dx$$

area della superficie di rotazione

$$L(x,y,z)=x\sqrt{1+z^2} \qquad \frac{\partial L}{\partial y}=0 \qquad \frac{\partial L}{\partial z}=x\frac{z}{\sqrt{1+z^2}}$$

E-L: $$0=\frac{d}{dx}\left(x\cdot\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)$$

E' la curvatura media della superficie lo vediamo nella prossima lezione

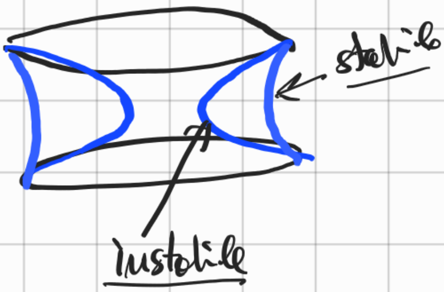
$$\frac{x\,u'}{\sqrt{1+(u')^2}}=c$$

$$x^2(u')^2=c^2\left(1+(u')^2\right)$$

$$(u')^2=\frac{c^2}{x^2-c^2} \qquad u'=\frac{1}{\sqrt{\left(\frac{x}{c}\right)^2-1}}$$

$$u(x)=c\cdot\operatorname{settcosh}\frac{x}{c}+k$$

grafico di u è una catenaria.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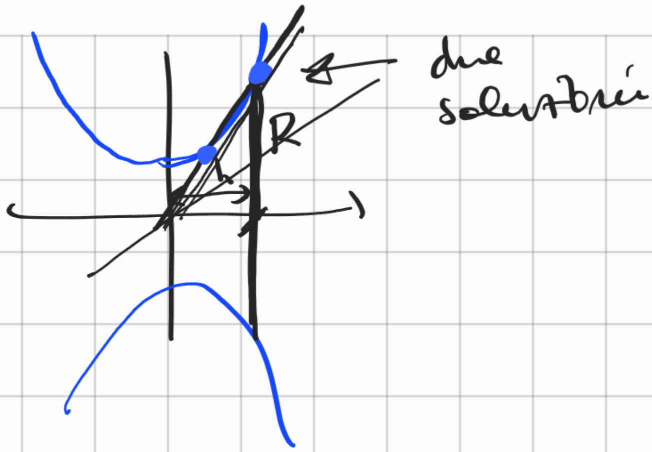
R
due soluzioni