ANALISI MATEMATICA B
LEZIONE 30 — 29.11.2023

Derivate

$$\left(g(f(x))\right)' = g'(f(x))\, f'(x)$$

$$f^{-1}(f(x)) = x$$

Se f e f^{-1} sono derivabili

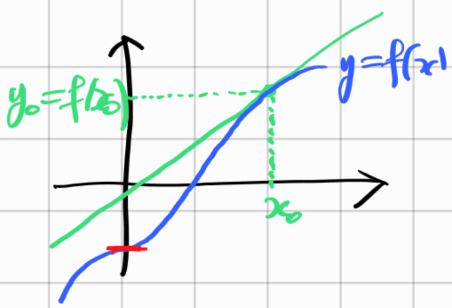
$$\left(f^{-1}\right)'(f(x)) \cdot f'(x) = 1$$

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)}$$

Teorema Se f è invertibile, f derivabile in x_0, $f'(x_0) \neq 0$ e se f^{-1} è continua in $f(x_0)$

Allora f^{-1} è derivabile in $f(x_0)$ e vale

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}$$



[Già visto:
$f: I \to \mathbb{R}$ monotona
$f(I)$ è un intervallo f è continua.]

$y = mx+q$
$x = \dfrac{y-q}{m}$

Se invece $f'(x_0) = 0$
f^{-1} non è derivabile in $y_0 = f(x_0)$.

dim

$$\lim_{y\to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x\to x_0} \frac{x - x_0}{f(x) - f(x_0)}$$

$y = f(x)$ o $x = f^{-1}(y)$

f^{-1} continua in y_0

$$= \lim_{x\to x_0} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)} \quad \square$$

$f'(x_0) \neq 0$

Notazioni

$$\left(f^{-1}(f(x_0))\right)' = 0$$

$$f^{-1}(f(\pi)) = 0$$

$$\left(f^{-1}(f(x))\right)' = 1$$

$$(x)' = 1$$

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

Operazioni con le derivate $\quad f, g$ derivabili

$$(f+g)' = f' + g'$$

[significa $(f+g)'(x) = f'(x) + g'(x)$]

D è lineare $\quad (c \cdot f)' = c \cdot f'$

costante

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$\left(\frac{f}{g}\right)' = \frac{f' g - f g'}{g^2}$$

$$(f-g)' = f' - g'$$

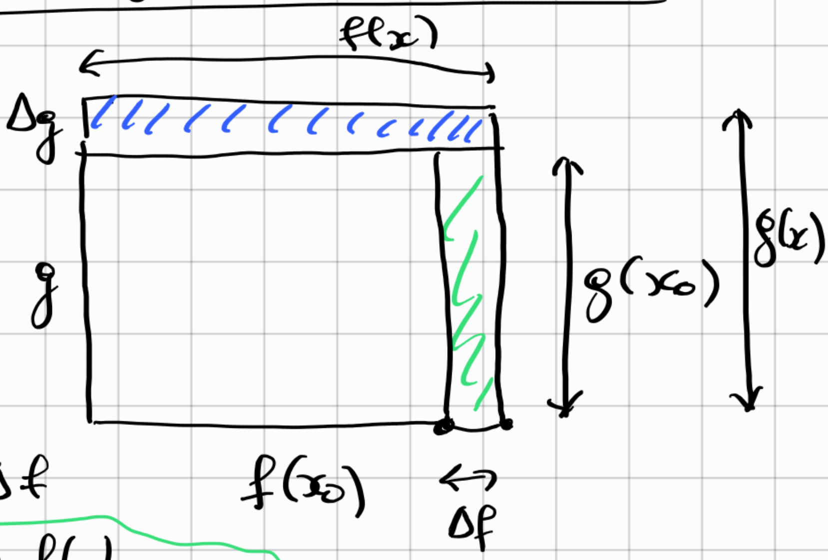
dim

$$\frac{(f \overset{(-)}{+} g)(x) - (f \overset{(-)}{+} g)(x_0)}{x - x_0} = \frac{f(x) \overset{(-)}{+} g(x) - (f(x_0) \overset{(-)}{+} g(x_0))}{x - x_0}$$

$$= \frac{f(x) - f(x_0)}{x - x_0} \overset{(-)}{+} \frac{g(x) - g(x_0)}{x - x_0} \longrightarrow f'(x_0) \overset{(-)}{+} g'(x_0)$$

$$\frac{(f \cdot g)(x) - (f \cdot g)(x_0)}{x - x_0} = \frac{f(x) \cdot g(x) - f(x_0) \cdot g(x_0)}{x - x_0} =$$

$$= \frac{f(x) g(x) - f(x) g(x_0) + f(x) g(x_0) - f(x_0) g(x_0)}{x - x_0}$$



$$= f(x) \cdot \underbrace{\frac{g(x) - g(x_0)}{x - x_0}}_{\Delta g} + \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\Delta f} \cdot g(x_0)$$

f è continua perché è derivabile.

$$\longrightarrow f(x_0) \cdot g'(x_0) + f'(x_0) \cdot g(x_0)$$

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)' = *$$

$h(y) = \frac{1}{y}$ $\qquad \frac{1}{g(x)} = h(g(x))$ $\qquad h'(y) = -\frac{1}{y^2}$

↑ già visto

$* = \frac{f'}{g} - f \cdot \frac{1}{g^2} \cdot g' = \frac{f'g - f \cdot g'}{g^2}$

↖ derivata della funzione composta.

$$\left[(h(g(x)))' = h'(g(x)) \cdot g'(x) \right]$$

Derivate delle funzioni elementari

$mx + q \xrightarrow{D} m$ $\qquad$ (banale)

$x^n \xrightarrow{D} n \cdot x^{n-1}$ $\qquad n \in \mathbb{Z} \setminus \{0\}$

$n \in \mathbb{N} \setminus \{0\}$ si dimostra per induzione:

$n = 1 \qquad x^1 = x \xrightarrow[D]{} 1 = 1 \cdot x^0$

$$(x^n)' = n x^{n-1} \overset{?}{\Rightarrow} (x^{n+1})' \overset{?}{=} (n+1) x^n$$

$$(x^{n+1})' = (x^n \cdot x)' = n x^{n-1} \cdot x + x^n \cdot 1 = (n+1) x^n \quad \underline{\text{ok}}$$

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = \frac{0 \cdot x^n - 1 \cdot n x^{n-1}}{(x^n)^2} = \frac{n x^{n-1}}{x^{2n}} =$$

$$= -\frac{n}{x^{n+1}} = -n \cdot x^{-(n+1)}$$

$m = -n \qquad (x^m)' = m\, x^{m-1}.$

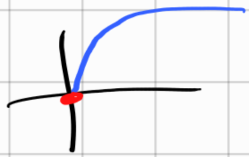
$$\sqrt[n]{x} \xrightarrow{D} \frac{1}{n\left(\sqrt[n]{x}\right)^{n-1}} \qquad (x \neq 0)$$

$$y = \sqrt[n]{x} \qquad y^n = x \qquad (y^n)' = n y^{n-1} \qquad \boxed{(f^{-1})'(x) = \frac{1}{f'(f(x))}}$$

$$\left(\sqrt[n]{x}\right)' = \frac{1}{n\left(\sqrt[n]{x}\right)^{n-1}}$$

$$\left(x^{\frac{1}{n}}\right)' = \frac{1}{n} \cdot x^{-\frac{(n-1)}{n}} = \frac{1}{n}\, x^{\frac{1-n}{n}} = \frac{1}{n} \cdot x^{\frac{1}{n}-1}$$

$$\sqrt{x} \xrightarrow{D} \frac{1}{2 \cdot \sqrt{x}} \qquad \text{se } x > 0$$



$$e^x \xrightarrow{D} e^x$$

$$\frac{e^{x+h} - e^x}{h} = e^x \cdot \frac{e^h - 1}{h} \xrightarrow{h \to 0} e^x \cdot 1 = e^x.$$

$$\sin x \xrightarrow{D} \cos x$$

$$\cos x \xrightarrow{D} -\sin x$$

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \sin x \cdot \underbrace{\frac{\cos h - 1}{h}}_{\downarrow \atop 0} + \cos x \cdot \underbrace{\frac{\sin h}{h}}_{\downarrow \atop 1} \xrightarrow[h \to 0]{} \cos x$$

$$\frac{\cos(x+h) - \cos x}{h} = \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \cos x \underbrace{\frac{\cos h - 1}{h}}_{\substack{\downarrow \\ 0}} - \sin x \underbrace{\frac{\sin h}{h}}_{\substack{\downarrow \\ 1}} \xrightarrow{h \to 0} -\sin x .$$

In campo complesso?

$$\frac{e^{z+h} - e^{z}}{h} = e^{z} \cdot \underbrace{\frac{e^{h} - 1}{h}}_{\substack{\downarrow \text{ per } h \to 0 \text{ in } \mathbb{C} \\ 1}} \longrightarrow e^{z}$$

[Tutte le regole di derivazione rimangono valide egualmente nel campo complesso]

$$(e^{ix})' = e^{ix} \cdot i \qquad \left[(e^{2x}) = e^{2x} \cdot 2 \right]$$

$$\parallel$$

$$(\cos x + i \operatorname{sen} x)' = (\cos x + i \operatorname{sen} x) \cdot i$$

$$\parallel$$

$$(\cos x)' + i (\operatorname{sen} x)' = -\operatorname{sen} x + i \cos x$$

$$\ln x \xrightarrow{D} \frac{1}{x} \qquad (x > 0)$$

$$\ln |x| \xrightarrow{D} \frac{1}{x} \qquad (x \neq 0)$$

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

$$\ln|x| = \begin{cases} \ln x & \text{se } x>0 \\ \ln(-x) & \text{se } x<0 \end{cases}$$

$$\left(\ln(-x)\right)' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$$

$$x^{\alpha} \xrightarrow{D} \alpha\, x^{\alpha-1}$$

$\alpha \in \mathbb{R}$, $x>0$

$$\left(x^{\alpha}\right)' = \left(e^{\alpha \ln x}\right)' = e^{\alpha \ln x}\cdot\alpha\cdot\frac{1}{x} = x^{\alpha}\cdot\alpha\cdot\frac{1}{x}$$

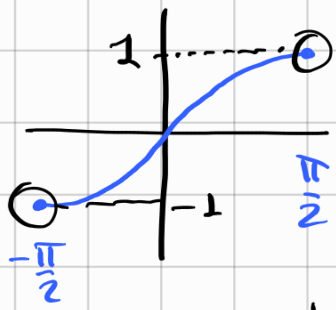
$$= \alpha x^{\alpha-1}$$

$$\operatorname{tg} x \xrightarrow{D} 1+\operatorname{tg}^2 x = \frac{1}{\cos^2 x}$$

$$\left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x}$$

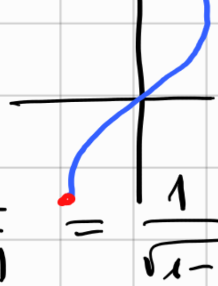
$$= \frac{\cos^2 x + \operatorname{sen}^2 x}{\cos^2 x} = \begin{cases} \dfrac{1}{\cos^2 x} \\ 1+\operatorname{tg}^2 x \end{cases}$$

$$\arcsin x \xrightarrow{D} \frac{1}{\sqrt{1-x^2}}$$



$D \sin x = \cos x = 0$ se $x = \pm\frac{\pi}{2}$

$\arcsin x$ è definita su $[-1, 1]$
ma è derivabile solo su $(-1, 1)$

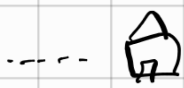


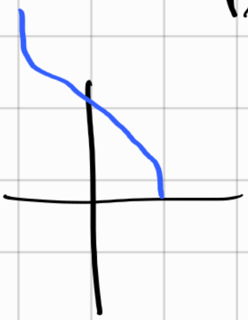
$$\left(\arcsin x\right)' = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}}$$

$$\cos y = \underset{?}{\pm}\sqrt{1-\sin^2 y}$$

$+$ se siamo in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\arccos x \xrightarrow{D} \frac{1}{-\sqrt{1-x^2}}$$





$$\text{arctg}\, x \xrightarrow{D} \frac{1}{1+x^2}$$

$$(\text{arctg}\, x)' = \frac{1}{1+\text{tg}^2(\text{arctg}\, x)} = \frac{1}{1+x^2}$$

$$\log_a x = \frac{\log_c x}{\log_c a} = \frac{\ln x}{\ln a}$$

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{x} \cdot \frac{1}{\ln a}$$

$$a^x = e^{x \ln a}$$

$$(a^x)' = e^{x \ln a} \cdot \ln a = \ln a \cdot a^x$$

Esercizio Calcolare la derivata di

$$f(x) = \sqrt{2 - \frac{1+\sin x}{\ln x}}$$

$$f'(x) = \frac{-\dfrac{\cos x \cdot \ln x - (1+\sin x)\frac{1}{x}}{\ln^2 x}}{2\sqrt{2 - \dfrac{1+\sin x}{\ln x}}}$$

se $2 - \dfrac{1+\sin x}{\ln x} > 0$

cosa succede se $\ldots = 0$? per ora non lo sappiamo

[esempio: $\sqrt[3]{x^3} = x$ è derivabile anche per $x=0$]

$$|x| \xrightarrow{D} \frac{x}{|x|} \qquad x \neq 0$$