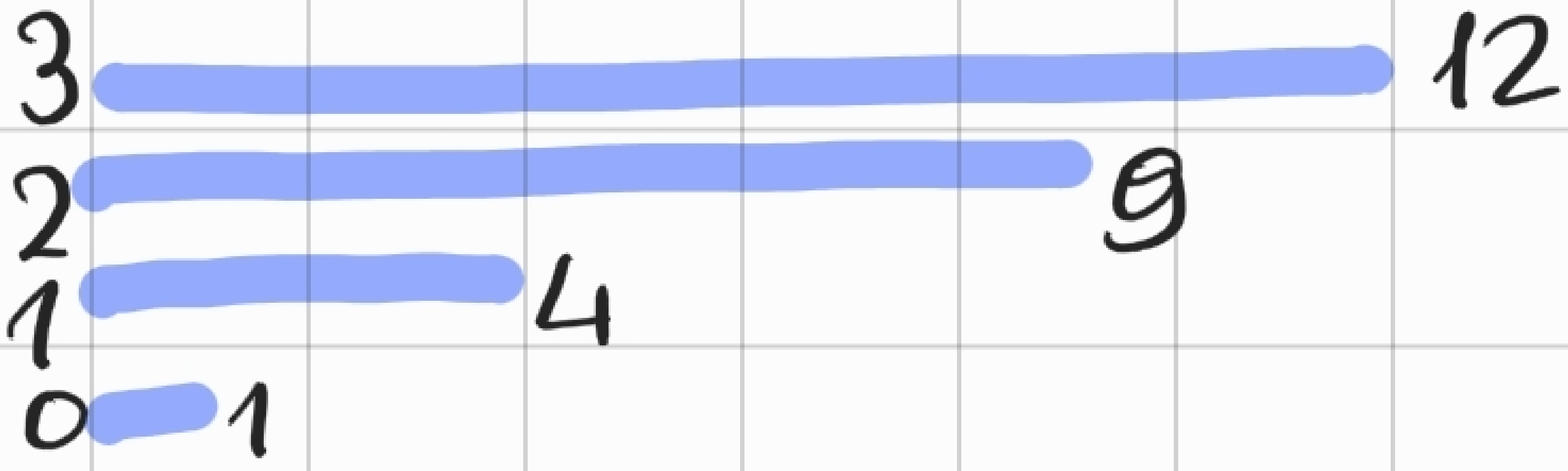
ANALISI MATEMATICA B
LEZIONE 70 29.3.2021

test settimanale



Es. 3 Posto $I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$

calcolare

$$\int_{-\infty}^{+\infty} x^2 e^{-x^2}\,dx .$$

$$\int x e^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}$$

$$\int x^3 e^{-x^2}\,dx = \int \underline{x^2}\ \underline{x e^{-x^2}}\,dx$$

$$\int x^n e^{-x^2} \rightsquigarrow \int x^{n-2} e^{-x^2} \int x e^{-x^2}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-x^2}\,dx = \int_{-\infty}^{+\infty} x \cdot x e^{-x^2}\,dx$$

$$= \cancel{\left[x\left(-\frac{1}{2}e^{-x^2}\right)\right]_{-\infty}^{+\infty}} - \int_{-\infty}^{+\infty}\left(-\frac{1}{2}e^{-x^2}\right)dx$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \frac{1}{2}\, I.$$

Esercizio (intuitivamente)

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

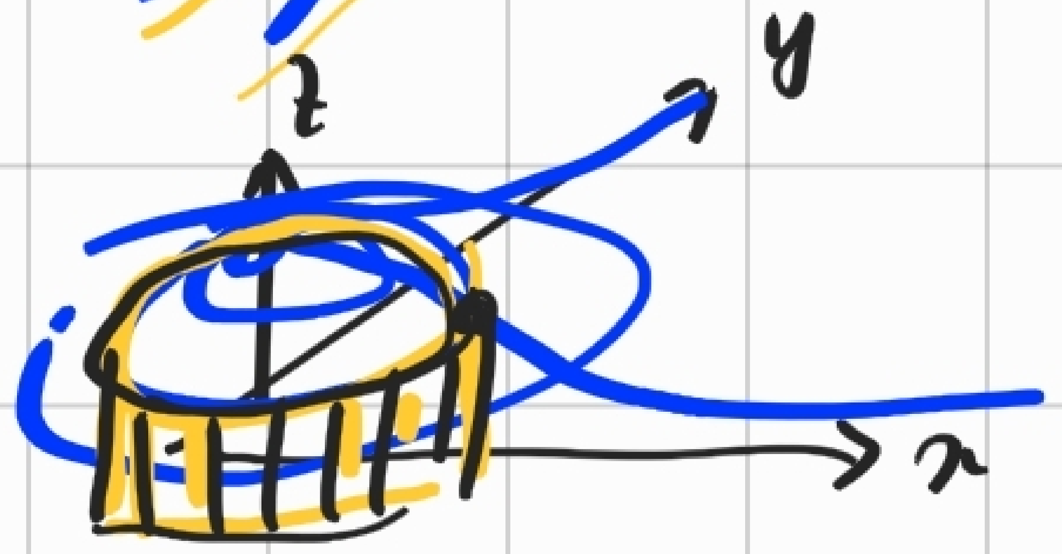
$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\cdot\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)\cdot e^{-x^2}\,dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-y^2} e^{-x^2}\,dy\,dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dy\,dx$$

$$z = e^{-(x^2+y^2)} = e^{-\rho^2}$$

$$\rho = \sqrt{x^2+y^2}$$



$$= \int_0^{+\infty} 2\pi\rho\, e^{-\rho^2}\,d\rho = \pi\left[-e^{-\rho^2}\right]_0^{+\infty} = \pi$$

Teorema Per $x \to x_0$ $a(x) \to a$, $b(x) \to a$

$f(x) \sim g(x)$ per $x \to a$, f, g continue.

Allora $$\int_{a(x)}^{b(x)} f(t)\,dt \sim \int_{a(x)}^{b(x)} g(t)\,dt \quad \text{per } x \to x_0$$

dim

$F' = f \quad G' = g$

$$\frac{\int_{a(x)}^{b(x)} f(t)\,dt}{\int_{a(x)}^{b(x)} g(t)\,dt} = \frac{\left[F(t)\right]_{a(x)}^{b(x)}}{\left[G(t)\right]_{a(x)}^{b(x)}} = \frac{F(b(x)) - F(a(x))}{G(b(x)) - G(a(x))}$$

$$= \frac{F'(c(x))}{G'(c(x))} = \frac{f(c(x))}{g(c(x))}$$

$$c(x) \in [a(x), b(x)] \qquad a(x) \to a, \ b(x) \to a$$

per $x \to x_0$ $y = c(x) \to a$

$$= \frac{f(y)}{g(y)} \to 1 \quad \text{per } y \to a \text{ ovvero}$$

per $x \to x_0$. $\square$

Esempio $$\lim_{x \to 0} \int_x^{2x} \frac{1}{t + \sin t}\,dt$$

$$= \lim_{x \to 0} \int_x^{2x} \frac{1}{2t}\,dt = \frac{1}{2}\left[\ln t\right]_x^{2x}$$

$$f(t) = \frac{1}{t+\sin t} = \frac{1}{t + t + o(t)} = \frac{1}{2t + o(t)} \quad \text{per } t \to 0$$

$$g(t) = \frac{1}{2t} \qquad \frac{f(t)}{g(t)} = \frac{\frac{1}{2t+o(t)}}{\frac{1}{2t}} = \frac{2t}{2t+o(t)}$$

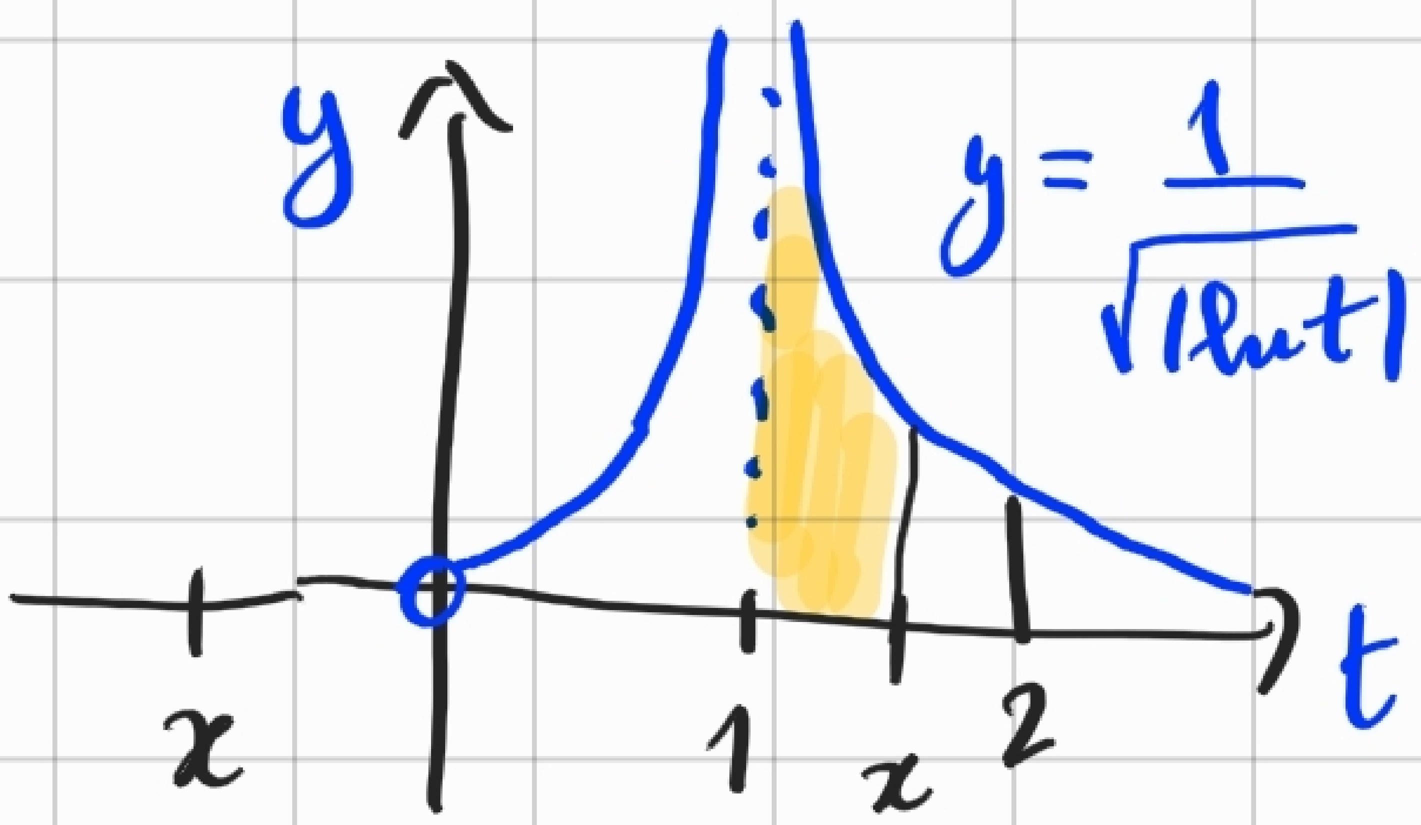
$$= \frac{1}{1 + \frac{1}{2}\underbrace{\cancel{\frac{o(t)}{t}}}_{0}} \longrightarrow 1 \quad \text{per } t \to 0.$$

$$\lim_{x\to 0} \int_x^{2x} \frac{1}{t+\sin t}\, dt$$

$$= \lim_{x\to 0} \int_x^{2x} \frac{1}{2t}\, dt = \frac{1}{2}\Big[\ln t\Big]_x^{2x} = \frac{1}{2}\left[\ln 2x - \ln x\right]$$

$$= \frac{1}{2} \ln 2 \qquad \square$$

Esercizio $\quad F(x) = \displaystyle\int_1^x \frac{1}{\sqrt{|\ln t|}}\, dt$



$$\ln t \sim t-1 \quad \text{per } t \to 1$$

$$\frac{1}{\sqrt{|\ln t|}} \sim \frac{1}{\sqrt{|t-1|}} \quad \text{per } t \to 1$$

$p = \frac{1}{2}$

Se $x \in [0, +\infty)$ $\quad \int_1^x \frac{1}{\sqrt{|\ln t|}}\,dt$

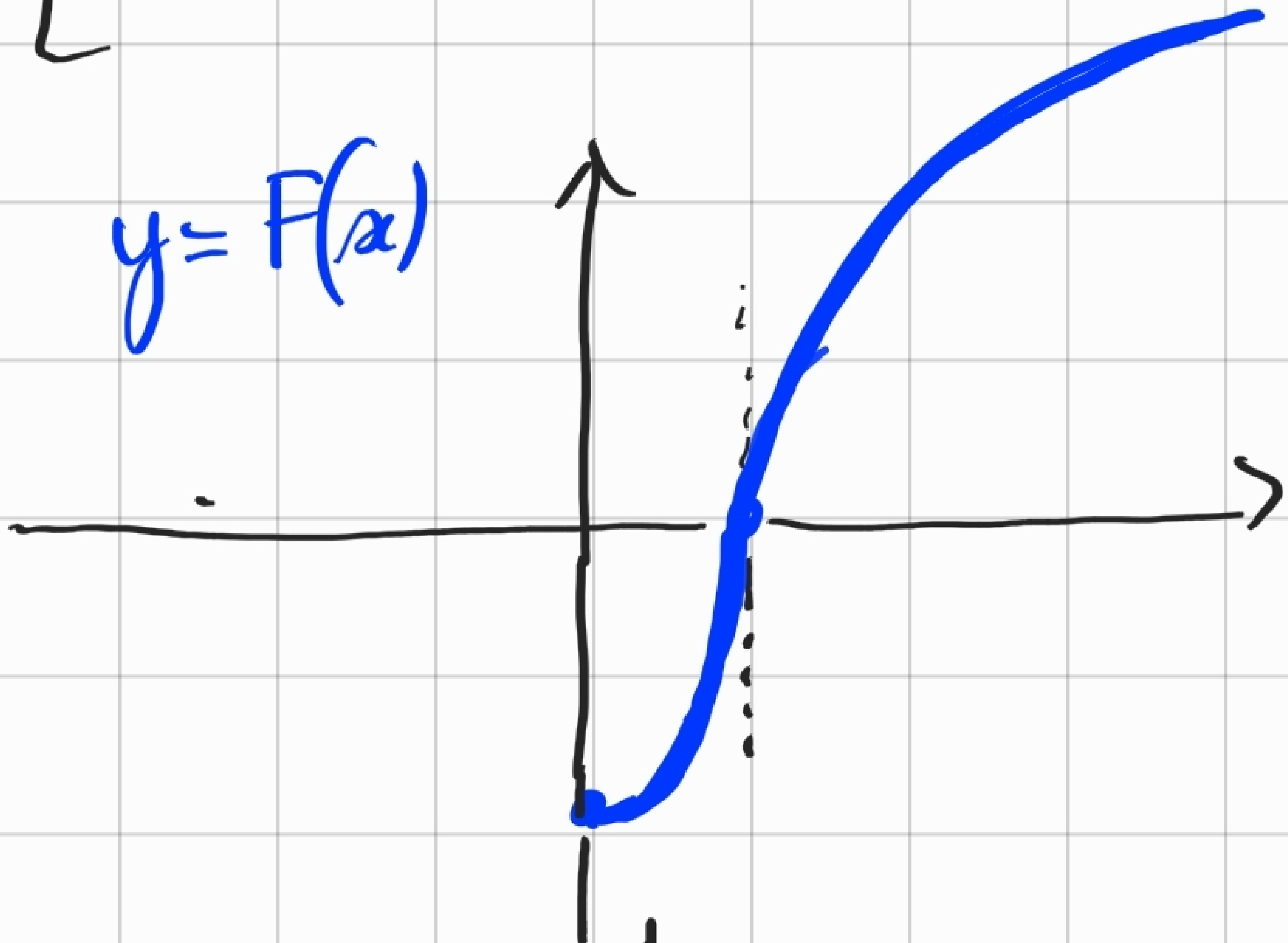
è un integrale improprio, convergente.

$F(1) = 0$ (per definizione)

$$\left[\; F(1) = \int_1^x f(t)\,dt = \int_1^2 f(t)\,dt - \int_x^2 f(t)\,dt \;\right]$$

per $x \to 1$

$$\int_1^2 f(t)\,dt - \int_x^2 f(t)\,dt \;\to\; \int_1^2 f(t)\,dt - \int_1^2 f(t)\,dt = 0$$



x:	0		1	
$F(x)$	$-$	$-$	0	$+$

$$F'(x) = \frac{1}{\sqrt{|\ln x|}}$$

per il teorema fondamentale del calcolo

$$\left(\; F(x) = \underbrace{\int_1^2 f(t)\,dt}_{\text{convergente}} + \underbrace{\int_2^x f(t)\,dt} \qquad (x>1) \;\right)$$

$$\left[\, F'(x) = f(x) \,\right]$$

$$F'(x) > 0 \qquad \forall x \neq 1.$$

$$F'(x) \longrightarrow +\infty \quad \text{per } x \to 1$$

$$\lim_{x \to 0} F'(x) = \lim_{x \to 0} \frac{1}{\sqrt{|\ln x|}} = 0$$

$$\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} \int_1^x \frac{1}{\sqrt{|\ln t|}}\, dt$$

$$= \int_1^{+\infty} \frac{1}{\sqrt{|\ln t|}}\, dt = +\infty$$

$$\ln t << t^p \qquad \forall p > 0$$

$$\sqrt{\ln t} << t^{p/2}$$

$$\frac{1}{\sqrt{\ln t}} >> \frac{1}{t^{p/2}} = \frac{1}{t} \quad \text{per } p = 2$$

$$\left[\, \int_1^{+\infty} \frac{1}{t}\, dt = +\infty \,\right]$$

$$F''(x) = \frac{d}{dx}\frac{1}{\sqrt{|\ln x|}} = \begin{cases} -\frac{1}{2}\,\frac{1}{x\,|\ln x|^{3/2}} & \text{se } x>1 \\[2ex] \frac{1}{2}\,\frac{1}{x\,|\ln x|^{3/2}} & \text{se } x<1 \end{cases}$$

x	0		1	
$F''(x)$		$+$	$\nexists$	$-$
F		$\cup$		$\cap$

Funzione Γ- di Eulero

$$\Gamma(x) = \int_0^{+\infty} e^{-t}\cdot t^{x-1}\cdot dt$$

Per quali x l'integrale converge?

per $t \to +\infty$ $\qquad e^{-t}\, t^{x-1} << \frac{1}{t^2}$

$$e^{-t} << t^{x+1}$$

l'integrale converge a $\to \infty$ $\quad \forall x \in \mathbb{R}$.

per $t \to 0^+$ $\quad e^{-t} \to 1 \quad e^{-t} t^{x-1} \sim t^{x-1}$

$$\int_0^1 t^{x-1}\, dt \quad \text{converge} \iff 1-x < 1$$

$$= \int_0^1 \frac{1}{t^{1-x}}\, dt \qquad \underline{\underline{x > 0}}$$

$$\Gamma : (0, +\infty) \to \mathbb{R}$$

$$\Gamma(1) = \int_0^{+\infty} e^{-t}\, dt = 1$$

$$\Gamma(x+1) = \int_0^{+\infty} e^{-t} t^{x}\, dt \qquad x > 0$$

$$= \cancel{\left[-e^{-t} \cdot t^{x}\right]_0^{+\infty}} - \int_0^{+\infty} (-e^{-t}) \cdot x\, t^{x-1}\, dt$$

$$= x \int_0^{+\infty} e^{-t} t^{x-1}\, dt = x\,\Gamma(x).$$

$$\Gamma(x+1) = x\,\Gamma(x)$$

$\Gamma(1) = 1$

$\Gamma(2) = 1$

$\Gamma(3) = 2 \cdot 1$

$\Gamma(4) = 3 \cdot 2 \cdot 1$

$\dots$

$$\Gamma(n) = (n-1)!$$

per induzione

$$\Gamma(n+1) = n!$$

$\Gamma(x+1)$ estende il fattoriale a tutti i numeri reali > -1

$$\boxed{\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$\Gamma\left(\frac{3}{2}\right) = ?$$

$$\Gamma\left(\frac{3}{2}\right) = \int_0^{+\infty} e^{-t}\, t^{\frac{3}{2}-1}\, dt$$

$$= \int_0^{+\infty} e^{-t} \sqrt{t}\, dt = \int_0^{+\infty} e^{-s^2} s\, 2s\, ds$$

$s = \sqrt{t}$

$t = s^2$

$dt = 2s\,ds$

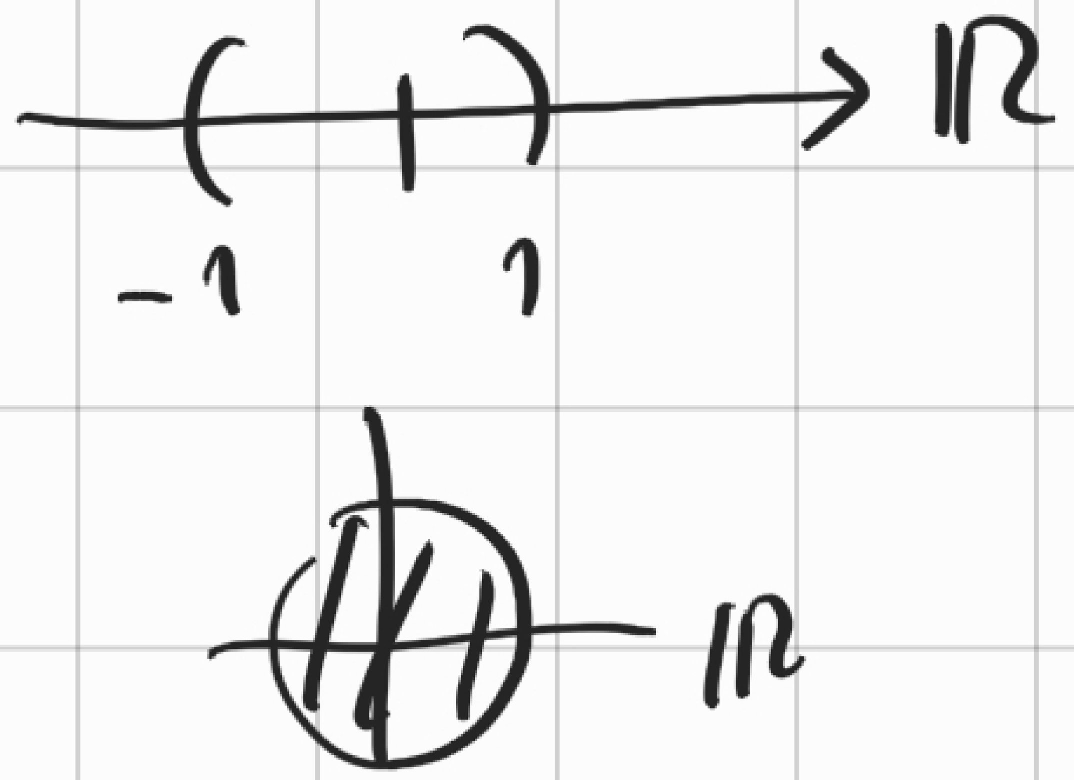
$$= 2\int_0^{+\infty} s^2 e^{-s^2}\, ds$$

$$= \cancel{2}\, \frac{\int_0^{+\infty} e^{-s^2}\, ds}{\cancel{2}}$$

$$= \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{1}{2}\right) = 2\Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}$$

Quanto misura la palla unitaria di $\mathbb{R}^n$? [ω_n]



n	ω_n
1	2 ✓
2	π ✓
3	$\frac{4}{3}\pi$ ✓
4	$\frac{\pi^2}{2}$

$$\boxed{\omega_n = \frac{\pi^{n/2}}{\left(\frac{n}{2}\right)!}}$$

$$\frac{\pi^{1/2}}{\left(\frac{1}{2}\right)!} = \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{2}\right)} = 2$$

$$\frac{\pi^{3/2}}{\left(\frac{3}{2}\right)!} = \frac{\pi\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}!}$$

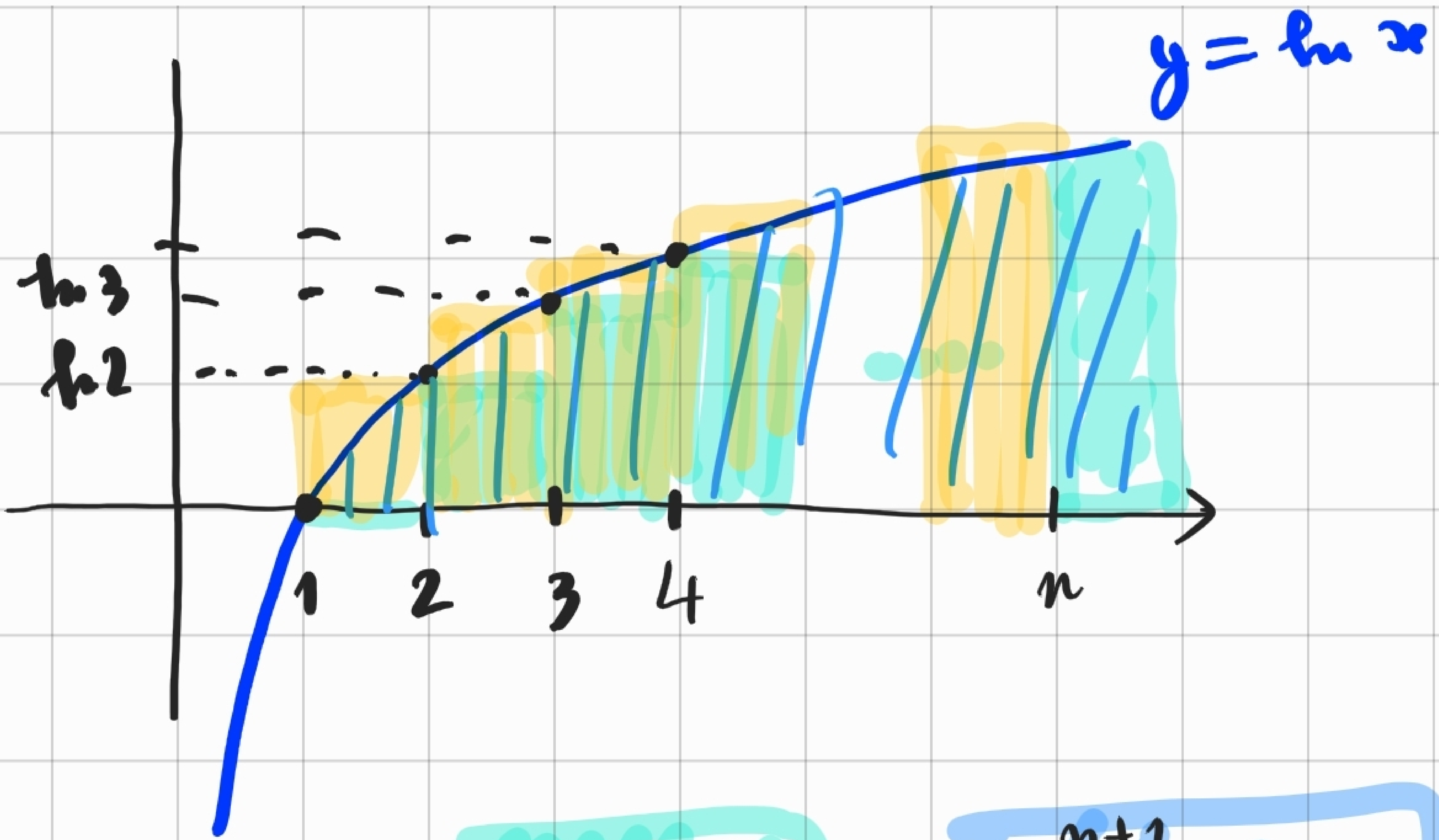
$$= \frac{\pi \cancel{\sqrt{\pi}}}{\frac{3}{2}\,\frac{\cancel{\sqrt{\pi}}}{2}} = \frac{4}{3}\pi$$

Esercizio: Mostriamo che $\ln(n!) \sim n \ln n$.

(Ma: $\underset{e^{\ln n!}}{\overset{\shortparallel}{n!}} << n^n = e^{n\ln n}$)

dim

$$\ln(n!) = \ln\left(n\cdot(n-1)\ldots 2\cdot 1\right) = \sum_{k=1}^{n} \ln k$$



$$\int_1^n \ln x \, dx \le \boxed{\sum_{k=1}^{n} \ln k} \le \boxed{\int_2^{n+1} \ln x \, dx}$$

$$\int_1^n \ln x \, dx = \left[x \ln x - x\right]_1^n = n \ln n - n + 1$$

$$\int_2^{n+1} \ln x \, dx = \left[x \ln x - x\right]_2^{n+1} = (n+1)\ln(n+1) - (n+1) - 2\ln 2 + 2$$

$$\frac{n \ln n - n + 1}{n \ln n} \le \frac{\displaystyle\sum_{k=1}^{n} \ln k}{n \ln n} \le \frac{(n+1)\ln(n+1) - (n+1) - 2\ln 2 + 2}{n \ln n}$$

$$\frac{n \ln n - n + 1}{n \ln n} \longrightarrow 1$$

$$\frac{(n+1)}{n} \cdot \frac{\ln(n+1)}{\ln n}, \qquad \frac{n+1}{n} \to 1, \qquad \frac{\ln(n+1)}{\ln n} \to 1$$

$$\frac{\ln(n+1)}{\ln n} = \frac{\ln\left(n\cdot\left(1+\frac{1}{n}\right)\right)}{\ln n} = \frac{\ln n + \ln\left(1+\frac{1}{n}\right)}{\ln n}$$

0