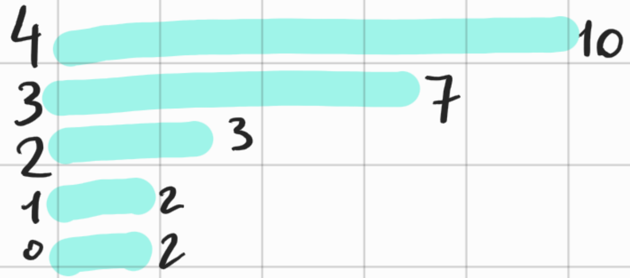
ANALISI MATEMATICA B
LEZIONE 59 - 1.3.2021

test settimanale

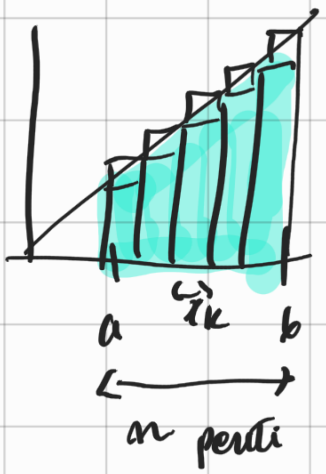


$$\underbrace{\int_a^b (mx+q)\,dx}_{} = m\underbrace{\int_a^b x\,dx}_{} + q\int_a^b 1\,dx$$

$$\int_a^b x\,dx$$

$$P_n = \{x_k : k = 0, \dots, n\}$$

$$x_k = a + \frac{b-a}{n}\cdot k$$



$$S^*_*(f,P) = \sum_{k=0}^{n-1} \frac{b-a}{n}\left(a + \frac{b-a}{n}(k+1)\right)$$

$$[x_k, x_{k+1})$$

$$= \frac{b-a}{n}\left[\sum_{k=0}^{n-1} a + \frac{b-a}{n}\sum_{k=0}^{n-1}(k+1)\right]$$

$$= \frac{b-a}{\cancel{n}}\left[\cancel{n}\cdot a + \frac{b-a}{n}\cancel{n}\cdot\frac{n\pm1}{2}\right]$$

$$= (b-a)\cdot a + (b-a)^2\,\frac{1 \pm \frac{1}{n}}{2} \xrightarrow[n\to+\infty]{} (b-a)\left(a+\frac{b-a}{2}\right)$$

$$\| $$

$$(b-a)\left(\frac{a+b}{2}\right)$$

$$\int_a^b mx+q = m\,(b-a)\left(\frac{a+b}{2}\right) + q\,(b-a)$$

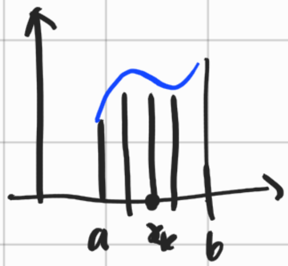
$$= (b-a)\left[\frac{m(a+b)}{2} + q\right]$$

$$= \frac{1}{2}(b-a)\left[ma + mb + 2q\right]$$

Integrabilità delle funzioni continue.

<u>Teorema</u> Se $f:[a,b]\to\mathbb{R}$ ~~uniformemente~~ continua allora f è limitata e Riemann-integrabile.

<u>dim</u>



[f è limitata per Weierstraß]

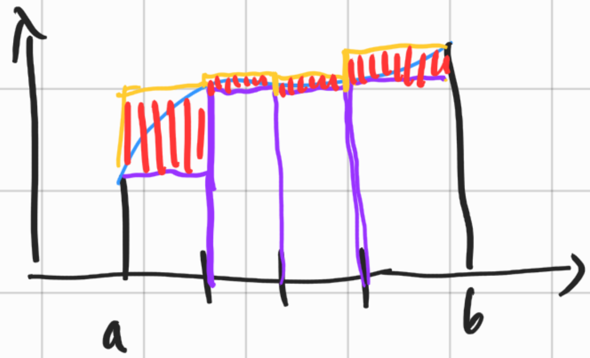
dividiamo $[a,b]$ in n parti:

$$P_n = \left\{ x_k = a + \frac{b-a}{n}\cdot k \;:\; k=0,1,\dots n \right\}$$

$$S^*_{*}(f,P_n) = \sum_{k=0}^{n-1} \frac{b-a}{n} \cdot \underset{x\in[x_k,x_{k+1}]}{\overset{\inf}{\sup}} f(x)$$

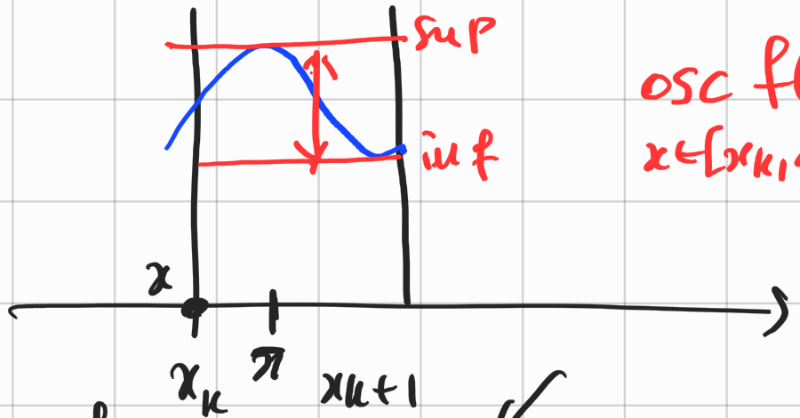
$\forall \varepsilon'>0\ \exists M:$

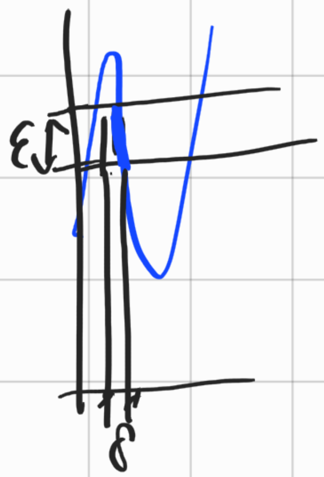
Criterio di integrabilità: $S^*(f,P_n) - S_*(f,P_n) \leq \varepsilon'$



$$S^*(f,P_n) - S_*(f,P_n) = \sum_{k=0}^{n-1} \frac{b-a}{n}\Big[\overbrace{\sup_{x\in[x_k,x_{k+1}]} f(x) \quad - \inf_{x\in[x_k,x_{k+1}]} f(x)}^{2\varepsilon}\Big]$$

$\operatorname{osc}\limits_{x\in[x_k,x_{k+1}]} f(x)$





[Se f unif. continua in x_k:

$$|x - x_k| < \delta \Rightarrow |f(x) - f(x_k)| < \varepsilon$$]

se $x_{k+1} - x_k < \delta$ $\qquad \forall x \in [x_k, x_{k+1}]$

$$\boxed{\frac{b-a}{n} < \delta} \qquad \overline{\quad |f(x) - f(x_k)| < \varepsilon \quad}$$

$$f(x_k) - \varepsilon \le f(x) \le f(x_k) + \varepsilon$$

$$\sup f(x) \le f(x_k) + \varepsilon$$

$$\inf f(x) \geq f(x_n) - \varepsilon$$

$$\boxed{\sup f(x) - \inf f(x) \leq 2\varepsilon.}$$

Se fosse

$$S^*(f, P_n) - S_*(f, P_n) \leq \sum_{k=0}^{n-1} \frac{b-a}{n} 2\varepsilon$$

$$= (b-a) \cdot 2\varepsilon \sum_{k=0}^{n-1} \frac{1}{n}$$

$$= (b-a) \cdot 2\varepsilon \leq \varepsilon' \quad \square$$

$$\varepsilon = \boxed{\frac{\varepsilon'}{2(b-a)}}$$

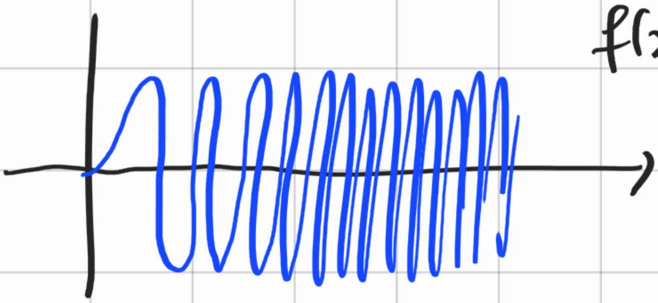
Prendo δ dall'uniforme continuità ($\forall \varepsilon : \exists \delta$)

$\boxed{\text{prendo } n \text{ t.c. } \frac{b-a}{n} < \delta}$

$$n > \frac{(b-a)}{\delta}$$

Questo lo posso fare solo se $\delta = \delta(\varepsilon)$ non dipende dal punto x.

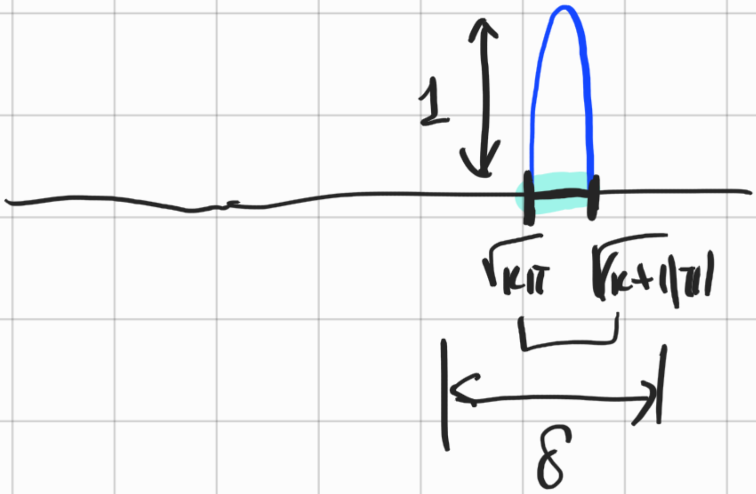
Esempio



$f(x) = \sin(x^2)$

$x^2 = k\pi$

$x = \sqrt{k\pi}$



Affinché la dimostrazione sia valida f deve soddisfare la seguente proprietà:

def [uniforme continuità]

$f : A \subseteq \mathbb{R} \to \mathbb{R}$ si dice essere **uniformemente continua** se

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in A : \forall y \in A : |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

def $f : A \subseteq \mathbb{R} \to \mathbb{R}$ è **continua** se:

$\forall x \in A$ f è continua in x cioè:

$$\forall x \in A : \forall \varepsilon > 0 \; \exists \delta > 0 : \forall y \in A : |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

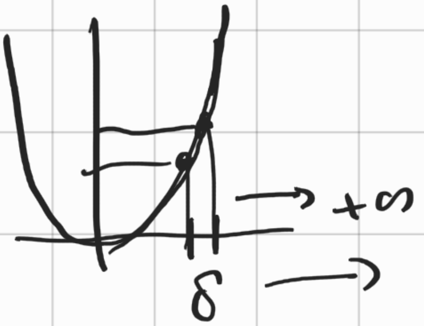
$\exists \delta : \forall x : \dots$ — δ non dipende da x

$\Downarrow$

$\forall x : \exists \delta : \dots$ — δ può dipendere da x

Esempio 2 $\quad f(x) = x^2$ è continua su $\mathbb{R}$.

ma non uniformemente continua



$\varepsilon = 1 \quad \exists ? \delta > 0$ t.c.

$$|x-y| < \delta \overset{?}{\Rightarrow} |x^2 - y^2| < 1$$

se prendo: $\quad y = x + \frac{\delta}{2}$

$$|x^2 - y^2| = \left(x + \frac{\delta}{2}\right)^2 - x^2 = \delta \cdot x + \frac{\delta^2}{4}$$

se $x \to +\infty \quad \delta \cdot x \to +\infty$.

Teorema (Heine-Cantor)

Se $f : [a,b] \to \mathbb{R}$ è continua (intervallo chiuso e limitato)

allora f è uniformemente continua

dim Hyp: f continua

Th: f uniformemente continua.

Per assurdo supponiamo che f non sia uniformemente continua:

$\neg\ \forall \varepsilon>0\ \exists \delta>0\ \forall x\in A\ \forall y \in A : |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$\exists \varepsilon>0\ \forall \delta>0\ \exists x\in A\ \exists y\ A : |x-y|<\delta \wedge |f(x)-f(y)|\geq\varepsilon$

$\exists \varepsilon>0\ \forall n\in\mathbb{N} : \delta=\frac{1}{n}\ \exists x_n\in A, \exists y_n \in A :$

$$x_n-y_n\to 0 \leftarrow \begin{cases} |x_n-y_n|<\frac{1}{n} \\ |f(x_n)-f(y_n)|\geq\varepsilon. \end{cases}$$

$A=[a,b]$ Applico Bolzano-Weierstraß

$\exists n_k \quad x_{n_k}\to\bar{x} \quad \bar{x}\in[a,b]$

$|y_{n_k}-x_{n_k}|\to 0 \quad y_{n_k}\to\bar{x}$

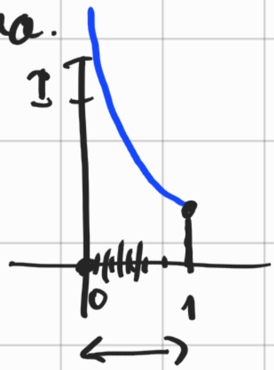
f è continua in $\bar{x}$ $\quad f(x_{n_k})\to f(\bar{x})$

$f(y_{n_k})\to f(\bar{x})$

$$|f(x_{n_k})-f(y_{n_k})|\longrightarrow|f(\bar{x})-f(\bar{x})|=0$$

Assurdo perché
$$|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$$
$\square$

<u>ES</u> $f:(0,1] \to \mathbb{R}$ $f(x) = \frac{1}{x}$ è continua ma non è uniformemente continua.



Se lo fosse: $\varepsilon = 1$

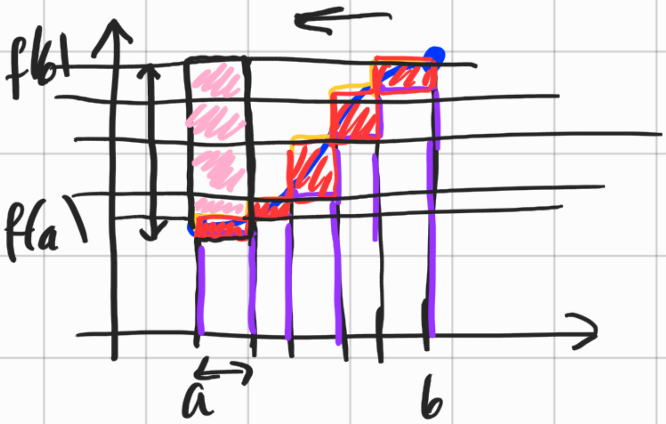
Qualunque sia $\delta \leq 1$

se prendo $x = \frac{\delta}{2}$, $y = \frac{\delta}{4}$

$$|f(x) - f(y)| = \left| \frac{2}{\delta} - \frac{4}{\delta} \right| = \frac{2}{\delta} > 2 > \varepsilon.$$
$\square$

<u>Teorema</u> Se $f:[a,b] \to \mathbb{R}$ è monotona allora f è limitata ed è Riemann-integrabile.

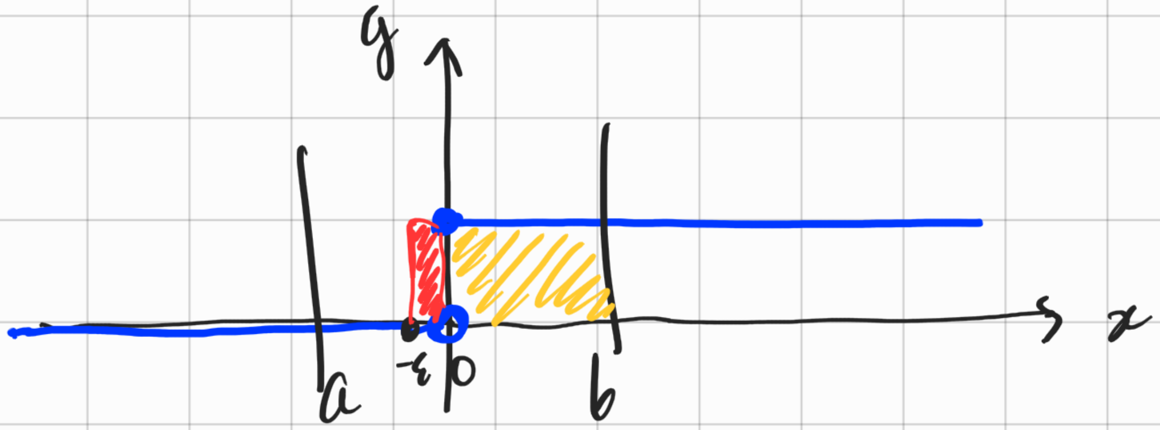
<u>dim</u> Supponiamo che f sia crescente.



per $n \to +\infty$

$$S^*(f, P_n) - S_*(f, P_n) = \frac{b-a}{n} \cdot (f(b) - f(a)) \longrightarrow 0$$
$\square$

ES $\quad f(x) = \begin{cases} 0 & \text{se } x < 0 \\ 1 & \text{se } x \geq 0 \end{cases}$



f è R-integrabile in ogni $[a,b]$.

esercizio | $\displaystyle\int_a^b f(x)\,dx = b \quad \begin{pmatrix} \text{se } b \geq 0 \\ \text{e } a \leq 0 \end{pmatrix}$