ANALISI MATEMATICA B
LEZIONE 53 9.2.2022

Polinomi di Taylor funzioni elementari.

$$e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + o(x^n) \quad \text{per } x \to 0$$

$$\sin x = x - \frac{x^3}{3!} + \dots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + o(x^{2n+1})$$

$$\cos x = 1 - \frac{x^2}{2} + \dots + \frac{(-1)^n}{(2n)!} x^{2n} + o(x^{2n})$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{(-1)^{n+1}}{n} x^n + o(x^n)$$

$$\operatorname{arctg}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{(-1)^n}{2n+1} x^{2n+1} + o(x^{2n+1})$$

$$\operatorname{tg} x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + o(x^5)$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$f(x) = \boxed{(1+x)^\alpha}$ $\qquad \alpha \in \mathbb{R}$

x^α

x^{α} $\qquad x \to 7 \qquad t = x - 7$

$$x^{\alpha} = (7+t)^{\alpha} = 7^{\alpha}\left(1+\frac{t}{7}\right)^{\alpha}$$

$\ln(x) \qquad x \to 7 \qquad t = x-7$

$$\ln(x) = \ln(7+t) = \ln\left(7\cdot\left(1+\frac{t}{7}\right)\right) = \ln 7 + \ln\left(1+\frac{t}{7}\right)$$

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per $x \to 0$

$$f(x) = (1+x)^{\alpha} \qquad f(0) = 1$$

$$f'(x) = \alpha(1+x)^{\alpha-1} \qquad f'(0) = \alpha$$

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \qquad f''(0) = \alpha(\alpha-1)$$

$\vdots$

$$f^{(n)}(x) = \underbrace{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}_{n}(1+x)^{\alpha-n}$$

$$f^{(n)}(0) = \alpha(\alpha-1)\dots(\alpha-n+1)$$

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

$$\dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + o(x^n)$$

<u>Def</u>

$$\binom{\alpha}{n} \overset{\text{def}}{=} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$$

se $\alpha = k \in \mathbb{N}$ $\qquad \binom{k}{n} = \dfrac{k!}{n!\,(n-k)!}$

<u>Es</u>

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2 + o(x^2)$$

$$\left[\quad \begin{aligned} (1+x)^3 &= 1 + 3x + 3x^2 + o(x^2) \\ (1+x)^n &= 1 + nx + \frac{n(n-1)}{2}x^2 + o(x^2) \end{aligned} \quad\right]$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)$$

<u>Es</u> $\qquad \displaystyle\lim_{x\to 0} \frac{\sqrt[3]{1+x} + \sqrt[3]{1-x} - 2}{x^2}$

$$\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}} = 1 + \frac{x}{3} + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)}{2}x^2 + o(x^2)$$

$$= 1 + \frac{x}{3} - \frac{x^2}{9} + o(x^2)$$

$$\sqrt[3]{1-x} = 1 - \frac{x}{3} - \frac{x^2}{9} + o(x^2)$$

$$\frac{\sqrt[3]{1+x} + \sqrt[3]{1-x} - 2}{x^2} = \frac{\cancel{1} + \cancel{\frac{x}{3}} - \frac{x^2}{9} + \cancel{1} - \cancel{\frac{x}{3}} - \frac{x^2}{9} - \cancel{2} + o(x^2)}{x^2}$$

$$= \frac{-\frac{2}{9}x^2 + o(x^2)}{x^2} = -\frac{2}{9} + \frac{o(x^2)}{x^2}$$

$$\frac{3x^2+o(x^2)}{2x^2+o(x^2)} = \frac{\cancel{x^2}\left(3+\frac{o(x^2)}{x^2}\right)}{\cancel{x^2}\left(2+\frac{o(x^2)}{x^2}\right)} \to \frac{3}{2}$$

Es $$\frac{x^2}{2x^2+o(x^2)} = x^2\cdot\left(2x^2+o(x^2)\right)^{-1}$$

$$= x^2\cdot(2x^2)^{-1}\left(1+\frac{o(x^2)}{2x^2}\right)^{-1}$$

$$= \frac{\cancel{x^2}}{2\cancel{x^2}}\left(1+o(1)\right)^{-1} = \frac{1}{2}\left(1+o(1)+o(\cancel{o}(1))\right)$$

$$= \frac{1}{2}+o(1).$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1+(-1)x+o(x) = 1-x+o(x)$$

$$\left[\begin{array}{l} o(o(1)) \overset{?}{=} o(1) \\ \frac{o(o(1))}{1} = \frac{o(o(1))}{\cancel{o(1)}}\cdot \cancel{o(1)} \to 0 \end{array}\right]$$

Nota $$\frac{1}{1+x} = (1+x)^{-1} = 1-x+x^2-x^3 \dots$$

$$\frac{1}{1-x} = 1+x+x^2+x^3+\dots$$

Es $\dfrac{1}{\sqrt{\cos x}}$ pol. di Taylor <u>secondo ordine</u> $x \to 0$

$$\frac{1}{\sqrt{\cos x}} = (\cos x)^{-\frac{1}{2}} = \left(1 - \frac{x^2}{2} + o(x^2)\right)^{-\frac{1}{2}}$$

$$= (1+y)^{-\frac{1}{2}}$$

$y = -\frac{x^2}{2} + o(x^2)$

$y \to 0$ se $x \to 0$

$$= 1 - \frac{1}{2}y + \frac{3}{8}y^2 + o(y^2)$$

$$= 1 - \frac{1}{2}\left(-\frac{x^2}{2}\right) + \cancel{\frac{3}{8}\left(-\frac{x^2}{2}\right)^2} + o(x^2)$$

$$= 1 + \frac{x^2}{4} + o(x^2)$$

Es Pol. di Taylor di $f(x) = \sqrt[5]{e^{\sin x}}$ per $x \to 0$ di ordine 2.

$$\sqrt[5]{1+x} = 1 + \frac{1}{5}x - \frac{2}{25}x^2 + o(x^2)$$

$$\sqrt[5]{e^{\sin x}} = 1 + \frac{1}{5}\left(e^{\sin x} - 1\right) - \frac{2}{25}\left(e^{\sin x} - 1\right)^2 + o\left(\left(e^{\sin x} - 1\right)^2\right)$$

$$e^{x} = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2} + o(\sin^2 x)$$

$$= 1 + \left(x + o(x^2)\right) + \frac{1}{2}\left(x + o(x^2)\right)^2 + o(x^2)$$

$$= \boxed{1 + x + \frac{1}{2}x^2 + o(x^2)}$$

$$e^{\sin x} - 1 = x + \frac{1}{2}x^2 + o(x^2) = x + o(x) \sim x$$

$$\sqrt[5]{e^{\sin x}} = 1 + \frac{1}{5}\left(x + \frac{1}{2}x^2 + o(x^2)\right) - \frac{2}{25}\left(x + o(x)\right)^2 + o(x^2)$$

$$= 1 + \frac{x}{5} + \frac{x^2}{10} - \frac{2}{25}x^2 + o(x^2)$$

$$= 1 + \frac{x}{5} + \frac{x^2}{50} + o(x^2)$$

Es $\sqrt[5]{e^{\sin x} + 1} = \sqrt[5]{2 + x + \frac{x^2}{2} + o(x^2)}$

$$= \sqrt[5]{2\left(1+\frac{x}{2}+\frac{x^2}{4}+o(x^2)\right)}$$

$$= \sqrt[5]{2}\cdot\left(1+\boxed{\frac{x}{2}+\frac{x^2}{4}+o(x^2)}\right)^{\frac{1}{5}}$$

$$= \sqrt[5]{2}\cdot\left(1+\frac{1}{5}\left(\frac{x}{2}+\frac{x^2}{4}\right)-\frac{4}{50}\left(\frac{x}{2}\right)^2+o(x^2)\right)$$

$$= \sqrt[5]{2}\left(1+\frac{x}{10}+\left(\frac{1}{20}-\frac{1}{50}\right)x^2+o(x^2)\right)$$

$$= \sqrt[5]{2}\left(1+\frac{x}{10}+\frac{3}{100}x^2+o(x^2)\right)$$

Come si formalizza la notazione o-piccolo ?
O-grande ?

$$o(x)-o(x)=o(x)$$

$$x^2 = o(x) = o(1) \qquad \text{per } x\to 0$$

$$o(1) \overset{\text{NO!}}{=} o(x) \overset{\text{NO!}}{=} x^2$$

$$o(x) = \left\{ f : \frac{f(x)}{x}\to 0 \text{ per } x\to 0 \right\}$$

$$o(g(x)) = \left\{ f(x) : \frac{f(x)}{g(x)}\to 0 \right\}$$

$$o(x) - o(x) = A - A = \{x - y : x \in A, y \in A\}$$

$$= o(x)$$

$$\Rightarrow x^2 = o(x) = o(1)$$

$$\Rightarrow x^2 \in o(x) \subseteq o(1)$$

$$o(1) \cancel{=} o(x) \cancel{=} x^2$$

Es

$$o(f) \cdot o(g) \subseteq o(f \cdot g)$$

Es

$$o(f) \cdot o(g) \supseteq o(f \cdot g) \quad ?$$