ANALISI MATEMATICA B
LEZIONE 52 — 12.2.2024

la funzione Γ di Eulero

$$\Gamma(x) = \int_0^{+\infty} e^{-t}\, t^{x-1}\, dt \quad , \quad \Gamma : (0,+\infty) \to \mathbb{R}$$

verifichiamo che è ben definita.

$+\infty$ è un "punto cattivo" $\quad e^{-t}\, t^{x-1} << \frac{1}{t^2}$ per $t \to +\infty$

l'integrale $\int_1^{+\infty}$ è convergente

0 è un "punto cattivo" se $x < 1$

ma
$$e^{-t}\, t^{x-1} \sim t^{x-1} \quad \text{per } t \to 0^+$$

$\int_0^1 t^{x-1}\, dt$ converge sse $1 - x < 1 \iff x > 0$.

cosa succede se integro per parti

$$\Gamma(x+1) = \int_0^{+\infty} \underset{\substack{\uparrow \\ \text{integro}}}{e^{-t}}\, \underset{\substack{\uparrow \\ \text{derivo}}}{t^{x}}\, dt = \left[-e^{-t}\, t^{x} \right]_0^{+\infty} - \int_0^{+\infty} \left(-e^{-t}\right) x\, t^{x-1}\, dt$$

$x > 0$

$$= 0 - 0 + x \cdot \Gamma(x)$$

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

analogo a
$$n! = n \cdot (n-1)!$$

$$\Gamma(n+1) \overset{?}{=} n!$$

$$\Gamma(1) = \int_0^{+\infty} e^{-t}\, t^{0}\, dt = \left[-e^{-t} \right]_0^{+\infty} = 0 - (-1) = 1.$$

$$\begin{cases} \Gamma(n+1) = n! & \text{per } n = 0 \\ \Gamma(n+1) = n \cdot \Gamma(n) & \text{per } n > 0 \end{cases} \Rightarrow \Gamma(n+1) = n!$$

$$\Gamma(x) = (x-1)! \quad \text{se } x \in \mathbb{N},\ x > 0$$

Mettere in relazione: $\Gamma\left(\frac{1}{2}\right) \longleftrightarrow \int_0^{+\infty} e^{-x^2}\,dx$

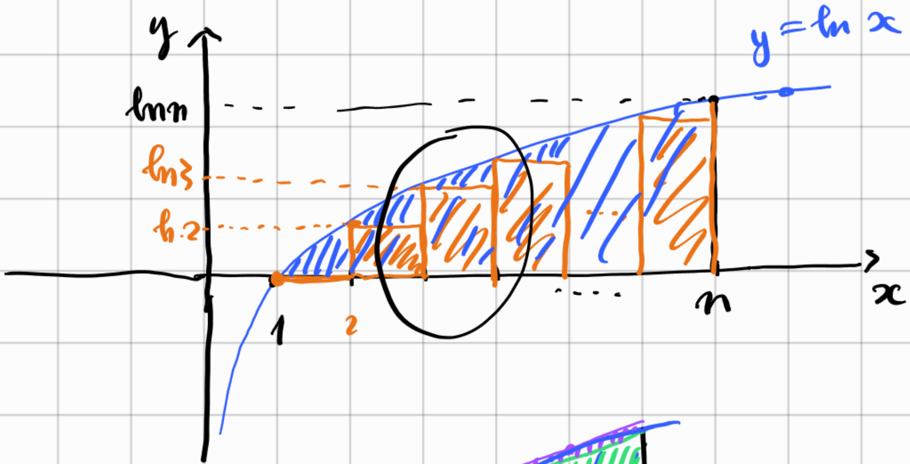
$$\Rightarrow \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right), \qquad \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right), \ldots$$

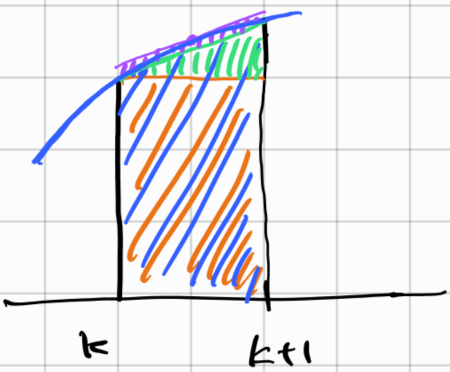
FORMULA di STIRLING

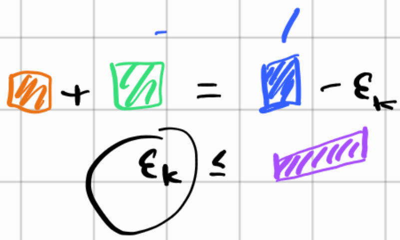
$$\boxed{n! \sim \sqrt{2\pi n}\cdot\frac{n^n}{e^n}} \qquad \text{per } n \to +\infty$$

$$\ln n! = \ln \prod_{k=1}^{n} k = \boxed{\sum_{k=1}^{n} \ln k}$$

assomiglia a $\int_1^{n+1} \ln x\,dx$







Vogliamo dimostrare che $\sum_{k=1}^{\infty} \varepsilon_k$ converge.

Passo 1 $\quad n! \sim C\cdot\sqrt{n}\,\frac{n^n}{e^n}$ se $\sum \varepsilon_k$ converge, c esiste finito.

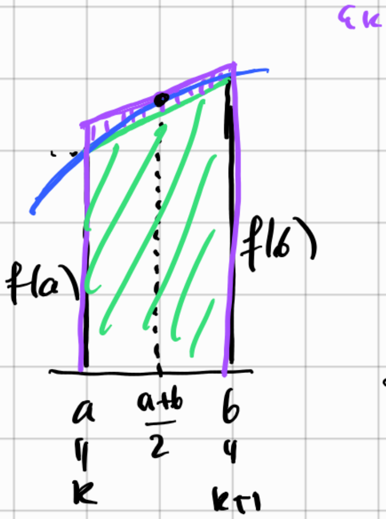
$$C = \lim_{n\to+\infty} \frac{n!}{\sqrt{n}\,\frac{n^n}{e^n}}$$

$$\ln c = \lim_{n\to+\infty} \left[\sum_{k=1}^{n} \ln k - \frac{1}{2}\ln n - n\ln n + n\right]$$

ε_k

$$\boxed{} = \int_k^{k+1} \ln x \, dx = \Big[x\ln x - x\Big]_k^{k+1} = (k+1)\ln(k+1) - (\cancel{k}+1) - k\ln k + \cancel{k}$$

$$= (k+1)\ln(k+1) - k\ln k - 1$$



In generale sia $f:[a,b] \to \mathbb{R}$ f concava. (il $\ln$ è concavo $\ln'' x = -\frac{1}{x^2} < 0$

$$\frac{f(a)+f(b)}{2}\cdot(b-a) \le \int_a^b f(x)\,dx \le f\left(\frac{a+b}{2}\right)\cdot(b-a)$$

$a=k$, $b=k+1$, $f=\ln$:

$$\frac{\ln(k)+\ln(k+1)}{2} \overset{(*)}{\le} \int_k^{k+1} \ln x \, dx \overset{(**)}{\le} \ln\left(k+\frac{1}{2}\right)$$

$$\overset{(*)}{0 \le} \varepsilon_k = \int_k^{k+1} \ln x\,dx - \frac{\ln k + \ln(k+1)}{2} \overset{(**)}{\le} \ln\left(k+\frac{1}{2}\right) - \frac{\ln k + \ln(k+1)}{2}$$

Voglio dimostrare che $\sum \varepsilon_k$ converge, infatti

$$\sum_{k=1}^{n-1} \varepsilon_k = \sum_{k=1}^{n-1} \int_k^{k+1} \ln x\,dx - \frac{1}{2}\sum_{k=1}^{n-1} \ln k - \frac{1}{2}\sum_{k=1}^{n-1} \ln(k+1)$$

$$= \int_1^n \ln x\,dx - \sum_{k=2}^{n-1} \ln k - \frac{1}{2}\ln n$$

$$= \Big[x\ln x - x\Big]_1^n - \sum_{k=1}^{n} \ln k + \frac{1}{2}\ln n$$

$$= n\ln n - n + 1 - \underbrace{\ln n!} + \frac{1}{2}\ln n$$

$$\ln n! = n\ln n - n + 1 + \frac{1}{2}\ln n - \sum_{k=1}^{n-1} \varepsilon_k$$

$$n! = \frac{n^n}{e^n}\cdot e \cdot \sqrt{n} \cdot e^{-\sum_{k=1}^{n-1} \varepsilon_k}$$

Per completare il passo 1 basta mostrare che $\sum \varepsilon_k$ converge. Abbiamo visto prima che:

$$0 \le \varepsilon_k \le \ln\left(k+\frac{1}{2}\right) - \frac{\ln k + \ln(k+1)}{2} = b_k$$

$$b_k = \ln\left(k+\frac{1}{2}\right) - \frac{1}{2}\ln k(k+1)$$

$$= \frac{1}{2}\left[\ln\frac{\left(k+\frac{1}{2}\right)^2}{k(k+1)}\right] = \frac{1}{2}\ln\frac{k^2+k+\frac{1}{4}}{k^2+k}$$

$$= \frac{1}{2}\ln\frac{1+\frac{1}{k}+\frac{1}{4k^2}}{1+\frac{1}{k}} = \frac{1}{2}\ln\left(1+\frac{\frac{1}{4k^2}}{1+\frac{1}{k}}\right)$$

$\ln(1+x) \sim x$ per $x \to 0$

$$\sim \frac{1}{2}\frac{\frac{1}{4k^2}}{1+\frac{1}{k}} \sim \frac{1}{8k^2}$$

$\sum \frac{1}{8k^2} = \frac{1}{8}\sum\frac{1}{k^2}$ converge.

$\square$ Passo 1.

Passo 2 trovare la costante. Ci serve la formula di Wallis.

Prodotto di Wallis:

$$\frac{\pi}{2} = \prod_{k=1}^{\infty}\frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdots$$

[SI INTENDE

$$\lim_{n\to+\infty}\prod_{k=1}^{n}\frac{(2k)^2}{(2k-1)(2k+1)} = \lim_{n\to\infty} e^{\sum_{k=1}^{n}\ln\frac{(2k)^2}{(2k-1)(2k+1)}} = e^{\sum_{k=1}^{\infty}\ln\frac{(2k)^2}{(2k-1)(2k+1)}}$$

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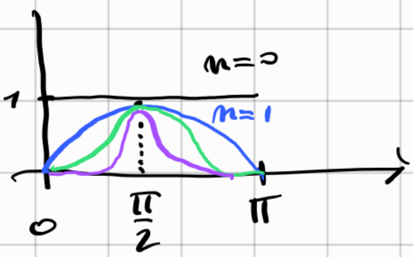
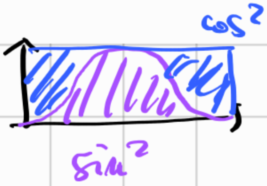
dim

$$I_n = \int_0^{\pi}\sin^n(x)\,dx$$



[En passant: $\int_0^{\pi} \sin^2 x \, dx = \frac{1}{2}$, 

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$n > 1$

$$I_n = \int_0^{\pi} \sin^n(x)\,dx = \int_0^{\pi} \underbrace{\sin^{n-1}(x)}_{\text{derivo}} \cdot \underbrace{\sin(x)}_{\text{integro}} \, dx \overset{\text{per parti}}{=}$$

$$= \left[\cancel{\sin^{n-1}(x)\cdot(-\cos x)}\right]_0^{\pi} - \int_0^{\pi} (n-1)\sin^{n-2}(x)\cos x \cdot (-\cos x)\,dx$$

$$= (n-1)\int_0^{\pi} \sin^{n-2}(x)\cdot\left(1-\sin^2(x)\right)dx$$

$$= (n-1)\, I_{n-2} - (n-1)\, I_n$$

$$n\, I_n = (n-1)\, I_{n-2}$$

$$\boxed{I_n = \frac{n-1}{n} I_{n-2}} \quad (*)$$

$$I_0 = \int_0^{\pi} 1 = \pi$$

$$I_1 = \int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = 2$$

$$I_{2n} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\overbrace{\frac{3}{4}\cdot\overbrace{\frac{1}{2}\cdot I_0}^{I_2}}^{I_4} \qquad (*)$$

$$I_{2n+1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\underbrace{\frac{4}{5}\cdot\underbrace{\frac{2}{3}\cdot I_1}_{I_3}}_{I_5} \qquad (*)$$

1 per $n \to \infty$

$$\frac{2n+1}{2n+2} \overset{(*)}{=} \frac{I_{2n+2}}{I_{2n}} \le \frac{I_{2n+1}}{I_{2n}} \le \frac{I_{2n}}{I_{2n}} = 1$$

$$1 \leftarrow \frac{I_{2n+1}}{I_{2n}} = \frac{2n(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 5\cdot 3}\,\underbrace{\frac{I_1}{I_0}}_{\frac{2}{\pi}}\,\frac{2n(2n-2)\cdots 4\cdot 2}{(2n-1)\cdots 3\cdot 1}$$

$$\frac{\pi}{2} = \frac{(2n)^2(2n-2)^2\cdots 4^2\cdot 2^2}{(2n+1)(2n-1)^2\cdots 5^2\cdot 3^2\cdot 1}$$

D

scrivere il prodotto di Wallis usando i semifattoriali $n!!$

$$\frac{\pi}{2} \leftarrow \frac{((2n)!!)^2}{(2n+1)!!\,(2n-1)!!} \quad \text{per } n\to\infty$$

$$(2n)!! = 2^n\cdot n!$$

$$(2n+1)!! = \frac{(2n+1)!}{(2n)!!} = \frac{(2n+1)!}{2^n\, n!}$$

scrivere $\frac{\pi}{2}$ come limite dei fattoriali e rimpiazzare i fattoriali con la formula di Stirling: $n! \sim c\sqrt{n}\,\frac{n^n}{e^n}$ per trovare $c=\sqrt{2\pi}$.