ERGODIC THEORY OF GEODESIC FLOWS

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Abstract. These are the notes of the course I gave for the PhD school in Mathematics at the University of Pisa in spring 2025.

The aim of the notes is to give an introduction to the ergodic theoretic properties of the geodesic flow on Riemannian negatively curved surfaces, a classical example of a "hyperbolic" flow which is a flourishing research area in dynamical systems. The statistical distribution of the orbits of the geodesic flow was already studied in the first half of the last century, first for the case of surfaces with constant negative curvature and then in the general case of variable curvature, and it was established the existence of a very rich dynamics making the geodesic flow as a prototypical example of a "chaotic" system. However, it was not until 1998 that the first quantitative result appeared about the speed of the decay of correlations for the case of compact surfaces with variable curvature, a result lately sharpened by applying the modern techniques of the dynamical systems theory.

After recalling the classical results, the notes focus on the more recent results about the decay of correlations. Finally, we introduce the thermodynamic formalism approach to dynamical systems and apply it to count the number of prime closed geodesics.

Contents

1. The geodesic flow on Riemannian surfaces

In this section, we first recall some basic notions of differential geometry of surfaces from the intrinsic point of view. The reader can find more details on books on differential geometry, two classical references we suggest are [dC76] and [Hi65].

Let Σ be a smooth connected Riemannian surface with metric tensor $< \cdot, \cdot >$. The tangent space at $z \in \Sigma$ is denoted by $T_z\Sigma$, and the tangent bundle of Σ is $T\Sigma$.

Definition 1.1 (Levi-Civita connection)**.** The *Levi-Civita connection* on Σ is the unique affine connection D which is compatible with the metric tensor and is torsion-free, that is, for all vector fields X, Y, Z (i.e. smooth sections of the tangent bundle) the following is true

$$Z(< X, Y >) =< D_Z X, Y > + < X, D_Z Y >,$$
$$D_X Y - D_Y X = [X, Y],$$

where $[X, Y]$ is the Lie bracket.

Definition 1.2 (Gaussian curvature)**.** Let the *Riemann curvature tensor* R be defined for vector fields X, Y, Z by

$$R(X, Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z .$$

The *Gaussian curvature of* Σ is the function K given by

$$K := \frac{< X, R(X, Y)Y >}{< X, X >< Y, Y > - < X, Y >^2}$$

for any vector fields X, Y for which the denominator does not vanish.

Definition 1.3 (Geodesics)**.** Let $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ be a smooth parametrized curve with $\gamma(0) = z \in \Sigma$ and $\dot\gamma(0) = v \in T_z\Sigma$. The curve γ is *the geodesic for* $(z, v) \in T\Sigma$ if $D_{\dot\gamma}\dot\gamma = 0$ for all $t \in (-\varepsilon, \varepsilon)$[1]. We use the notation $\gamma_{(z,v)}$ for the geodesic for (z, v).

Proposition 1.4. *Given $z \in \Sigma$ and $v \in T_z\Sigma$, $v \neq 0$, there exist $\varepsilon > 0$ and a unique smooth parametrized curve $\gamma_{(z,v)} : (-\varepsilon, \varepsilon) \to \Sigma$ which is the geodesic for (z, v).*

Definition 1.5 (Complete Riemannian surface)**.** A Riemannian surface Σ is called *geodesically complete* if all the geodesics have maximal interval of definition given by $\mathbb{R}$.

Proposition 1.6. *Let $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ be a geodesic with $\dot\gamma(0) \neq 0$, then $|\dot\gamma(t)|$ is constant.*

Proof. Let $s(t) := |\dot\gamma(t)|$ and let $\varepsilon' \in (0, \varepsilon)$ such that $s(t) > 0$ for all $t \in (-\varepsilon', \varepsilon')$. Then, define the vector field X along γ for $t \in (-\varepsilon', \varepsilon')$ as $X := \dot\gamma/s$. It holds $< X, X >= 1$, therefore

$$0 = X(< X, X >) = 2 < D_X X, X >,$$

hence $D_X X$ is orthogonal to X. Since $D_{\dot\gamma}\dot\gamma = 0$ and $\dot\gamma = sX$, by the properties of a connection[2] we have

$$0 = D_{\dot\gamma}\dot\gamma = s\, D_X(sX) = s^2\, D_X X + s\, X(s)\, X .$$

In conclusion, being $D_X X$ orthogonal to X and $s > 0$, both terms vanish, therefore $D_X X = 0$ and $X(s) = 0$. $\square$

Definition 1.7 (Jacobi field)**.** A vector field J defined along a geodesic γ is called a *Jacobi field* if

$$D_{\dot\gamma} D_{\dot\gamma} J = R(\dot\gamma, J)\dot\gamma .$$

[1]Intuitively, the component of the acceleration of the curve which is tangential to Σ vanishes.

[2]For all $f, g \in C^\infty(\Sigma)$ and vector fields X, Y, it holds $D_{fX} Y = f D_X Y$ and $D_X(gY) = g D_X Y + X(g)Y$.

Proposition 1.8. *Given a geodesic $\gamma : [0,1] \to \Sigma$, for each $u, v \in T_{\gamma(0)}\Sigma$ there exists a Jabobi field J along γ such that $J(0) = u$ and $D_{\dot\gamma}J(0) = v$. Moreover, if $u, v \in T_{\gamma(0)}\Sigma$ are orthogonal to $\dot\gamma(0)$, then J and $D_{\dot\gamma}J$ remain orthogonal to $\dot\gamma$ for all t.*

Proof. The first statement follows by thinking of u, v as initial conditions for the second-order differential equation defining a Jacobi field. The second statement follows from

$$\dot\gamma(< D_{\dot\gamma}J, \dot\gamma >) =< D_{\dot\gamma}D_{\dot\gamma}J, \dot\gamma >=< R(\dot\gamma, J)\dot\gamma, \dot\gamma >=< R(\dot\gamma, \dot\gamma)\dot\gamma, J >= 0\,,$$

where we have used the symmetries of the Riemann curvature tensor, and by

$$\dot\gamma(< J, \dot\gamma >) =< D_{\dot\gamma}J, \dot\gamma > \;.$$

□

Proposition 1.9. *Given a geodesic $\gamma : [0,1] \to \Sigma$ and a geodesic variation h_γ, namely a smooth function $h_\gamma : (-\varepsilon, \varepsilon) \times [0,1] \to \Sigma$ such that $h_\gamma(0,t) = \gamma(t)$ for all $t \in [0,1]$ and $t \mapsto h_\gamma(s,t)$ is a geodesic for all $s \in (-\varepsilon, \varepsilon)$, then $\partial h_\gamma/\partial s(0,t)$ is a Jacobi field along γ.*

Proof. Let's denote with $\dot h_\gamma$ the partial derivative of h_γ with respect to t. Since $t \mapsto h_\gamma(s,t)$ is a geodesic for all $s \in (-\varepsilon, \varepsilon)$, it holds $D_{\dot h_\gamma}\dot h_\gamma = 0$. Therefore, for all $(s,t) \in (-\varepsilon, \varepsilon) \times [0,1]$,

$$R\left(\dot h_\gamma, \frac{\partial h_\gamma}{\partial s}\right)\dot h_\gamma = D_{\dot h_\gamma}D_{\partial h_\gamma/\partial s}\dot h_\gamma - D_{\partial h_\gamma/\partial s}D_{\dot h_\gamma}\dot h_\gamma - D_{[\dot h_\gamma, \partial h_\gamma/\partial s]}\dot h_\gamma = D_{\dot h_\gamma}D_{\dot h_\gamma}\frac{\partial h_\gamma}{\partial s}\,,$$

where we have also used that $[\dot h_\gamma, \partial h_\gamma/\partial s] = 0$ and that the connection D is torsion-free. It is then enough to put $s = 0$ in the previous equality. □

Proposition 1.10. *Let $\gamma : [0,1] \to \Sigma$ be a unit speed geodesic and J an orthogonal Jacobi field along γ, that is $J(0)$ and $D_{\dot\gamma}J(0)$ are orthogonal to $\dot\gamma(0)$. Denote by N the orthonormal field to $\dot\gamma$, that is $N \in T_\gamma\Sigma$ and $< N, \dot\gamma >= 0$, then the function*

$$\rho(t) := \frac{\dot\gamma(< J, N >)}{< J, N >}$$

satisfies the Riccati differential equation

$$\dot\rho(t) = -K(\gamma(t)) - \rho^2(t)\,, \quad \forall\, t \in (0,1)\,, \tag{1.1}$$

where K is the Gaussian curvature.

Proof. Since $D_{\dot\gamma}N = 0$, we have

$$\dot\gamma(< J, N >) =< D_{\dot\gamma}J, N > \quad \text{and} \quad \dot\gamma(< D_{\dot\gamma}J, N >) =< D_{\dot\gamma}D_{\dot\gamma}J, N >=< R(\dot\gamma, J)\dot\gamma, N >\,,$$

where we have used that J is a Jacobi field. Moreover, since J is orthogonal and $|\dot\gamma| = 1$, we have

$$K(\gamma) = \frac{< \dot\gamma, R(\dot\gamma, J)J >}{< J, J >} = -\frac{< J, R(\dot\gamma, J)\dot\gamma >}{< J, J >} = -\frac{< N, R(\dot\gamma, J)\dot\gamma >}{< J, N >}\,.$$

Hence,

$$\dot\rho = \frac{\dot\gamma(< D_{\dot\gamma}J, N >)}{< J, N >} - \frac{< D_{\dot\gamma}J, N >^2}{< J, N >^2} = -K(\gamma) - \rho^2\,.$$

□

Definition 1.11 (Length of smooth curves)**.** Let $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ be a smooth parametrized curve, its *length* is defined as

$$\ell(\gamma) := \int_{-\varepsilon}^{\varepsilon} |\dot\gamma(t)|\, dt\,.$$

Theorem 1.12 (Hopf-Rinow)**.** *Let $d : \Sigma \times \Sigma \to \mathbb{R}_0^+$ be the geodesic distance defined by*

$$d(z, z') := \inf \left\{ \ell(\alpha) \, : \, \alpha \textit{ is a piecewise smooth curve connecting } z \textit{ and } z' \right\} .$$

Then Σ is geodesically complete if and only if (Σ, d) is a complete metric space.

Proposition 1.13. *For all $z \in \Sigma$ there exists a neighborhood $U(z)$ in Σ and $r > 0$ such that for all $w, w' \in U$ there exists a unique smooth geodesic $\gamma : [0, 1] \to \Sigma$ such that: $\gamma(0) = w$; $\gamma(1) = w'$; $\ell(\gamma) < r$; $\ell(\gamma) \leq \ell(\alpha)$ for any piecewise smooth curve α connecting w and w', and if $\ell(\gamma) = \ell(\alpha)$ then α and γ have the same image.*

The properties of the geodesics on a complete Riemannian surface imply that it is well defined the geodesic flow on the three-dimensional unit tangent bundle.

Definition 1.14 (Geodesic flow)**.** Let Σ be a smooth complete Riemannian surface and let $T^1\Sigma$ denote its unit tangent bundle, i.e. all (z, v) where $z \in \Sigma$ and $v \in T_z\Sigma$ with $|v| = 1$. Then the *geodesic flow on* Σ is the dynamical systems given by the action of $\mathbb{R}$ on $T^1\Sigma$ by

$$\mathbb{R} \times T^1\Sigma \ni (t, (z, v)) \mapsto \varphi_t(z, v) := (\gamma_{(z,v)}(t), \dot{\gamma}_{(z,v)}(t)) \in T^1\Sigma \, .$$

For each $t \in R$, the map $\varphi_t : T^1\Sigma \to T^1\Sigma$ is smooth.

2. Surfaces with constant curvature

A well-known result states that if Σ is simply connected with constant Gaussian curvature then it is isometric to one of the standard surfaces.

Theorem 2.1 (Killing-Hopf). *Let Σ be a connected, simply connected, complete Riemannian surface with constant Gaussian curvature K. Then, Σ is isometric to the Euclidean plane $\mathbb{R}^2$ if $K = 0$, to the sphere $\mathbb{S}^2$ if $K > 0$, and to the hyperbolic half-plane $\mathbb{H}^2$ if $K < 0$.*
Dropping the simple connectedness, Σ is isometric to a quotient of $\mathbb{R}^2$, $\mathbb{S}^2$, or $\mathbb{H}^2$, by a group acting freely and properly discontinuously.

2.1. **The Euclidean plane.** For the Euclidean plane $\mathbb{R}^2$, it is well-known that the geodesics are given by the straight lines, and for surfaces with constant vanishing Gaussian curvature, the geodesics are given by the projection of the straight lines. This follows from the fact that on the plane $D_{\dot\gamma}\dot\gamma = \ddot\gamma$, therefore the geodesic $\gamma_{(z,v)}$ with $(z, v) \in T^1\mathbb{R}^2$ is the solution of the Cauchy problem

$$\begin{cases} \ddot\gamma(t) = 0 \\ \gamma(0) = z \\ \dot\gamma(0) = v \end{cases}$$

hence $\gamma_{(z,v)}(t) = z + tv$.

In particular, the geodesic flow on a flat surface has simple dynamical properties. We state the following facts, which should be compared to their counterparts in the other cases:

(flat-a) On $\mathbb{R}^2$, there are no closed geodesics and all geodesics are unbounded. On $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, a geodesic is closed or dense on $\mathbb{T}^2$, but not on $T^1\mathbb{T}^2$.

(flat-b) On $\mathbb{R}^2$, for all $z \in \mathbb{R}^2$ and all $v, v' \in T^1_z\mathbb{R}^2$ with $v \neq v'$, we have

$$d(\gamma_{(z,v)}(t), \gamma_{(z,v')}(t)) = 2t \sin\alpha\,, \quad \forall\, t \in \mathbb{R}\,,$$

where $2\alpha \in (0, \pi)$ is the angle between v and v'. On a compact quotient of $\mathbb{R}^2$, this relation holds for small times.
On any flat surface, the projection of the geodesic flow on Σ is an isometry. That is, for all $z, z' \in \mathbb{R}^2$ and all v of norm 1, we have

$$d(\gamma_{(z,v)}(t), \gamma_{(z',v)}(t)) = d(z, z')\,, \quad \forall\, t \in \mathbb{R}\,.$$

2.2. **The sphere.** It is well-known that the geodesics on the sphere are all and only the great circles. We give a proof of this results which mixes the analytical and geometrical points of view.

Proposition 2.2. *The geodesics on $\mathbb{S}^2$ are all and only the great circles parametrised with constant velocity.*

Proof. First, we show that all the great circles are geodesics parametrised with constant velocity. The sphere $\mathbb{S}^2$ as the subset of the Euclidean space $\mathbb{R}^3$ of equation $x^2 + y^2 + \omega^2 = 1$ in Euclidean coordinates is a surface with constant Gaussian curvature $K = 1$. It is immediate to prove that the curve $\Gamma = \mathbb{S}^2 \cap \{\omega = 0\}$ parametrised by

$$\gamma(t) = (\cos t\,, \sin t\,, 0)\,, \quad t \in [0, 2\pi]\,,$$

is a geodesic. Indeed, $|\dot\gamma(t)| = 1$, and $\ddot\gamma(t)$ is in the plane $\{\omega = 0\}$ and is orthogonal to $\dot\gamma(t)$ for all t. In addition, all the other great circles may be obtained as the image of Γ for an isometry of $\mathbb{S}^2$. Since by Proposition 1.13, the geodesics are locally the unique minimisers of the geodesic distance, it follows that all the great circles parametrised with constant velocity are geodesics.

Analogously, let $z \in \mathbb{S}^2$ and $z' \in U(z)$ such that $d(z, z') < r$, with $U(z)$ and $r > 0$ given by Proposition 1.13. Let $\alpha(t)$ be a geodesic with $\alpha(0) = z$, $\alpha(1) = z'$. Since there is a great circle through z and z', by the uniqueness result in Proposition 1.13 it follows that $\alpha(t)$ is part of a great circle. $\square$

Concerning the dynamical properties of the geodesic flow on $\mathbb{S}^2$ we can then make the following remarks:

(pos-a) On $\mathbb{S}^2$, all the geodesics are closed.
(pos-b) For all $(z, v), (z', v') \in \mathbb{S}^2$, the geodesics $\gamma_{(z,v)}(t)$ and $\gamma_{(z',v')}(t)$ eventually intersect.

2.3. **The hyperbolic half-plane.** We consider the Poincaré upper half-plane model $\mathbb{H}^2$ of the hyperbolic two-dimensional space. Let

$$\mathbb{H}^2 := \{z = x + iy \in \mathbb{C} \,:\, y > 0\}$$

with $T\mathbb{H}^2 \cong \mathbb{H}^2 \times \mathbb{C}$. The connected half-plane $\mathbb{H}^2$ becomes a Riemannian surface with constant Gaussian curvature $K = -1$ when endowed with the metric tensor defined for each $z = x + iy \in \mathbb{H}^2$ by

$$< v, v' >:= \frac{1}{y^2}\, \Re(v\, \overline{v'})\,, \quad \forall\, v, v' \in T_z\mathbb{H}^2\,, \tag{2.1}$$

where $\Re(v\, \bar{v'})$ coincides with the Euclidean scalar product for v, v' seen as vectors in $\mathbb{R}^2$.

We now proceed as in Section 2.2 to find all the geodesics on $\mathbb{H}^2$. The argument here follows [FH19, Section 2.1].

Lemma 2.3. *The imaginary axis $I := \{z \in \mathbb{H}^2 \,:\, x = 0\}$ is a geodesic with unit-speed parametrization $\gamma_{(i,i)}(t) = ie^t$.*

Proof. First, it is immediate that $\dot\gamma_{(i,i)}(t) = ie^t$ satisfies

$$|\dot\gamma_{(i,i)}(t)| = \sqrt{< \dot\gamma_{(i,i)}(t), \dot\gamma_{(i,i)}(t) >} = \frac{1}{e^t}\sqrt{\Re(ie^t\, \overline{ie^t})} = 1\,, \quad \forall\, t \in \mathbb{R}.$$

Then, let $y_0, y_1 \in (0, +\infty)$ and $\alpha(t) = x(t) + iy(t)$ be a piecewise smooth curve connecting $z_0 = iy_0$ to $z_1 = iy_1$ for $t \in [t_0, t_1]$. Then

$$\ell(\alpha) = \int_{t_0}^{t_1} |\dot\alpha(t)|\, dt = \int_{t_0}^{t_1} \sqrt{\frac{\dot x^2(t) + \dot y^2(t)}{y^2(t)}}\, dt \ge \int_{t_0}^{t_1} \frac{\dot y(t)}{y(t)}\, dt\,,$$

where the right-most term is the length of the imaginary axis between z_0 and z_1. Hence by the length minimizing property of the geodesics, we find that the imaginary axis is a geodesic and the geodesic distance between z_0 and z_1 on I is $d(iy_0, iy_1) = \log(y_1/y_0)$. $\square$

To argue as in Proposition 2.2, we need to have a characterization of the group of isometries of $\mathbb{H}^2$. We begin by introducing the *Möbius transformations*.

Lemma 2.4. *Let $GL_+(2, \mathbb{R})$ be the group of real 2×2 matrices with positive determinant, and let Ψ be the map which associates to each $g \in GL_+(2, \mathbb{R})$ the transformation of $\mathbb{H}^2$ given by*

$$GL_+(2, \mathbb{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi_g\,, \quad \textit{with} \quad \psi_g(z) = \frac{az + b}{cz + d}\,, \quad \forall\, z \in \mathbb{H}^2.$$

Then, $\Psi(GL_+(2, \mathbb{R}))$ is a group under composition and it is a subgroup of the isometries of $\mathbb{H}^2$. Moreover, Ψ is a group homomorphism with kernel $\mathbb{R}^0 \cdot Id$, where $\mathbb{R}^0 := \mathbb{R} \setminus \{0\}$.

Proof. The result follows from standard computations showing that for all $g, g' \in GL_+(2,\mathbb{R})$ we have $\psi_g \circ \psi_{g'} = \psi_{gg'}$ and that $\psi_{\lambda g} = \psi_g$ for all $\lambda \in \mathbb{R}^0$, and by proving that ψ_g is an isometry of $\mathbb{H}^2$. First,

$$\Im(\psi_g(z)) = \frac{\det(g)\,\Im(z)}{|cz+d|^2} > 0\,, \quad \forall\, z \in \mathbb{H}^2\,,\; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_+(2,\mathbb{R})\,, \tag{2.2}$$

hence $\psi_g(\mathbb{H}^2) \subseteq \mathbb{H}^2$. Then, $d_z\psi_g(v) = \psi_g'(z)v$ for all $(z,v) \in T\mathbb{H}^2$, where

$$\psi_g'(z) = \frac{\det(g)}{(cz+d)^2} \tag{2.3}$$

is the derivative of $\psi_g(z)$ with respect to z as a complex number. Hence, for all $z \in \mathbb{H}^2$ and all $v, v' \in T_z\mathbb{H}^2$, we have

$$< d_z\psi_g(v), d_z\psi_g(v) >= \frac{1}{(\Im(\psi_g(z)))^2}\,\Re(\psi_g'(z)v\,\overline{\psi_g'(z)v'}) = \frac{1}{(\Im(z))^2}\,\Re(v\,\overline{v'}) =< v, v' >$$

for all $g \in GL_+(2,\mathbb{R})$. $\square$

The subgroup of isometries $\mathcal{M} := \Psi(GL_+(2,\mathbb{R}))$ is called the group of Möbius transformations, and it is isomorphic to the quotient $PSL(2,\mathbb{R}) := GL_+(2,\mathbb{R})/(\mathbb{R}^0 \cdot Id) = SL(2,\mathbb{R})/(\pm Id)$.

Definition 2.5. The elements of $\mathcal{M} \cong PSL(2,\mathbb{R})$ are of three types:

- *Elliptic* if the matrix $g \in SL(2,\mathbb{R})$ satisfies $|tr(g)| < 2$ or ψ_g has exactly one fixed point in $\mathbb{H}^2$. An important example of an elliptic element is the *inversion* given by
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \psi_S(z) = -\frac{1}{z}\,;$$
- *Parabolic* if the matrix $g \in SL(2,\mathbb{R})$ satisfies $|tr(g)| = 2$ or ψ_g has exactly one fixed point which lies in $\partial\mathbb{H}^2$. Important examples of parabolic elements are the *translation* given for $b \in \mathbb{R}$ by
$$\tau_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \psi_{\tau_b}(z) = z + b\,;$$
- *Hyperbolic* if the matrix $g \in SL(2,\mathbb{R})$ satisfies $|tr(g)| > 2$ or ψ_g has exactly two fixed points both on $\partial\mathbb{H}^2$. Important examples of hyperbolic elements are the *homotheties* (or *squeezings*) given for $a \in \mathbb{R} \setminus \{0, \pm 1\}$ by
$$\lambda_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{or} \quad \psi_{\lambda_a}(z) = a^2 z\,.$$

All parabolic and hyperbolic elements are conjugate to τ_b or to λ_a, respectively, for some $b \in \mathbb{R}$ or $a \in \mathbb{R} \setminus \{0, \pm 1\}$.

We can now prove the fundamental result to find all the geodesics on $\mathbb{H}^2$.

Lemma 2.6. *For all $g \in SL(2,\mathbb{R})$, the image $\psi_g(I)$ of the imaginary axis $I := \{z \in \mathbb{H}^2 : x = 0\}$ is a vertical line or a semicircle with center on the real line $\{y = 0\}$. Viceversa, let C be any vertical line or any semicircle with center on the real line $\{y = 0\}$, then there exists $g \in SL(2,\mathbb{R})$ such that $\psi_g(I) = C$.*

Proof. The first part is a simple verification for which we use an important property. Namely, every $g \in SL(2,\mathbb{R})$ can be written as a composition of translations, a homothety, and the inversion. Indeed,

$$SL(2,\mathbb{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \tau_{ba} \circ \lambda_a \,, & \text{if } c = 0 \text{ and } d = a^{-1}; \\ \tau_{a/c} \circ S \circ \tau_{cd} \circ \lambda_c \,, & \text{if } c \neq 0. \end{cases} \tag{2.4}$$

Since the the homotheties preserve I and the translations send I to a vertical line, $\psi_g(I)$ is a vertical line if $c = 0$. If $c \neq 0$, $(\tau_{cd} \circ \lambda_c)(I)$ is a vertical line, equal to I if $d = 0$. Now, the inversion S preserves I and sends a vertical line $\{x = x_0\}$ in $\mathbb{H}^2$ with $x_0 \neq 0$ to the semicircle with center in $z = -1/(2x_0)$, hence $\psi_g(I)$ is again a vertical line or a semicircle with center on the real line.

The second part of the statement can be proved by constructing ψ_g. If C is a vertical line $\{x = x_0\}$ then $\psi_{\tau_{x_0}}(I) = C$. If C is a semicircle with endpoints x_0 and $x_0 + r > x_0$ on the line $\{y = 0\}$, then

$$g = \tau_{x_0} \circ \lambda_{\sqrt{r}} \circ \tau_1 \circ S \circ \tau_1 \quad \text{gives} \quad \psi_g(I) = C\,.$$

It is enough to check that $\tau_1 \circ S \circ \tau_1$ sends I to the semicircle with endpoints in 0 and 1, and that the homothety $\lambda_{\sqrt{r}}$ and the translation τ_{x_0} give the wanted semicircle C. □

Proposition 2.7. *The geodesics on $\mathbb{H}^2$ are all and only the vertical lines $\{x = x_0\}$ and the semicircles with center on the real line $\{y = 0\}$.*

Proof. We proceed as in the proof of Proposition 2.2. Given that by Lemma 2.3 the imaginary axis I is a geodesic, all the vertical lines and all the semicircles with center on the real line are image of a geodesic for an isometry $\psi_g \in \mathcal{M}$ of $\mathbb{H}^2$. Hence, by Proposition 1.13, they are geodesics.

On the contrary, all points $z_1, z_2 \in \mathbb{H}^2$ may be connected by a curve C, which is either a vertical line or a semicircle with center on the real line. By Lemma 2.6, there exists an isometry which sends C to I, therefore again by the length minimizing property of the geodesics, the only geodesic connecting z_1 and z_2 is C. □

We can now state the first property of the geodesic flow on $\mathbb{H}^2$.

(neg-a) On $\mathbb{H}^2$, there are no closed geodesics and all geodesics are unbounded. Given $(z, v) \in T^1\mathbb{H}^2$, the geodesic $\gamma_{(z,v)}$ has endpoints

$$\gamma^{\pm}_{(z,v)} := \lim_{t \pm \infty} \gamma_{(z,v)}(t)\,. \tag{2.5}$$

If the geodesic is a semicircle then $\gamma^{\pm}_{(z,v)} \in \{y = 0\} \subset \partial\mathbb{H}^2$, if the geodesic is a vertical line then one of the endpoints lies on $\{y = 0\}$ and the other one is $\infty \in \partial\mathbb{H}^2$.
On the quotients of $\mathbb{H}^2$ it is possible to find closed geodesics and dense on $T^1\mathbb{H}^2$ ones, but other behaviors are possible.

For the other dynamical properties we need to extend the algebraic approach to $T^1\mathbb{H}^2$.

Proposition 2.8. *The action of $\mathcal{M} \cong PSL(2,\mathbb{R})$ on $\mathbb{H}^2$ may be lifted to an action on $T^1\mathbb{H}^2$, which is simply transitive. In particular, $PSL(2,\mathbb{R})$ is homeomorphic to $T^1\mathbb{H}^2$.*

Proof. The action of $\mathcal{M} \cong PSL(2,\mathbb{R})$ on $T^1\mathbb{H}^2$ is obtained by using the differential of the Möbius transformations. That is we consider

$$PSL(2,\mathbb{R}) \times T^1\mathbb{H}^2 \ni (g, (z, v)) \mapsto d\psi_g(z, v) := (\psi_g(z), \psi'_g(z)v) \in T^1\mathbb{H}^2\,, \tag{2.6}$$

where $|\psi'_g(z)v| = |v| = 1$ because ψ_g is an isometry.

It is now enough to show that for each $(z,v) \in T^1\mathbb{H}^2$ there exists a unique $g \in PSL(2,\mathbb{R})$ for which $d\psi_g(i,i) = (z,v)$. First, given $z = x+iy$, it is a simple calculation to show that

$$g_0 = \tau_x\,\lambda_{\sqrt{y}} = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \quad \text{gives} \quad \psi_{g_0}(i) = z\,. \tag{2.7}$$

Then, a unit vector $v \in T_z\mathbb{H}^2$ may be written as the complex number

$$v = i\,y\,(\cos\theta + i\sin\theta)\,, \tag{2.8}$$

where $y = \Im(z)$ and θ is the angle between v and the vertical vector pointing upward in the plane measured counterclockwise. So, for example $i \in T_i\mathbb{H}^2$ has $\theta = 0$ and $y = 1$. Using (2.3), it is a straightforward computation to verify that,

$$d\psi_{g_0}(i,v) = (z, yv)\,. \tag{2.9}$$

Hence g_0 does not change the orientation of any vector $v \in T^1_i\mathbb{H}^2$, and the same holds when acting on any $(z',v) \in T^1\mathbb{H}^2$.

Let us consider the action on (i,i) of the following subset of Möbius transformations

$$\rho_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Notice that $S = \rho_{-\pi}$ in $PSL(2,\mathbb{R})$. Using (2.3) as before, one can verify that

$$d\psi_{\rho_\theta}(i,i) = \Big(i, i(\cos\theta + i\sin\theta)\Big)\,. \tag{2.10}$$

In particular, the subgroup $K := \{\rho_\theta\}_\theta$ is contained in the the stabilizer of $z = i$ for the action of $\mathcal{M}$ on $\mathbb{H}^2$ and acts on $T_i\mathbb{H}^2$ as a rotation of angle θ.

Given any $(z,v) \in T^1\mathbb{H}^2$ with $z = x+iy$ and $v = i\,y\,(\cos\theta + i\sin\theta)$, we are now ready to show that $d\psi_g(i,i) = (z,v)$ for[3]

$$g = g_{(z,v)} := \tau_x\,\lambda_{\sqrt{y}}\,\rho_\theta = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{2.11}$$

It is enough to put together (2.9) and (2.10).

So far we have proved that the action of $\mathcal{M}$ on $T^1\mathbb{H}^2$ is transitive. We now have to prove that it is free, and it is enough to prove that the element $g_{(z,v)}$ defined in (2.11) is the unique for which $d\psi_g(i,i) = (z,v)$. This follows by showing that the stabilizer of (i,i) is $\{\pm Id\}$.

Let $g \in PSL(2,\mathbb{R})$ with $d\psi_g(i,i) = (i,i)$. Then

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad \frac{ai+b}{ci+d} = i \quad \text{and} \quad \frac{i}{(ci+d)^2} = i\,.$$

The first condition on the right implies that the stabilizer of i for the action on $\mathbb{H}^2$ is in fact the subgroup K, and the two conditions together give that $d\psi_g(i,i) = (i,i)$ implies $g \in \{\pm Id\}$. We have thus proved that (2.11) gives a map from $T^1\mathbb{H}^2$ to $PSL(2,\mathbb{R})$ that has an inverse, which is given by

$$PSL(2,\mathbb{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\frac{ai+b}{ci+d}\,,\, i\,\frac{\cos\theta + i\sin\theta}{c^2+d^2}\right) \in T^1\mathbb{H}^2\,, \tag{2.12}$$

where $\theta = -2\arctan(c/d)$ if $d \neq 0$, and $\theta = \pi \pmod{2\pi}$ if $d = 0$.

[3] This is known as the *Iwasawa decomposition* of $SL(2,\mathbb{R})$.

Finally, thanks to the topological structures of $PSL(2,\mathbb{R})$ and $T^1\mathbb{H}^2$, we have defined a homeomorphism between the two spaces. $\square$

The geodesic flow of Definition 1.14 is the dynamical system on $T^1\mathbb{H}^2$ given by the action of $\mathbb{R}$ which makes a point (z,v) to flow along the unique geodesic it defines. Since $T^1\mathbb{H}^2$ is homeomorphic to $PSL(2,\mathbb{R})$, the geodesic flow on $T^1\mathbb{H}^2$ may be seen as the action of a one-dimensional subgroup of $PSL(2,\mathbb{R})$ on the group $PSL(2,\mathbb{R})$.

Proposition 2.9. *Let $(z,v) \in T^1\mathbb{H}^2$ with $z = x+iy$ and $v = i\,y\,(\cos\theta + i\sin\theta)$, and let $g_{(z,v)}$ be as in (2.11). Then, $\varphi_t(z,v)$ is represented in $PSL(2,\mathbb{R})$ by the matrix*

$$g_{\varphi_t(z,v)} = g_{(z,v)}\,\lambda_{e^{t/2}} = g_{(z,v)} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Hence, the homeomorphism between $T^1\mathbb{H}^2$ and $PSL(2,\mathbb{R})$ is a topological conjugacy between the geodesic flow on $T^1\mathbb{H}^2$ and the right action of the one-dimensional group $\{\lambda_a\}_{a>0}$ on $PSL(2,\mathbb{R})$.

Proof. First, let's prove the result for (i,i). We have $g_{(i,i)} = Id$ and, by Lemma 2.3, $\varphi_t(i,i) = (ie^t, ie^t)$. It follows that $g_{\varphi_t(i,i)} = \lambda_{e^{t/2}}$.

Let now $(z,v) \in T^1\mathbb{H}^2$ with $z = x+iy$ and $v = i\,y\,(\cos\theta + i\sin\theta)$. By Lemma 2.6 and Proposition 2.8, there exists a unique $g = g_{(z,v)}$ such that $d\psi_g(i,i) = (z,v)$ and $\varphi_t(z,v) = d\psi_g(\varphi_t(i,i))$. It follows that $g_{\varphi_t(z,v)} = g_{(z,v)}\,g_{\varphi_t(i,i)} = g_{(z,v)}\,\lambda_{e^{t/2}}$. $\square$

Remark 2.10. Given $(z,v) \in T^1\mathbb{H}^2$, we can write the endpoints $\gamma^{\pm}_{(z,v)}$ (see (2.5)) of the geodesic for (z,v) in terms of the entries of the matrix $g_{(z,v)}$. Applying the geodesic flow as the right multiplication by $\lambda_{e^{t/2}}$, one finds

$$g_{(z,v)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad \gamma^{+}_{(z,v)} = \frac{a}{c}, \quad \gamma^{-}_{(z,v)} = \frac{b}{d},$$

with the convention $1/0 = \infty \in \partial\mathbb{H}^2$.

We are now ready to study the properties in (neg-b). First, we read the properties in (flat-b) by using the differential structure of $T^1\mathbb{R}^2 \cong \mathbb{R}^2 \times S^1$. The unit tangent bundle $T^1\mathbb{R}^2$ is a three-dimensional manifold with $T_{(z,v)}(T^1\mathbb{R}^2) \cong \mathbb{R}^2 \times \mathbb{R}$ for all $(z,v) \in T^1\mathbb{R}^2$, with a possible basis of the tangent space given by $\{v,u,1\}$ where $v,u \in \mathbb{R}^2$ satisfy $< v,u >= 0$. This basis identifies three directions along which to move the initial condition (z,v) of the geodesic flow. The first direction, the variation along v in the position z, corresponds to the motion along the geodesic itself. The second direction, the variation along u in the position z, corresponds to consider initial conditions $(z+\delta u, v)$ for $\delta \in \mathbb{R}$. The third direction, the variation in the direction of v, corresponds to consider initial conditions (z,v') with $v' \neq v$. Thus, the natural variations considered in (flat-b) correspond to the variations along a basis of the tangent space $T_{(z,v)}(T^1\mathbb{R}^2)$.

We can now repeat the same argument to obtain the statements in (neg-b). First, we have to characterize the tangent spaces $T_{(z,v)}(T^1\mathbb{H}^2)$. For this, we use the homeomorphism with $PSL(2,\mathbb{R})$, which is a three-dimensional Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. A possible basis of the Lie algebra is given by

$$X := \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad H_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.13}$$

Therefore, given $(z,v) \in T^1\mathbb{H}^2$ as the initial condition of a geodesic flow, we look at the variations of the initial condition along the three directions identified by X, H_+, H_- and using right actions.

We find that the first direction gives the point (z', v') for which

$$g_{(z',v')} = g_{(z,v)} \exp(\delta X) = g_{(z,v)} \lambda_{e^{\delta/2}},$$

that is $(z', v') = \varphi_\delta(z, v)$, hence it corresponds to the motion along the geodesic $\gamma_{(z,v)}$.

Let us now consider the variations along $H_\pm$ beginning with initial condition (i, i). Since

$$\exp(\delta H_+) = \tau_\delta \quad \text{and} \quad \exp(\delta H_-) = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}, \tag{2.14}$$

we find

$$W^+(i,i) := \left\{ g_{(i,i)} \exp(\delta H_+) \right\}_{\delta \in \mathbb{R}} = \left\{ (x+i, i) \in T^1\mathbb{H}^2 \right\}_{x \in \mathbb{R}}, \tag{2.15}$$

that is the horizontal line $\{y = 1\}$ with tangent vectors $i \in T_{x+i}\mathbb{H}^2$ for all $x \in \mathbb{R}$, and

$$\begin{aligned} W^-(i,i) &:= \left\{ g_{(i,i)} \exp(\delta H_-) \right\}_{\delta \in \mathbb{R}} = \\ &= \left\{ (x+iy, v) \in T^1\mathbb{H}^2 \,:\, x^2 + \left(y - \frac{1}{2} \right)^2 = \frac{1}{4}\,,\ \theta = -2 \arctan\left(\frac{x}{y} \right) \right\}, \end{aligned} \tag{2.16}$$

that is the circle tangent to the line $\{y = 0\}$ at $x = 0$ and passing through i, and with unit tangent vectors orthogonal to the circle and pointing outwards.

Definition 2.11 (Horocycles)**.** Let $(z, v) \in T^1\mathbb{H}^2$ and let $\gamma_{(z,v)}$ be the geodesic for (z, v).

- If $\gamma_{(z,v)}$ is a vertical line with v pointing upwards, then we call *positive horocycle of* (z, v) the set $W^+(z, v) \subset T^1\mathbb{H}^2$ given by the horizontal line $\{y = \Im(z)\}$ with tangent vectors $i\Im(z) \in T_{x+i\Im(z)}\mathbb{H}^2$ for all $x \in \mathbb{R}$, and we call *negative horocycle of* (z, v) the set $W^-(z, v) \subset T^1\mathbb{H}^2$ given by the circle tangent to the line $\{y = 0\}$ at $x = \Re(z)$ and passing through z, and with unit tangent vectors orthogonal to the circle and pointing outwards.
- If $\gamma_{(z,v)}$ is a vertical line with v pointing downwards, then we call *positive horocycle of* (z, v) the set $W^+(z, v) \subset T^1\mathbb{H}^2$ given by the circle tangent to the line $\{y = 0\}$ at $x = \Re(z)$ and passing through z, and with unit tangent vectors orthogonal to the circle and pointing inwards, and we call *negative horocycle of* (z, v) the set $W^-(z, v) \subset T^1\mathbb{H}^2$ given by the horizontal line $\{y = \Im(z)\}$ with tangent vectors $-i\Im(z) \in T_{x+i\Im(z)}\mathbb{H}^2$ for all $x \in \mathbb{R}$.
- If $\gamma_{(z,v)}$ is a semicircle with center and endpoints $\gamma^\pm_{(z,v)}$ on the line $\{y = 0\}$, then we call *positive horocycle of* (z, v) the set $W^+(z, v) \subset T^1\mathbb{H}^2$ given by circle tangent to the line $\{y = 0\}$ at $x = \gamma^+_{(z,v)}$ and passing through z, and with unit tangent vectors orthogonal to the circle and pointing inwards, and we call *negative horocycle of* (z, v) the set $W^-(z, v) \subset T^1\mathbb{H}^2$ given by circle tangent to the line $\{y = 0\}$ at $x = \gamma^-_{(z,v)}$ and passing through z, and with unit tangent vectors orthogonal to the circle and pointing outwards.

Proposition 2.12. *For all* $(z, v) \in T^1\mathbb{H}^2$ *we have* $W^\pm(z, v) = d\psi_{g_{(z,v)}}(W^\pm(i, i))$.

Proof. It follows from the homeomorphism between $PSL(2, \mathbb{R})$ and $T^1\mathbb{H}^2$, since

$$g_{(z,v)} \exp(\delta H_\pm) \quad \longleftrightarrow \quad d\left(\psi_{g_{(z,v)}} \circ \psi_{\exp(\delta H_\pm)} \right)(i, i)\,.$$

□

Definition 2.13 (Distance on $T^1\mathbb{H}^2$)**.** The geodesic distance $d(\cdot, \cdot)$ on $\mathbb{H}^2$ can be extended to a distance $d_h(\cdot, \cdot)$ on $T^1\mathbb{H}^2$ by

$$d_h((z, v), (z', v')) = \sqrt{d^2(z, z') + (\measuredangle(v, v'))^2}\,, \quad \forall\, (z, v), (z', v') \in T^1\mathbb{H}^2,$$

where $\measuredangle(v, v')$ is the angle in $[0, 2\pi]$ between v' and the parallel transport of v from $T_z\mathbb{H}^2$ to $T_{z'}\mathbb{H}^2$ along the geodesic connecting z and z'.

Lemma 2.14. *For all $x \in \mathbb{R}$, consider the point $(x+i, i) \in W^+(i, i)$. We have*

$$\lim_{t\to+\infty} e^t\, d_h(\varphi_t(i, i), \varphi_t(x+i, i)) = \sqrt{2}\, |x| \,.$$

For all $(z, v) \in W^-(i, i)$,

$$\lim_{t\to-\infty} e^{-t}\, d_h(\varphi_t(i, i), \varphi_t(z, v)) = \sqrt{2} \left| \frac{\Re(z)}{\Im(z)} \right| .$$

Proof. For the first part, we need to compute the distance d_h between the points $(z, v) = \varphi_t(i, i) = (ie^t, ie^t)$ and $(z', v') = \varphi_t(x+i, i) = (x+ie^t, ie^t)$. The geodesic between z and z' is the semicircle centered on the line $\{y = 0\}$ at $(x/2, 0)$ of radius $r_t = \sqrt{x^2/4 + e^{2t}}$. It follows that

$$d(z, z') \sim |x|\, e^{-t}\,, \qquad \left|\measuredangle(v, v')\right| \sim |x|\, e^{-t}\,, \quad \text{as } t \to +\infty,$$

hence we obtain the first result.

For the second result, we start observing that

$$d_h(\varphi_t(i, i), \varphi_t(z, v)) = d_h(\varphi_{-t}(i, -i), \varphi_{-t}(z, -v))\,,$$

and $(z, -v) \in W^+(i, -i)$. Then, by Proposition 2.12, the transformation $d\psi_S$ given by the inversion S sends $W^+(i, -i)$ to $W^+(i, i)$, hence $d\psi_S(z, -v) \in W^+(i, i)$, and we can apply the first part. It follows that

$$\begin{aligned} d_h(\varphi_t(i, i), \varphi_t(z, v)) = d_h(\varphi_{-t}(i, -i), \varphi_{-t}(z, -v)) &= d_h(d\psi_S(\varphi_{-t}(i, -i)), d\psi_S(\varphi_{-t}(z, -v))) = \\ &= d_h(\varphi_{-t}(i, i), \varphi_{-t}(d\psi_S(z, -v)))\,, \end{aligned}$$

and

$$\lim_{t\to-\infty} e^{-t}\, d_h(\varphi_t(i, i), \varphi_t(z, v)) = \lim_{t\to+\infty} e^t\, d_h(\varphi_t(i, i), \varphi_t(d\psi_S(z, -v))) = \sqrt{2}\, |\Re(\psi_S(z))|\,.$$

Finally, from (2.16), we obtain

$$(z, v) = (x+iy, v) \in W^-(i, i) \;\Rightarrow\; x^2 + y^2 = y \;\Rightarrow\; \Re(\psi_S(z)) = \Re\left(-\frac{1}{z}\right) = -\frac{x}{y}\,.$$

□

Applying Proposition 2.12, we can state the general behavior for the orbits of variations of the initial conditions.

(neg-b) For all $(z, v) \in T^1\mathbb{H}^2$ the tangent space $T_{(z,v)}(T^1\mathbb{H}^2)$ can be written as the sum of three one-dimensional vector spaces,

$$E^c := \mathrm{Span}(X)\,, \quad E^s := \mathrm{Span}(H_+)\,, \qquad E^u := \mathrm{Span}(H_-)\,. \tag{2.17}$$

If we move the point (z, v) along E^c, we obtain points on the geodesic $\gamma_{(z,v)}$.

If we move the point (z, v) along E^s, we obtain points (z', v') of the positive horocycle $W^+(z, v)$, and

$$d_h(\varphi_t(z, v), \varphi_t(z', v')) \sim_{t\to+\infty} const(z, z') \cdot e^{-t}\,, \quad \forall\, (z', v') \in W^+(z, v)\,.$$

If we move the point (z, v) along E^u, we obtain points (z', v') of the negative horocycle $W^-(z, v)$, and

$$d_h(\varphi_t(z, v), \varphi_t(z', v')) \sim_{t\to-\infty} const(z, z') \cdot e^{t}\,, \quad \forall\, (z', v') \in W^-(z, v)\,.$$

We end this section by introducing the *positive* and *negative horocycle flows* on $\mathbb{H}^2$. Given $(z,v) \in T^1\mathbb{H}^2$ and its positive (negative) horocycle $W^+(z,v)$ $(W^-(z,v))$, the positive (negative) horocycle flow $\{u_s^+(z,v)\}_{s\in\mathbb{R}}$ $(\{u_s^-(z,v)\}_{s\in\mathbb{R}})$ slides the unit vector $v \in T_z\mathbb{H}^2$ along $W^+(z,v)$ $(W^-(z,v))$.

By mimicking the argument in Proposition 2.9 we can prove the following algebraic representation of the horocycle flows.

Proposition 2.15. *Let $(z,v) \in T^1\mathbb{H}^2$ with $z = x + iy$ and $v = i\,y\,(\cos\theta + i\sin\theta)$, and let $g_{(z,v)}$ be as in (2.11). Then, $u_s^+(z,v)$ is represented in $PSL(2,\mathbb{R})$ by the matrix*

$$g_{u_s^+(z,v)} = g_{(z,v)}\,\exp(sH_+) = g_{(z,v)} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Hence, the homeomorphism between $T^1\mathbb{H}^2$ and $PSL(2,\mathbb{R})$ is a topological conjugacy between the positive horocycle flow on $T^1\mathbb{H}^2$ and the right action of the one-dimensional group $\{\tau_b\}_{b\in\mathbb{R}}$ on $PSL(2,\mathbb{R})$.

Analogously, $u_s^-(z,v)$ is represented in $PSL(2,\mathbb{R})$ by the matrix

$$g_{u_s^-(z,v)} = g_{(z,v)}\,\exp(sH_-) = g_{(z,v)} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

Hence, the homeomorphism between $T^1\mathbb{H}^2$ and $PSL(2,\mathbb{R})$ is a topological conjugacy between the negative horocycle flow on $T^1\mathbb{H}^2$ and the right action of the one-dimensional group $\{\exp(\delta H_-)\}_{\delta\in\mathbb{R}}$ on $PSL(2,\mathbb{R})$.

Proposition 2.16. *For all $t, s \in \mathbb{R}$, the geodesic and the horocycle flows on $\mathbb{H}^2$ satisfy the following commutation rules*

$$\varphi_t \circ u_s^\pm = u_{se^{\mp t}}^\pm \circ \varphi_t\,.$$

Proof. Using Propositions 2.9 and 2.15, it's enough to prove that

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & s\,e^{-t} \\ 0 & 1 \end{pmatrix}$$

for the positive horocycle flow, and the analogous relation for the negative horocycle flow. ☐

3. Hyperbolic dynamics

As shown in the previous section, the geodesic flow on the Euclidean plane and on the sphere exhibits quite a "simple" dynamics. Either all orbits are periodic or little variations in the initial conditions lead to "slow" separation of the orbits. The situation is different on the hyperbolic half-plane.

In this section, we introduce the notion of *Anosov flow*, a class of flows including the geodesic flow on surfaces with constant negative curvature, for which the dynamical properties have been studied extensively over the last years.

Definition 3.1 (Anosov flow)**.** Let M be a smooth connected manifold and $\Phi = \{\varphi_t\}$ a smooth flow on M without fixed points[4]. A compact Φ-invariant set $\Lambda \subset M$ is *hyperbolic* if there exist a Φ-invariant splitting $TM|_\Lambda = E^c \oplus E^s \oplus E^u$ and constants $C \geq 1$, $\lambda \in (0,1)$, and $\mu > 1$ such that for all $p \in \Lambda$ we have:

(i) $E^c_p = \mathrm{Span}(\dot\varphi_t(p)|_{t=0})$;

(ii) for all $w \in E^s_p$, $|d_p\varphi_t(w)| \leq C\,\lambda^t\,|w|$ for all $t > 0$;

(iii) for all $w \in E^u_p$, $|d_p\varphi_t(w)| \leq C\,\mu^t\,|w|$ for all $t < 0$.

A smooth flow Φ on M is said to be an *Anosov flow* if M is closed[5] and hyperbolic.

An Anosov flow is an example of a uniformly hyperbolic dynamical system, an important class of systems with many interesting properties, some of which will not be considered in these notes. We refer to [FH19] and references therein for a more detailed discussion of Anosov flows.

That the geodesic flow on a compact hyperbolic surface is an Anosov flow follows from the properties we have stated in Section 2.3,(neg-b) (see also [FH19, Remark 2.2.2] for a discussion using the Lie bracket of vector fields). However the geodesic flow may be Anosov also on closed surfaces with non-positive curvature. The first result was proved in [An69] for manifolds of any dimension, and in [Eb73] by using Jacobi fields (see also the exposition for surfaces in [Ya22]). In these notes we give a sketch of the proof by the *Alekseev cone field criterion.*

Definition 3.2 (Cones)**.** Given linear spaces E, F and a constant $\beta \in (0,1)$, the *β-cone* $C_\beta(E,F)$ is the subset of $E \oplus F$ given by

$$C_\beta(E,F) = \{v + w \,:\, v \in E,\, w \in F,\, |w| < \beta|v|\}\ .$$

Proposition 3.3 ([FH19])**.** *Given a smooth connected manifold M and a smooth flow $\Phi = \{\varphi_t\}$ on M without fixed points, a compact Φ-invariant set $\Lambda \subset M$ is hyperbolic if and only if there exist constants $C \geq 1$, $\beta, \lambda \in (0,1)$, and a decomposition $T_pM = S_p \oplus E^c_p \oplus U_p$ for all $p \in \Lambda$ such that we have:*

(i) $E^c_p = \mathit{Span}(\dot\varphi_t(p)|_{t=0})$;

(ii) $d_p\varphi_t(\overline{C_\beta(U_p, E^c_p \oplus S_p)}) \subset C_\beta(U_{\varphi_t(p)}, E^c_{\varphi_t(p)} \oplus S_{\varphi_t(p)})$ *for all* $t > 0$;

(iii) $d_p\varphi_t(\overline{C_\beta(S_p, E^c_p \oplus U_p)}) \subset C_\beta(S_{\varphi_t(p)}, E^c_{\varphi_t(p)} \oplus U_{\varphi_t(p)})$ *for all* $t < 0$;

(iv) for all $w \in C_\beta(S_p, E^c_p \oplus U_p)$, $|d_p\varphi_t(w)| \leq C\,\lambda^t\,|w|$ *for all* $t > 0$;

(v) for all $w \in C_\beta(U_p, E^c_p \oplus S_p)$, $|d_p\varphi_t(w)| \leq C\,\lambda^{-t}\,|w|$ *for all* $t < 0$.

[4] This condition can be dropped, it is satisfied by geodesic flows.

[5] That is, compact and without boundary.

We just remark that for the “if” part of the proof of the proposition it is enough to consider the construction

$$E^s_p := \bigcap_{t<0} d_{\varphi_{-t}(p)}\varphi_t\big(\overline{C_\beta(S_{\varphi_{-t}(p)}, E^c_{\varphi_{-t}(p)} \oplus U_{\varphi_{-t}(p)})}\big)\,;$$

$$E^u_p := \bigcap_{t>0} d_{\varphi_{-t}(p)}\varphi_t\big(\overline{C_\beta(U_{\varphi_{-t}(p)}, E^c_{\varphi_{-t}(p)} \oplus S_{\varphi_{-t}(p)})}\big)\,.$$

Theorem 3.4 ([Eb73])**.** *Let Σ be a closed connected oriented Riemannian surface with non-positive curvature. If every geodesic in Σ contains a point where the curvature is negative, then the geodesic flow on Σ is Anosov.*

Proof. We sketch the proof in [Ko18] (see also [FH19, Theorem 5.2.8]). The idea is to construct a cone field on $T^1\Sigma$ and verify the cone criterion.

First, it is useful to consider for $(z,v) \in T^1\Sigma$ the isomorphism

$$T_{(z,v)}(T^1\Sigma) \cong \mathcal{J}(z,v) \oplus E^c_{(z,v)}\,,$$

where $E^c_{(z,v)} = \mathrm{Span}(v)$, and $\mathcal{J}(z,v)$ is the two-dimensional vector space of Jacobi fields along the geodesic $\gamma_{(z,v)}$ for which J and $D_{\dot\gamma_{(z,v)}}J$ are orthogonal to $\dot\gamma_{(z,v)}$. By Proposition 1.8, the bundle $\{\mathcal{J}(z,v)\}_{(z,v)}$ is invariant, namely

$$d_{(z,v)}\varphi_t(\mathcal{J}(z,v)) = \mathcal{J}(\varphi_t(z,v))\,, \quad \forall\, t \in \mathbb{R}\,.$$

Then, for $\varepsilon > 0$, one considers the following cones

$$C^{\pm}_0 := \left\{(\xi,\eta) \in \mathbb{R}^2 \,:\, \pm\xi\eta > 0\right\}\,, \quad C^{\pm}_\varepsilon := \left\{(\xi,\eta) \in \mathbb{R}^2 \,:\, \varepsilon\eta \le \pm\xi \le \frac{\eta}{\varepsilon}\right\}\,,$$

and the cone fields which to each $(z,v) \in T^1\Sigma$ associate $C^{\pm}_0, C^{\pm}_\varepsilon \subset \mathcal{J}(z,v)$.

The thesis is a consequence of the following steps.

Step 1. There exists $\kappa > 0$ and $t_0 > 0$ such that for every unit speed geodesic γ we have

$$\int_0^{t_0} K(\gamma(t))\,dt \le -\kappa\,. \tag{3.1}$$

If the statement is false, let (γ_n) be a sequence of unit speed geodesics such that each γ_n is defined on $[0,n]$ and

$$0 \ge \int_0^n K(\gamma_n(t))\,dt \ge -\frac{1}{n}\,.$$

By compactness and a diagonal argument, one finds a subsequence converging uniformly on each $[0,n]$ to a geodesic γ defined on $\mathbb{R}$ for which $\int_{\mathbb{R}} K(\gamma(t))\,dt = 0$. This is a contradiction with the assumption of the theorem.

Step 2. There exists $m > 0$ and $t_1 > 0$ such that, for any unit speed geodesic γ, the solution of the Riccati differential equation $\dot u(t) = -K(\gamma(t)) - u^2(t)$ with $u(0) = 0$ satisfies $u(t_1) \ge m$.

Given a unit speed geodesic γ, let κ and t_0 as in step 1. First of all, by comparison with the differential equation $\dot v = -v^2$, we have that $u(t) \ge 0$ for all $t \ge 0$. Then, choose $t_1 > t_0$ and, for $a > \max\{1, t_1\kappa\}$, let $m = \kappa(1/a - t_1\kappa/a^2)$. Define

$$t^* := \sup\left\{t \in [0,t_1] \,:\, u(t) \ge \frac{\kappa}{a}\right\}\,.$$

If $t^* = 0$ then, by step 1,

$$u(t_1) = u(0) + \int_0^{t_1} \dot u(t)\,dt = \int_0^{t_1} (-K(\gamma(t)) - u^2(t))\,dt \geq \int_0^{t_0} \Big(- K(\gamma(t))\Big)\,dt + \int_0^{t_1} \left(-\frac{\kappa^2}{a^2}\right)\,dt$$

$$\geq \kappa - t_1\frac{\kappa^2}{a^2} > m\,.$$

On the contrary,

$$u(t_1) = u(t^*) + \int_{t^*}^{t_1} \dot u(t)\,dt \geq u(t^*) + \int_{t^*}^{t_1} (-u^2(t))\,dt \geq \frac{\kappa}{a} + \int_0^{t_1} \left(-\frac{\kappa^2}{a^2}\right)\,dt$$

$$\geq \frac{\kappa}{a} - t_1\frac{\kappa^2}{a^2} = m\,.$$

Step 3. Let $(z,v)\in T^1\Sigma$ and $C_0^\pm, C_\varepsilon^\pm$ the cone fields defined above. Let $\rho(t)$ as in Proposition 1.10 which satisfies the Riccati equation (1.1)*. Then, for t_1 found in step 2,*

$$d_{(z,v)}\varphi_{t_1}(C_0^+) \subseteq C_\varepsilon^+ \quad \textit{and} \quad d_{(z,v)}\varphi_{-t_1}(C_0^-) \subseteq C_\varepsilon^-\,.$$

See [Ko18, Lemma 5.7].

Step 4. We can apply Proposition 3.3.

Since φ_t preserves the volume form on $T^1\Sigma$ (see Theorem 4.5) and the surface Σ is compact, the cone fields induce a decomposition of $T_{(z,v)}T^1\Sigma$ satisfying the properties in Proposition 3.3 (see [Ko18, Theorem 3.3 and Lemma 5.8]). ☐

For our aims, the most important property of an Anosov flow is the existence of the *strong stable* and *strong unstable foliations.*

Theorem 3.5 (Stable and Unstable Manifold Theorem (e.g. [FH19]))**.** *Let M be a smooth connected closed manifold and $\Phi = \{\varphi_t\}$ a C^r, $r\geq 1$, Anosov flow on M with constants λ,μ as in Definition 3.1. Then for each $p\in M$ there is a pair of embedded C^r-disks, $W^s_{loc}(p)$ and $W^u_{loc}(p)$, depending continuously on p in the C^1 topology and called the* local strong stable manifold *and the* local strong unstable manifold *of p, respectively, such that*

(i) $T_pW^s_{loc}(p) = E^s_p$, $T_pW^u_{loc}(p) = E^u_p$;

(ii) $\varphi_t(W^s_{loc}(p)) \subseteq W^s_{loc}(\varphi_t(p))$ for all $t>0$ and $\varphi_t(W^u_{loc}(p)) \subseteq W^u_{loc}(\varphi_t(p))$ for all $t<0$;

(iii) for every $\delta >$ there exists $C_\delta > 0$ such that

$$d(\varphi_t(p),\varphi_t(q)) < C_\delta\,(\lambda+\delta)^t\,d(p,q)\,,\quad \forall\, q\in W^s_{loc}(p),\ t>0\,,$$

$$d(\varphi_t(p),\varphi_t(q)) < C_\delta\,(\mu-\delta)^t\,d(p,q)\,,\quad \forall\, q\in W^u_{loc}(p),\ t<0\,;$$

(iv) there exists a continuous family of neighborhoods U_p of $p\in M$ such that

$$W^s_{loc}(p) = \left\{q\in U_p\,:\,\varphi_t(q)\in U_{\varphi_t(p)}\,\forall\, t>0\,,\ d(\varphi_t(p),\varphi_t(q))\to 0 \text{ as } t\to+\infty\right\}\,,$$

$$W^u_{loc}(p) = \left\{q\in U_p\,:\,\varphi_t(q)\in U_{\varphi_t(p)}\,\forall\, t<0\,,\ d(\varphi_t(p),\varphi_t(q))\to 0 \text{ as } t\to-\infty\right\}\,.$$

Then, it is possible to look at the time evolution of the local strong stable and unstable manifolds, to obtain the sets

$$\begin{aligned} W^s(p) &:= \bigcup_{t<0} \varphi_t(W^s_{loc}(\varphi_{-t}(p))) = \{q\in M\,:\,d(\varphi_t(p),\varphi_t(q))\to 0 \,\text{as}\, t\to+\infty\}\,,\\ W^u(p) &:= \bigcup_{t>0} \varphi_t(W^u_{loc}(\varphi_{-t}(p))) = \{q\in M\,:\,d(\varphi_t(p),\varphi_t(q))\to 0 \,\text{as}\, t\to-\infty\}\,, \end{aligned} \tag{3.2}$$

which are smoothly injectively immersed manifolds called the *strong stable* and *strong unstable manifolds*, respectively. Finally, we call *weak stable* and *weak unstable manifolds*, the manifolds

$$W^{cs}(p) := \bigcup_{t\in\mathbb{R}} \varphi_t(W^s(p)) \quad \text{and} \quad W^{cu}(p) := \bigcup_{t\in\mathbb{R}} \varphi_t(W^u(p))\,. \tag{3.3}$$

Example 3.6. For the geodesic flow on the hyperbolic half-plane $\mathbb{H}^2$, the results in (neg-b) can be restated by saying that for all $(z,v) \in T^1\mathbb{H}^2$,

$$W^s(z,v) = W^+(z,v) \quad \text{and} \quad W^u(z,v) = W^-(z,v)\,.$$

Moreover, the linear splitting of $T_{(z,v)}T^1\mathbb{H}^2$ showing hyperbolicity of the geodesic flow does not depend on (z,v).

For an Anosov flow, every single leaf of the strong foliations is a regular embedded manifold in M, whose regularity is given by the regularity of the flow. However, the regularity of the foliations is a different problem and, in general, one should not expect it to be more than continuous[6]. A better result holds for geodesic flows on surfaces.

Theorem 3.7 ([Ho40, HP75])**.** *Let Σ be a closed connected Riemannian surface with negative curvature. Then the geodesic flow admits C^1 strong stable and unstable foliations.*

[6]For example, if the weak foliations are very smooth then the manifold is algebraic (see [Gh87]).

4. Ergodicity

In this section, we begin to study the ergodic properties of the geodesic flow on smooth connected complete orientable Riemannian surfaces Σ with negative curvature. The hyperbolic structure constructed in Section 3 will play a major role.

We start by recalling the basic notions of ergodic theory.

Definition 4.1 (Invariant measure)**.** Let (M, μ) be a probability space and $T : M \to M$ a (Borel-) measurable map. The map T is *measure preserving* (or μ is *T-invariant*) if $\mu(T^{-1}(A)) = \mu(A)$ for all measurable $A \subset M$.

A measurable flow $\Phi = \{\varphi_t\}$ on (M, μ) is *measure preserving* if $\varphi_t : M \to M$ is measure preserving for all $t \in \mathbb{R}$.

A fundamental result for probability preserving maps and flows is the *Birkhoff Theorem.*

Theorem 4.2. *Let $\Phi = \{\varphi_t\}$ be a measurable measure preserving flow on the probability space (M, μ). Then, for all $f \in L^1(M, \mu)$ there exists a Φ-invariant*[7] *$f_\Phi \in L^1(M, \mu)$ such that*

$$f_\Phi(p) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(\varphi_t(p))\, dt = \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(\varphi_{-t}(p))\, dt\,, \quad \textit{for } \mu\textit{-a.e. } p,$$

and $\int_M f\, d\mu = \int_M f_\Phi\, d\mu$.

Definition 4.3 (Ergodic measure)**.** A measurable flow $\Phi = \{\varphi_t\}$ on (M, μ) is said to be *ergodic* with respect to μ (or μ is *ergodic* with respect to Φ) if for any measurable $A \subset M$ with $\varphi_t(A) = A$ for all $t \in \mathbb{R}$ either $\mu(A) = 0$ or $\mu(M \setminus A) = 0$.

It is immediate that a measurable flow $\Phi = \{\varphi_t\}$ is ergodic if a single time-t map φ_t is ergodic. Moreover, ergodicity can be reformulated in various ways.

Proposition 4.4. *For a measurable flow $\Phi = \{\varphi_t\}$ on (M, μ), the following are equivalent:*

(i) Φ is ergodic with respect to μ.
(ii) Any Φ-invariant $f \in L^1(M, \mu)$ is constant μ-a.e.
(iii) Assuming that Φ is measure preserving, if $\nu << \mu$ is a Φ-invariant measure on M then $\mu = \nu$.
(iv) μ is ergodic for the time-t map φ_t for all but countably many $t \in \mathbb{R}$.
(v) Assuming that Φ is measure preserving, for all $f \in L^1(M, \mu)$ the function f_Φ in Theorem 4.2 is μ-a.e. equal to the constant $\int_M f\, d\mu$.

Let's now consider the geodesic flow $\Phi = \{\varphi_t\}$ on $T^1\Sigma$. First, we show that there exists an invariant measure for Φ.

Theorem 4.5. *The geodesic flow $\Phi = \{\varphi_t\}$ on a smooth connected complete orientable Riemannian surface Σ is a* contact flow*. Namely, there exists a 1-form α on $T^1\Sigma$ such that $\alpha \wedge d\alpha$ is non-degenerate, and the geodesic vector field $X := d/dt|_{t=0}\, \varphi_t \in T(T^1\Sigma)$ satisfies: $\alpha(X) = 1$, $d\alpha(X, \xi) = 0$ for all $\xi \in T(T^1\Sigma)$.*

In particular, the geodesic flow preserves the Liouville measure*, or volume form, $\mu = m \times \ell$ on $T^1\Sigma$. Here, m denotes the volume form on Σ, and ℓ the Lebesgue measure on S^1. If Σ is closed, then μ is a probability measure.*

Proof. Here we sketch the main steps of the proof. For all the details we refer to [Pa99, Section 1.3].

First, we define the 1-form α. Let $\pi : T\Sigma \to \Sigma$ be the canonical projection, $\pi(z, v) = z$ for all $(z, v) \in T\Sigma$. Then, define

$$\alpha_{(z,v)}(\xi) :=< d_{(z,v)}\pi(\xi), v >\,, \quad \forall\, \xi \in T_{(z,v)}(T\Sigma)\,. \tag{4.1}$$

[7] A measurable $f : M \to \mathbb{R}$ is said to be Φ-*invariant* if $f \circ \varphi_t = f$ μ-a.e. for all $t \in \mathbb{R}$.

Then, $d\alpha$ is a symplectic form on $T\Sigma$ for which the geodesic flow is the Hamiltonian flow of $H(z,v) = 1/2\,|v|^2$, that is

$$dH(\xi) = -d\alpha(X,\xi)\,, \quad \forall\, \xi \in T(T\Sigma)\,.$$

It follows that, for all $(z,v) \in T^1\Sigma$,

$$\alpha_{(z,v)}(X(z,v)) =< d_{(z,v)}\pi(X(z,v)), v >=< v, v >= 1\,,$$

and

$$d\alpha(X,\xi) = -dH(\xi) = 0\,, \quad \forall\, \xi \in T(T^1\Sigma)\,,$$

since H is constant on $T^1\Sigma$.

It follows that the flow Φ preserves α and $d\alpha$, in particular it preserves $\alpha \wedge d\alpha$, hence the Liouville measure. Indeed,

$$L_X\alpha = i_X(d\alpha) + d(i_X\alpha) = 0\,,$$

since $i_X\alpha = \alpha(X) = 1$ and $i_X(d\alpha)(\xi) = d\alpha(X,\xi) = 0$ for all $\xi \in T(T^1\Sigma)$. □

Example 4.6. The geodesic flow $\Phi = \{\varphi_t\}$ on the flat torus $\mathbb{T}^2$ is not ergodic. For all $(z,v) \in T^1\mathbb{T}^2$, we have $\varphi_t(z,v) = (\gamma_{(z,v)}(t), v)$ for all $t \in \mathbb{R}$. Hence, if $[a,b] \subset S^1$ denotes a finite arc of directions of positive Lebesgue measure, the set $A_{a,b} = \mathbb{T}^2 \times [a,b] \subset T^1\mathbb{T}^2$ is Φ-invariant, and $\mu(A_{a,b}) = m(\mathbb{T}^2)\cdot \ell([a,b]) \neq 0$.

The same problem arises for the geodesic flow on the sphere $\mathbb{S}^2$ with constant positive curvature.

In the following of this section, we prove that the geodesic flow on a closed surface with negative curvature is ergodic with respect to the Liouville measure. First, we prove this result in the case of constant negative curvature using the algebraic framework introduced in Section 2.3. Actually, we prove a more general result.

Theorem 4.7. *Let $\Gamma < SL(2,\mathbb{R})$ be a* lattice*, that is a discrete subgroup such that $\Sigma = \Gamma\backslash\mathbb{H}^2$ has finite volume. Then, the right action of all non-elliptic elements $g \in SL(2,\mathbb{R})$ is measure preserving and ergodic with respect to the Liouville measure μ on $T^1\Sigma$.*

Proof. The Liouville measure μ is invariant by the left action of all elements $g \in SL(2,\mathbb{R})$ since they are isometries. That the right action of g is measure preserving with respect to μ follows from the fact that $SL(2,\mathbb{R})$ is unimodular (see [EW11, Proposition 9.19]).

It remains to prove that the right action of non-elliptic elements of $SL(2,\mathbb{R})$ is ergodic with respect to μ. By Proposition 4.4, it is enough to show that all g-invariant functions in $L^2(T^1\Sigma)$ are constant μ-a.e. We adapt a more general argument from Margulis (see [EW11, Proposition 11.18]).

First, let $g \in SL(2,\mathbb{R})$ be a hyperbolic element. Then, up to conjugacy, we let $g = \lambda_a$ with $a \in \mathbb{R} \setminus \{0,\pm 1\}$ (see Definition 2.5) and $|a| > 1$. The case $|a| < 1$ follows analogously. We now prove the following claim: If $f \in L^2(T^1\Sigma)$ is λ_a-invariant, then it is constant μ-a.e.

By rescaling f, we assume $\|f\|_2 = 1$. Since the action $SL(2,\mathbb{R}) \ni h \mapsto f\circ h \in L^2(T^1\Sigma)$ is unitary and continuous, we obtain that the map

$$SL(2,\mathbb{R}) \ni h \mapsto p(h) :=< f\circ h\,,\, f > \tag{4.2}$$

is continuous. Moreover, for all $n, m \in \mathbb{Z}$,

$$p(\lambda_a^n\, h\, \lambda_a^{-m}) =< f\circ(\lambda_a^n\, h)\,,\, f\circ \lambda_a^m >= p(h) \tag{4.3}$$

since f is λ_a-invariant. Let us now consider the one-dimensional subgroup $\{\tau_b\}_{b\in\mathbb{R}}$. For all τ_b, we have by a direct computation that $\lambda_a^n\,\tau_b\,\lambda_a^{-n} \to Id$ for $n \to -\infty$. Hence, by (4.3) and the continuity of $p(\cdot)$,

$$p(\tau_b) = \lim_{n\to-\infty} p(\lambda_a^n\,\tau_b\,\lambda_a^{-n}) = p(Id) = 1\,.$$

Then, by the Cauchy-Schwarz inequality, $1 = |p(\tau_b)| \le \|f\circ\tau_b\|_2\,\|f\|_2$, and it follows that $f\circ\tau_b = f$ μ-a.e. for all $b \in \mathbb{R}$.

In the same way, one can prove that if $f \in L^2(T^1\Sigma)$ is λ_a-invariant, then it is invariant under the action of $\exp(\delta H_-)$ for all $\delta \in \mathbb{R}$ (see (2.14)). Recalling that $SL(2,\mathbb{R})$ is generated by the subgroups $\{\tau_b\}_b$ and $\{\exp(\delta H_-)\}_\delta$, it follows that if $f \in L^2(T^1\Sigma)$ is λ_a-invariant, it is $SL(2,\mathbb{R})$-invariant. Hence, f is constant μ-a.e. because the action of $SL(2,\mathbb{R})$ on $\mathbb{H}^2$ is transitive.

Let's consider now the case of a parabolic $g \in SL(2,\mathbb{R})$. Up to conjugacy, we let $g = \tau_b$ with $b \in \mathbb{R}$. As before, we prove that: If $f \in L^2(T^1\Sigma)$ is τ_b-invariant, then it is constant μ-a.e.

We follow the same ideas as before. Fixed $b \in \mathbb{R}$, let $\{\varepsilon_k\}$ be a vanishing sequence in $\mathbb{R}^+$ such that $b\varepsilon_k \in \mathbb{R}\setminus\mathbb{Q}$ for all $k \in \mathbb{N}$. Then, classical results in Diophantine approximation of real numbers guarantee the existence of sequences $\{n_k\}$ and $\{m_k\}$ in $\mathbb{Z}$ such that

$$|n_k\, b\varepsilon_k - 1| < \varepsilon_k \quad \text{and} \quad \left| m_k\, b\varepsilon_k + \frac{n_k\, b\varepsilon_k}{1 + n_k\, b\varepsilon_k} \right| < \varepsilon_k\,.$$

Let now consider the elements $h_k = \exp(\varepsilon_k H_-)$. If $f \in L^2(T^1\Sigma)$ is τ_b-invariant, using the function $p(h)$ defined in (4.2), by (4.3) we have

$$p(\tau_b^{n_k}\, h_k\, \tau_b^{m_k}) = p(h_k) \to p(Id) = 1\,, \quad \text{as } k \to +\infty,$$

At the same time,

$$\tau_b^{n_k}\, h_k\, \tau_b^{m_k} = \begin{pmatrix} 1 + n_k\, b\varepsilon_k & (1 + n_k\, b\varepsilon_k)m_k\, b + n_k\, b \\ \varepsilon_k & 1 + m_k\, b\varepsilon_k \end{pmatrix} \to \begin{pmatrix} 2 & r \\ 0 & \frac{1}{2} \end{pmatrix} =: \bar{h}\,, \quad \text{as } k \to +\infty,$$

up to the choice of as subsequence $\{k_j\}$. Therefore, the continuity of the function $p(h)$ implies

$$p(\tau_b^{n_k}\, h_k\, \tau_b^{m_k}) \to p(\bar{h})\,, \quad \text{as } k \to +\infty,$$

and, as before, from $p(\bar{h}) = 1$, we obtain that $f \circ \bar{h} = f$ μ-a.e.

Finally, since $\bar{h}$ is a hyperbolic element in $SL(2,\mathbb{R})$, it is conjugate to some λ_a with $a \in \mathbb{R}\setminus\{0,\pm 1\}$. Hence, we have proved that if $f \in L^2(T^1\Sigma)$ is τ_b-invariant for any $b \in \mathbb{R}$, it is λ_a-invariant for some $a \in \mathbb{R}\setminus\{0,\pm 1\}$. As proved above, this shows that f is constant μ-a.e. and the proof is finished. □

Corollary 4.8. *Let $\Gamma < SL(2,\mathbb{R})$ be a lattice and $\Sigma = \Gamma\backslash\mathbb{H}^2$. Then, the geodesic flow and the horocycle flows on Σ are measure preserving and ergodic with respect to the Liouville measure μ on $T^1\Sigma$.*

We now consider the non-constant curvature case.

Theorem 4.9 ([Ho39, Ho40])**.** *Let Σ be a smooth closed connected orientable Riemannian surface with negative curvature. Then, the geodesic flow is ergodic with respect to the Liouville measure $\mu = m \times \ell$ on $T^1\Sigma$.*

Proof. We follow the argument by Hopf as exposed in [Br95] for manifolds of arbitrary finite dimension. The proof is divided into steps.

Step 1. Let $f : T^1\Sigma \to \mathbb{R}$ be a measurable function invariant for the geodesic flow $\Phi = \{\varphi_t\}$. Then, there exists sets N^s and N^u of null μ-measure, such that

$$f(z',v') = f(z,v) \quad \textit{for all } (z,v),(z',v') \in T^1\Sigma \setminus (N^s \cup N^u) \textit{ and } (z',v') \in W^s(z,v) \cup W^u(z,v).$$

Without loss of generality let f be a non-negative Φ-invariant function. Then, for all $C > 0$, the function $f_C := \min\{f, C\}$ is in $L^1(T^1\Sigma,\mu)$ and Φ-invariant. Hence, there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ of continuous functions on $T^1\Sigma$ such that $\|f_C - g_n\|_1 < 1/n$.

For each $n \in \mathbb{N}$, Theorem 4.2 implies that there exists $(g_n)_\Phi$ such that

$$(g_n)_\Phi(z,v) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T g_n(\varphi_t(z,v))\,dt$$

for μ-a.e. $(z, v) \in T^1\Sigma$. Then, if $(z', v') \in W^s(z, v)$ and $(g_n)_\Phi(z', v')$ exists, $(g_n)_\Phi(z', v') = (g_n)_\Phi(z, v)$ because g_n is uniformly continuous.

Finally, using that the Liouville measure μ is Φ-invariant, we obtain

$$\frac{1}{n} > \int_{T^1\Sigma} |f_C(z,v) - g_n(z,v)| \, d\mu = \int_{T^1\Sigma} |f_C(\varphi_t(z,v)) - g_n(\varphi_t(z,v))| \, d\mu$$
$$= \int_{T^1\Sigma} |f_C(z,v) - g_n(\varphi_t(z,v))| \, d\mu \,,$$

from which

$$\int_{T^1\Sigma} \left| f_C(z,v) - \frac{1}{T} \int_0^T g_n(\varphi_t(z,v)) \, dt \right| d\mu \le \frac{1}{T} \int_0^T \int_{T^1\Sigma} |f_C(z,v) - g_n(\varphi_t(z,v))| \, d\mu \, dt < \frac{1}{n} \,.$$

Passing to the limit for $n \to \infty$, we obtain that $\|f_C - (g_n)_\Phi\|_1 \to 0$.

Therefore, there exists a set N_n^s of null μ-measure, such that for all $(z, v), (z', v') \in T^1\Sigma \setminus N_n^s$ and $(z', v') \in W^s(z, v)$ the value $(g_n)_\Phi(z', v')$ exists and is equal to $(g_n)_\Phi(z, v)$. Hence, $f_C(z', v') = f_C(z, v)$ for all $(z, v), (z', v') \in T^1\Sigma \setminus N_n^s$ and $(z', v') \in W^s(z, v)$ where $N^s = \cup_{n \in \mathbb{N}} N_n^s$.

The same argument works for the unstable manifold.

Step 2. The weak stable and weak unstable manifolds give absolutely continuous foliations of $T^1\Sigma$. That is, for each $(z, v) \in T^1\Sigma$, let U be an open set for which $U \cap W^{cs}(z, v)$ is a union of local stable manifolds, i.e. there exists $T > 0$ for which

$$U \cap W^{cs}(z,v) = \bigcup_{t \in (-T,T)} (W^s_{loc}(\varphi_t(z,v)) \cap U) \,.$$

Then, there is a measurable family of positive measurable functions $h_t : W^s_{loc}(\varphi_t(z, v)) \cap U \to \mathbb{R}$ such that for any measurable set $A \subset U$

$$\mu_{cs}(A \cap W^{cs}(z,v)) = \int_{-T}^{T} \Big(\int_{W^s_{loc}(\varphi_t(z,v)) \cap U} \chi_A(z',v') \, h_t(z',v') \, d\mu_s(z',v') \Big) \, dt \,,$$

where μ_{cs} and μ_s denote the two and one dimensional volumes induced from μ on $W^{cs}(z, v)$ and $W^s(z, v)$, respectively. And the analogous result holds for the weak unstable manifold.

This follows putting together that: the geodesic flow is smooth; the local stable and unstable manifolds are smooth; the strong stable and unstable foliations are C^1 (see Theorem 3.7).

Step 3. Let $f : T^1\Sigma \to \mathbb{R}$ be a measurable function invariant for the geodesic flow $\Phi = \{\varphi_t\}$. Then, f is constant μ-a.e.

From Step 1, a Φ-invariant measurable function f is constant on the strong stable and unstable manifolds up to a set of null μ-measure. Since the strong foliations are C^1, we can repeat Step 2 using the weak and the strong foliations to obtain an absolutely continuous foliation of the manifold, on which the function f is then constant μ-a.e. $\square$

The proof of Theorem 4.9 can be adapted to finite volume surfaces with curvature bounded and bounded away from 0.

5. A detour: flows under a function

A classical method to study the dynamics of a flow is the identification of a *Poincaré section* for the flow and the definition of a *Poincaré map*. The result of this classical procedure is modelling the flow by a *flow under a function*.

Definition 5.1. Given a set M and a map $T : M \to M$, consider a function $\tau : M \to [0, +\infty)$ and define the space

$$M_\tau := \{(p, s) \in M \times \mathbb{R} \,:\, 0 \le s \le \tau(p)\} \,/ \sim$$

with $(p, \tau(p)) \sim (T(p), 0)$. The function τ is called the *roof function of* M_τ and the map $T : M \to M$ is called the *base map*.
The *semi-flow under the function* τ is the "vertical" flow $V = \{V_t\}$ defined on M_τ for all $t \in [0, +\infty)$ as

$$M_\tau \ni (p, s) \mapsto V_t(p, s) = (p, s + t) \in M_\tau \,.$$

If the map T is invertible, the maps V_t can be defined for all $t \in \mathbb{R}$ and define the *flow under the function* τ.

When the map T is the Poincaré map on a section X, then τ is the return time function for the flow.

Proposition 5.2. *Given a measure space (M, ν) and a ν-measure-preserving map $T : M \to M$, the semi-flow $V = \{V_t\}$ under a function on M_τ, with $\tau \in L^1(M, \nu)$, preserves the probability measure $\mu_\tau := (\nu \times \ell)/ \int_M \tau \, d\nu$.*

Proof. It's enough to control the μ_τ-measure of the sets of the form $A \times (s_1, s_2)$ for $A \subset M$ ν-measurable. Then, the results follows by checking that the μ_τ-measure doesn't change when the set goes through the roof of M_τ. $\square$

Proposition 5.3. *Given a probability space (M, ν) and a ν-measure-preserving map $T : M \to M$, the semi-flow $V = \{V_t\}$ under a function on M_τ, with $\tau \in L^1(M, \nu)$, is ergodic with respect to μ_τ if and only if T is ergodic with respect to ν.*

Proof. Let's assume that the base map T is ergodic on (M, ν) and let $f : M_\tau \to \mathbb{R}$ be in $L^2(M_\tau, \mu_\tau)$ and V-invariant, that is $f \circ V_t = f$ μ_τ-a.e. for all $t \in \mathbb{R}$. Then, there exists a set $N \subset M$ of null ν-measure such that

$$g(p) := \frac{1}{\tau(p)} \int_0^{\tau(p)} f(p, s) \, ds$$

is defined for all $p \in M \setminus N$, and $g(p) = f(p, 0)$ and $g(T(p)) = f(T(p), 0)$. Since $f(T(p), 0) = f(p, \tau(p)) = (f \circ V_{\tau(p)})(p, 0)$, it follows $g(p) = g(T(p))$ for ν-a.e. $p \in M$. By the ergodicity of the base map T, it follows that $g(p)$ is constant ν-a.e. Hence $f(p, s)$ is constant μ_τ-a.e.

On the other direction, let's assume that the flow V is ergodic on (M_τ, μ_τ). Given a function $g \in L^2(M, \nu)$ which is T-invariant, we define $f(p, s) = g(p)$ for all $s \in [0, \tau(p))$. Then, $f(p, s + t) = f(p, s)$ for all $t \in [-s, \tau(p) - s)$. For the other t's, we have

$$f(p, s + t) = f(T(p), s + t - \tau(p)) = g(T(p)) = g(p) = f(p, s)$$

for all $t \in [\tau(p) - s, \tau(T(p)) + \tau(p) - s)$, and the argument can be repeated for the other cases to show that f is V-invariant. Hence, $f(p, s)$ is μ_τ-a.e. constant, and the same holds for g for ν-a.e. $p \in M$. $\square$

Exercise 5.4. Consider $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the flow $\Phi = \{\varphi_t\}$ given by

$$[0, 1) \times [0, 1) \ni (x, y) \mapsto \varphi_t(x, y) = (x, y) + tv \in \mathbb{T}^2$$

for a fixed $v = (v_1, v_2) \in \mathbb{R}^2$ with $|v| = 1$, $v_1 \geq 0$, and $v_2 \neq 0$. Consider the section $M = \{0\} \times [0,1) \subset \mathbb{T}^2$, compute the return time function τ on M, and define the Poincaré map T on M. Find the conditions on v under which the Poincaré map is ergodic.

It turns out that flows under a function are an important class to study. For a proof of the next result see also [FH19, Theorem 3.6.2].

Theorem 5.5 ([Am41])**.** *Let Φ be a measure-preserving flow on a Lebesgue space with essentially no fixed points, then Φ is isomorphic to a flow under a function.*

5.1. **The modular surface $SL(2,\mathbb{Z})\backslash\mathbb{H}^2$.** The following material is from [BDI24, Appendix A]. Let us consider the quotient of $\mathbb{H}^2$ by the action of the lattice $SL(2,\mathbb{Z})$, in this section Σ denotes the *modular surface*, that is the quotient $SL(2,\mathbb{Z})\backslash\mathbb{H}^2$, with fundamental domain

$$\mathcal{F} = \left\{ z = x + iy \in \mathbb{H}^2 \,:\, |x| \leq \frac{1}{2},\ |z| \geq 1 \right\},$$

and let $\pi : \mathbb{H}^2 \to \Sigma$ be the standard projection. By Corollary 4.8, the geodesic flow on Σ is measure preserving and ergodic with respect to the Liouville measure μ. Here we construct a Poincaré section and write the geodesic flow as a flow under the return time function. This is similar in spirit to the results in [AF84, Se85] and uses essentially the same Poincaré section as in [Ma12].

Given $n \in \mathbb{Z}$, let $\mathcal{I}_n$ denote the vertical line in $\mathbb{H}^2$ given by

$$\mathcal{I}_n = \left\{ x + iy \in \mathbb{H}^2 \,:\, x = n \right\}$$

and let

$$\mathcal{I}^+ := \mathcal{I}_0 \cap \mathcal{F} = \left\{ z \in \mathbb{H}^2 \,:\, x = 0,\ y \geq 1 \right\} \quad \text{and} \quad \mathcal{I}^- := \mathcal{I}_0 \setminus \mathcal{I}^+ = \left\{ z \in \mathbb{H}^2 \,:\, x = 0,\ 0 < y < 1 \right\}.$$

Our Poincaré section will be the set

$$\mathcal{C} := \left\{ (z,v) \in T^1\Sigma \,:\, z \in \mathcal{I}^+,\ \theta(v) \neq 0, \pi \right\},$$

where θ is as in (2.8). Set moreover $C := \left\{ (z,v) \in T^1\mathbb{H}^2 \,:\, d\pi(z,v) \in \mathcal{C} \right\}$. Using $\psi_S(i) = i$ and $\psi_S(\mathcal{I}^+ \setminus \{i\}) = \mathcal{I}^-$, with S defined as in Definition 2.5, and the fact that the lift $d\psi_S : T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$ satisfies

$$\theta(d\psi_S(z,v)) = \theta(v) + \pi \pmod{2\pi}, \quad \forall\, (z,v) \in T^1\mathbb{H}^2 \text{ with } z \in \mathcal{I}_0,$$

for each $(z,v) \in \mathcal{C}$ there exists $(z',v') \in T^1\mathbb{H}^2$ with $z' \in \mathcal{I}_0$ and $\theta(v') \in (-\pi, 0)$ such that $d\pi(z',v') = (z,v)$. Hence in the following we identify $\mathcal{C}$ with the set of points on $\mathcal{I}_0$ with tangent unit vector v pointing towards the half-plane of points with strictly positive real part, in accordance with

$$\mathcal{C} = d\pi\Big(\left\{ (z,v) \in T^1\mathbb{H}^2 \,:\, z \in \mathcal{I}_0\,,\ \theta(v) \in (-\pi, 0) \right\} \Big). \tag{5.1}$$

Stated otherwise, when we write $(z,v) \in \mathcal{C}$ we think of it as a couple with $z \in \mathcal{I}_0$ and $\theta \in (-\pi, 0)$. Analogously we set

$$C = \left\{ (z,v) \in T^1\mathbb{H}^2 \,:\, \exists\, g \in SL(2,\mathbb{Z}) \text{ such that } \psi_g(z) \in \mathcal{I}_0\,,\ \theta(d\psi_g(z,v)) \in (-\pi, 0) \right\}. \tag{5.2}$$

Subsets of C which play an important role in the rest of the section are points with base on the lines $\mathcal{I}_n$ and on the half-circles

$$\mathcal{J}_n := \left\{ x + iy \in \mathbb{H}^2 \,:\, \left(x - n - \frac{1}{2} \right)^2 + y^2 = \frac{1}{4} \right\}.$$

Note that $\mathcal{I}_n = \psi_{\tau_1}^n(\mathcal{I}_0)$ and $\mathcal{J}_n = \psi_{\tau_1}^n(\psi_l(\mathcal{I}_0))$, where

$$l = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note moreover that for each $n \in \mathbb{Z}$ the lines $\mathcal{I}_n$, $\mathcal{J}_n$ and $\mathcal{I}_{n+1}$ are the sides of the hyperbolic triangle

$$\Delta_n := \left\{ x + iy \in \mathbb{H}^2 \,:\, n \le x \le n+1,\ \left(x - n - \frac{1}{2} \right)^2 + y^2 \ge \frac{1}{4} \right\}$$

with vertices in n, $n+1 \in \mathbb{R}$ and ∞ (each Δ_n contains three copies of the fundamental domain $\mathcal{F}$).

Let $(z,v) \in \mathcal{C}$ and $\gamma_{(z,v)}(t)$ be the corresponding geodesic. Since $\theta(v) \in (-\pi, 0)$ we have $\gamma^+_{(z,v)} \in \mathbb{R}^+$ and $\gamma^-_{(z,v)} \in \mathbb{R}^-$. We let

$$\mathbb{G} := \left\{ (\gamma, \eta) \,:\, \gamma, \eta \in \mathbb{R}^+ \right\}$$

and consider the map

$$\mathcal{G} : \mathcal{C} \to \mathbb{G}, \quad (z,v) \mapsto \mathcal{G}(z,v) = \left(\gamma^+_{(z,v)}, -\gamma^-_{(z,v)} \right).$$

Using the identification of $T^1\mathbb{H}^2$ with $PSL(2,\mathbb{R})$ in (2.11), we have

$$(z,v) = \mathcal{G}^{-1}(\gamma,\eta) \quad \Rightarrow \quad g_{(z,v)} = \begin{pmatrix} \left(\frac{\eta}{\gamma}\right)^{\frac{1}{4}} \frac{\gamma}{\sqrt{\gamma+\eta}} & -\left(\frac{\gamma}{\eta}\right)^{\frac{1}{4}} \frac{\eta}{\sqrt{\gamma+\eta}} \\ \left(\frac{\eta}{\gamma}\right)^{\frac{1}{4}} \frac{1}{\sqrt{\gamma+\eta}} & \left(\frac{\gamma}{\eta}\right)^{\frac{1}{4}} \frac{1}{\sqrt{\gamma+\eta}} \end{pmatrix}. \tag{5.3}$$

We now consider the return to $\mathcal{C}$ starting from a point $(z,v) \in \mathcal{C}$ and along $\gamma_{(z,v)}(t)$ for positive times t. Let $(\gamma,\eta) = \mathcal{G}(z,v)$. Given (z',v') the point in $\mathcal{C}$ of the first return, we can describe the Poincaré map as a map

$$\mathcal{P} : \mathbb{G} \to \mathbb{G}, \quad \mathcal{P}_g(\gamma,\eta) = (\gamma',\eta') := \mathcal{G}(z',v').$$

Our aim is now to describe the map $\mathcal{P}$ on the points of $\mathbb{G}$. The map $\mathcal{P}_g$ depends on the set of C defined in (5.2) on which the first return occurs. The proof of the following result is straightforward.

Lemma 5.6. *For $(z,v) \in \mathcal{C}$, let $(\gamma,\eta) = \mathcal{G}(z,v)$. If the first return to C along $\gamma_{(z,v)}(t)$ for positive t occurs on $\psi_g^{-1}(\mathcal{I}_0)$ for some $g \in SL(2,\mathbb{Z})$, then $(\gamma',\eta') = \mathcal{P}(\gamma,\eta)$ satisfies*

$$\gamma' = \psi_g(\gamma) \quad \text{and} \quad \eta' = -\psi_g(-\eta).$$

There are a couple of different cases to be considered. Given $(z,v) = \mathcal{G}^{-1}(\gamma,\eta)$, if $\gamma > 1$ the first return to C along $\gamma_{(z,v)}(t)$ for positive t occurs on $\mathcal{I}_1 = \psi_{\tau_1}(\mathcal{I}_0)$. On the other hand, if $\gamma < 1$ it occurs on $\mathcal{J}_0 = \psi_l(\mathcal{I}_0)$. Finally, when $\gamma = 1$ there are no returns to C along $\gamma_{(z,v)}(t)$ for positive t. However we can extend the map also to this case, and using Lemma 5.6 we obtain

$$(\gamma',\eta') = \mathcal{P}(\gamma,\eta) = \begin{cases} (\gamma - 1, \eta + 1), & \text{if } \gamma > 1; \\ \left(\frac{\gamma}{1-\gamma}, \frac{\eta}{1+\eta} \right), & \text{if } \gamma < 1; \\ (0, \infty), & \text{if } \gamma = 1. \end{cases} \tag{5.4}$$

We can now construct the flow under a function associated to $(\mathbb{G}, \mathcal{P})$. Let $(z,v) \in \mathcal{C}$ and apply the geodesic flow along $\gamma_{(z,v)}(t)$ for t positive. Letting $(\gamma,\eta) := \mathcal{G}(z,v)$, the first return of the geodesic flow on C occurs at a time $\bar{t}(z,v) > 0$, which can then be written as a function of (γ,η) as $\tau(\gamma,\eta) := \bar{t}(\mathcal{G}^{-1}(\gamma,\eta))$. Let us now compute $\tau(\gamma,\eta)$. When $\gamma > 1$, we have seen that the first return on C along $\gamma_{(z,v)}(t)$ for positive t occurs on $\mathcal{I}_1$. Hence, using the identification between $T^1\mathbb{H}^2$ and $PSL(2,\mathbb{R})$ and Proposition 2.9, we write

$$g_{\gamma_{(z,v)}(t)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \quad \text{where } g_{(z,v)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

from which it follows that $\bar{t}(z,v)$ is the positive solution of

$$\Re\left(\frac{a\,e^{\frac{t}{2}}i+b\,e^{-\frac{t}{2}}}{c\,e^{\frac{t}{2}}i+d\,e^{-\frac{t}{2}}}\right)=1\,.$$

A straightforward computation and (5.3) give

$$\bar{t}(z,v)=\frac{1}{2}\,\log\left(\frac{ac+d^2}{ac-c^2}\right)\quad\text{and}\quad\tau(\gamma,\eta)=\frac{1}{2}\,\log\left(\frac{1+\frac{1}{\eta}}{1-\frac{1}{\gamma}}\right)\ \text{if}\ \gamma>1.$$

When $\gamma<1$, we have seen that the first return on C along $\gamma_{(z,v)}(t)$ for positive t occurs on $\mathcal{J}_0$. Hence, as before it follows that $\bar{t}(z,v)$ is the positive solution of

$$\left|\left(\frac{a\,e^{\frac{t}{2}}i+b\,e^{-\frac{t}{2}}}{c\,e^{\frac{t}{2}}i+d\,e^{-\frac{t}{2}}}\right)-\frac{1}{2}\right|=\frac{1}{2}\,.$$

A straightforward computation and (5.3) give

$$\bar{t}(z,v)=\frac{1}{2}\,\log\left(\frac{b^2-bd}{ac-a^2}\right)\quad\text{and}\quad\tau(\gamma,\eta)=\frac{1}{2}\,\log\left(\frac{1+\eta}{1-\gamma}\right)\ \text{if}\ \gamma<1.$$

Now we can construct the flow under the roof function $\tau(\gamma,\eta)$ with base map $\mathcal{P}$ on $\mathbb{G}$. Given

$$\mathbb{G}_\tau:=\{(\gamma,\eta,s)\,:\,(\gamma,\eta)\in\mathbb{G}\,,\,\gamma\neq 1\,,\,0\leq s\leq\tau(\gamma,\eta)\}\,/\sim\,,$$

with $(\gamma,\eta,\tau(\gamma,\eta))\sim(\mathcal{P}(\gamma,\eta),0)$, we consider the vertical flow $V=\{V_t\}$ on $\mathbb{G}_\tau$ defined by

$$V_t(\gamma,\eta,s)=(\gamma,\eta,s+t)\,.$$

Proposition 5.7. *The following properties hold:*

(i) The map $\mathcal{P}:\mathbb{G}\to\mathbb{G}$ preserves the infinite measure ν which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{G}$ with density $h(\gamma,\eta)=(\gamma+\eta)^{-2}$.

(ii) The roof functions $\tau(\gamma,\eta)$ is in $L^1(\mathbb{G},\nu)$, hence the vertical flow $V=\{V_t\}$ preserves the probability measure $\mu_\tau=(\nu\times\ell)/\int_{\mathbb{G}}\tau\,d\nu$ on $\mathbb{G}_\tau$.

(iii) The vertical flow $V=\{V_t\}$ on $(\mathbb{G}_\tau,\mu_\tau)$ is isomorphic to the geodesic flow $\Phi=\{\varphi_t\}$ on $(T^1\Sigma,\mu)$, where μ is the Liouville measure $d\mu(x,y,\theta)=y^{-2}\,dx\,dy\,d\theta$, with $\theta=\theta(v)$.

6. Decay of correlations

We begin to study the "chaotic" properties of the geodesic flows. The chaos index we consider is *mixing* and its rate. Let's first recall the basic definitions.

Definition 6.1 (Mixing)**.** Let (M, μ) be a probability space and $\Phi = \{\varphi_t\}$ a measurable flow on M which preserves μ. The flow Φ is said to be *mixing* if for any two measurable sets $A, B \subset M$

$$\lim_{t \to +\infty} \mu(A \cap \varphi_{-t}(B)) = \mu(A)\, \mu(B)\,.$$

It is interesting to study the quantity

$$\mu(A \cap \varphi_{-t}(B)) = \int_M \chi_A\, (\chi_B \circ \varphi_t)\, d\mu$$

for more general observables.

Definition 6.2 (Koopman operator)**.** Let (M, μ) be a probability space and $\Phi = \{\varphi_t\}$ a measurable flow on M which preserves μ. The *Koopman operator* U_t associated to Φ for $t \in \mathbb{R}$ is the linear operator defined for all $p \geq 1$ by

$$U_t : L^p(M, \mu) \to L^p(M, \mu)\,, \quad U_t f = f \circ \varphi_t\,.$$

Definition 6.3 (Decay of correlations)**.** Let (M, μ) be a probability space and $\Phi = \{\varphi_t\}$ a measurable flow on M which preserves μ. Given $f, g \in L^2(M, \mu)$, we call *correlations of f and g with respect to the flow* Φ the quantities

$$C_\Phi(f, g, t) := \int_M f\, (U_t g)\, d\mu - \Big(\int_M f\, d\mu \Big) \Big(\int_M g\, d\mu \Big)\,.$$

We say that the flow Φ has *decay of correlations* if for all $f, g \in L^2(M, \mu)$

$$\lim_{t \to +\infty} C_\Phi(f, g, t) = 0\,.$$

Proposition 6.4. *For a measure-preserving flow $\Phi = \{\varphi_t\}$ on the probability space (M, μ), the following are equivalent:*

(i) Φ is mixing.
(ii) Φ has decay of correlations.

The question we want to answer is whether it is possible to obtain an estimate on the rate of the decay of correlations for observables in a functional subspace of L^2. We can interpret this rate of decay as a measure of chaoticity of the system.

The first results on a precise rate of decay of correlations for the geodesic flow on a negatively curved surface were obtained in the case of constant curvature in [Ra87]. The main advantage was clearly the possibility of using the algebraic structure. Here, we recall the proof that the geodesic flow on the hyperbolic surfaces is mixing.

Theorem 6.5. *Let $\Gamma < SL(2, \mathbb{R})$ be a lattice and $\Sigma = \Gamma \backslash \mathbb{H}^2$. Then, the geodesic flow on Σ is mixing with respect to the Liouville measure μ on $T^1\Sigma$.*

Proof. We follow [FH19, Theorem 3.4.32] and prove that the geodesic flow $\Phi = \{\varphi_t\}$ has decay of correlations. First, we remark that the Koopman operator U_t is an isometry on $L^2 = L^2(T^1\Sigma, \mu)$. Then, for all $g \in L^2$, the set $\{U_t g\}_t$ is bounded, hence there exist a sequence $t_k \to +\infty$ and $g_\infty \in L^2$ such that $U_{t_k} g$ converges weakly to g_∞. The proof of the theorem is finished if we show that, for all $g \in L^2$, we have $g_\infty = \int_{T^1\Sigma} g\, d\mu$ and the convergence holds for $U_t g$ as $t \to +\infty$.

Let 's consider a fixed $g \in L^2$ and a sequence $\{t_k\}$ for which $t_k \to +\infty$ and

$$U_{t_k} g \xrightharpoonup[k \to +\infty]{} g_\infty \in L^2\,.$$

Then, on one hand, using Proposition 2.16 and the isometric property of the Koopman operator, we have

$$\|U_{t_k} g \circ u_s^+ - U_{t_k} g\|_2 = \|U_{t_k}(g \circ u_{se^{-t_k}}^+) - U_{t_k} g\|_2 = \|g \circ u_{se^{-t_k}}^+ - g\|_2 \xrightarrow[k\to\infty]{} 0\,,$$

where $\{u_s^+\}$ is the positive horocycle flow. On the other hand, being the Liouville measure u_s^+-invariant by Corollary 4.8, we have

$$U_{t_k} g \circ u_s^+ \xrightharpoonup[k\to+\infty]{} g_\infty \circ u_s^+$$

for all fixed s. It follows that

$$\|g_\infty \circ u_s^+ - g_\infty\|_2 = 0\,.$$

Since the horocycle flow is ergodic with respect to the Liouville measure (see Corollary 4.8), then $g_\infty = \int_{T^1\Sigma} g\, d\mu$.

Finally, the same argument can be repeated for all sequences $\{t_k\}$ for which $t_k \to +\infty$, hence

$$U_t g \xrightharpoonup[t\to+\infty]{} \int_{T^1\Sigma} g\, d\mu$$

and the thesis follows. ☐

The argument in the proof of Theorem 6.5 can be repeated for the geodesic flow on a negatively curved closed surface Σ, by proving that for all $g \in L^2(T^1\Sigma, \mu)$ the weak-accumulation points of $\{U_t g\}$ are μ-a.e. constant along the strong stable and unstable foliations. Then, we obtain the following result, whose proof can be found in [FH19, Theorem 7.1.9].

Theorem 6.6. *Let Σ be a smooth closed connected orientable Riemannian surface with negative curvature. Then, the geodesic flow is mixing with respect to the Liouville measure μ on $T^1\Sigma$.*

We conclude this short discussion about the mixing property, by recalling the *Anosov alternative* for flows under a functions.

Theorem 6.7. *Let Φ be a measure preserving Anosov flow then:*

either *the strong stable and unstable foliations are ergodic and Φ is mixing;*
or *the flow is the flow under a function with constant roof function and base map given by an Anosov diffeomorphism.*

This gives another proof of Theorem 6.6.

Corollary 6.8. *Contact measure preserving Anosov flows are mixing.*

Proof. We show that a contact Anosov flow $\Phi = \{\varphi_t\}$ cannot be given by a flow under a function with constant roof. For simplicity, we consider flows on three dimensional manifolds.

Let α be the 1-form such that the flow vector field $X = d/dt|_{t=0}\varphi_t$ satisfies $\alpha(X) = 1$, $\alpha \wedge d\alpha$ is non-degenerate, and $d\alpha(X, \xi) = 0$ for all vector fields ξ. It follows by the Anosov property that $\ker\alpha = E^s \oplus E^u$. If the flow is a flow under a constant function, then $\ker\alpha$ is tangent to the foliation into level sets of time, hence $\alpha = a(s)ds$ for some smooth function a. But then, $d\alpha = 0$ contrarily to the non-degeneracy assumption on $\alpha \wedge d\alpha$. ☐

6.1. **Exponential decay of correlations.** Finally, we are ready to state and prove the estimate on the decay of correlations for geodesic flows.

Definition 6.9 (Rates of decay of correlations)**.** Let (M, μ) be a probability space and $\Phi = \{\varphi_t\}$ a measurable flow on M which preserves μ. Given a function $r : \mathbb{R} \to (0, +\infty)$ such that $r(t) \to 0$ as $t \to +\infty$ and two Banach spaces $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{B}', \|\cdot\|_{\mathcal{B}'})$ continuously embedded in $(L^2(M, \mu), \|\cdot\|_2)$, we say that the flow Φ has *decay of correlations with rate* $r(t)$ *on* $\mathcal{B}$ *and* $\mathcal{B}'$ if there exists a constant $C > 0$ such that for all $f \in \mathcal{B}$ and $g \in \mathcal{B}'$

$$|C_\Phi(f, g, t)| \le C \, \|f\|_{\mathcal{B}} \, \|g\|_{\mathcal{B}'} \, r(t) \,. \tag{6.1}$$

In particular, we say that flow Φ has *exponential decay of correlations on* $\mathcal{B}$ *and* $\mathcal{B}'$ if (6.1) holds with $r(t) = \vartheta^t$ for some $\vartheta \in (0, 1)$.

Theorem 6.10 ([Do98, Li04]). *Let* Σ *be a smooth closed connected orientable Riemannian surface with negative curvature. Then, the geodesic flow has exponential decay of correlations on couples of* C^1 *observables.*

The proof uses the *transfer operator* $\mathcal{L}_t$ associated to the geodesic flow $\Phi = \{\varphi_t\}$ on Σ, defined as the operator dual to the Koopman operator with respect to the Liouville measure μ, that is

$$\int_{T^1\Sigma} f \, (U_t g) \, d\mu = \int_{T^1\Sigma} (\mathcal{L}_t f) \, g \, d\mu \,, \quad \forall \, f, g \in L^2(T^1\Sigma, \mu) \,,$$

from which, since μ is Φ-invariant,

$$\mathcal{L}_t f = f \circ \varphi_{-t} \,. \tag{6.2}$$

It follows from (6.2) that the constant functions are fixed by the transfer operator, and this is equivalent to the Liouville measure being Φ-invariant. Hence, we restrict ourselves to the case of functions $f \in L^2(T^1\Sigma, \mu)$ with $\int_{T^1\Sigma} f d\mu = 0$. This can be done without loss of generality. Let $f = f_0 + c$ for $c \in \mathbb{R}$ and $\int_{T^1\Sigma} f_0 d\mu = 0$. Then, for all $g \in L^2(T^1\Sigma, \mu)$,

$$C_\Phi(f, g, t) = \int_{T^1\Sigma} f_0 \, (U_t g) \, d\mu + c \int_{T^1\Sigma} U_t g \, d\mu - c \int_{T^1\Sigma} g \, d\mu = C_\Phi(f_0, g, t) \,,$$

where we have used that μ is Φ-invariant. Then, we follow [Li04] in proving Theorem 6.10 as a corollary of the following result for the transfer operator $\mathcal{L}_t$. See Appendix A for an example of the application of the transfer operator method to discrete dynamical systems.

We need to define a Banach space $\mathcal{B}$ of functions on which we can control the action of $\mathcal{L}_t$. We begin by introducing a couple of *dynamical distances*. The geodesic flow on Σ is Anosov, hence we find constants $\lambda \in (0, 1)$ and $\mu > 1$ as in Definition 3.1.

Definition 6.11. Let $\sigma \in (0, +\infty)$ such that $e^\sigma \in (1, \min\{\lambda^{-1}, \mu\})$ and let

$$d_s((z, v), (z', v')) := \int_0^{+\infty} e^{\sigma t} \, d(\varphi_t(z, v), \varphi_t(z', v')) \, dt \,,$$

$$d_u((z, v), (z', v')) := \int_{-\infty}^0 e^{-\sigma t} \, d(\varphi_t(z, v), \varphi_t(z', v')) \, dt \,,$$

where $d(\cdot, \cdot)$ is the Riemannian metric on $T^1\Sigma$.

The two dynamical (pseudo-)distances d_s, d_u are finite and smooth only on points which belong to the same strong stable and unstable manifold, respectively. Moreover,

$$\begin{aligned} &d_s(\varphi_t(z, v), \varphi_t(z', v')) \le e^{-\sigma t} \, d_s((z, v), (z', v')) \,, \quad \forall \, t > 0 \,, \\ &d_u(\varphi_t(z, v), \varphi_t(z', v')) \le e^{\sigma t} \, d_u((z, v), (z', v')) \,, \quad \forall \, t < 0 \,. \end{aligned} \tag{6.3}$$

Then, we consider the spaces of Hölder functions with respect to these distances. For a fixed $\delta > 0$, let for $\beta > 0$ and $h \in C^1(T^1\Sigma)$

$$H_{s,\beta}(h) := \sup_{d_s((z,v),(z',v'))<\delta} \frac{|h(z, v) - h(z', v')|}{d_s((z, v), (z', v'))^\beta} \,, \quad |h|_{s,\beta} := \|h\|_\infty + H_{s,\beta}(h) \,. \tag{6.4}$$

Then, we denote by $C_s^\beta \subset C^0(T^1\Sigma)$ the closure of $C^1(T^1\Sigma)$ with respect to $|\cdot|_{s,\beta}$. The analogous definitions using d_u lead to quantities $H_{u,\beta}(h)$ and $|h|_{u,\beta}$, and to the space C_u^β.

We are now ready to introduce an auxiliary space $\mathcal{B}$ of Banach functions useful in the proof of the result.

Definition 6.12. Let $f \in C^1(T^1\Sigma)$ and $\beta \in (0,1)$. Let's consider the following quantities

$$|f|_w := \sup_{h \in C_s^1,\, |h|_{s,1} \leq 1} \int_{T^1\Sigma} f\, h\, d\mu\,,$$

$$\|f\|_{\mathcal{B}} := H_{u,\beta}(f) + \sup_{h \in C_s^\beta,\, |h|_{s,\beta} \leq 1} \int_{T^1\Sigma} f\, h\, d\mu\,.$$

Then, we define the space $\mathcal{B}$ to be the closure of $C^1(T^1\Sigma)$ with respect to $\|\cdot\|_{\mathcal{B}}$, and the space $\mathcal{B}_w$ to be the closure of $C^1(T^1\Sigma)$ with respect to $|\cdot|_w$.

Theorem 6.10 follows from:

Theorem 6.13 ([Li04])**.** *The transfer operators $\mathcal{L}_t$ defined in* (6.2) *form a strongly continuous group on $\mathcal{B}$. In addition, there exist constants $C > 0$ and $\theta \in (0,1)$ such that for all $f \in C^1(T^1\Sigma)$ with $\int_{T^1\Sigma} f d\mu = 0$ one has*

$$\|\mathcal{L}_t f\|_{\mathcal{B}} \leq C\, \|f\|_{C^1}\, \theta^t\,, \quad \forall\, t > 0\,. \tag{6.5}$$

Proof of Theorem 6.10. For all $f \in C^1$ with $\int_{T^1\Sigma} f d\mu = 0$ and $g \in C^1 \subset C_s^\beta$, with $\beta \in (0,1)$, we can apply Theorem 6.13 to obtain constants $C > 0$ and $\theta \in (0,1)$, independent on f and g, such that for all $t > 0$,

$$|C_\Phi(f,g,t)| = \Big| \int_{T^1\Sigma} (\mathcal{L}_t f)\, g\, d\mu \Big| \leq \|\mathcal{L}_t f\|_{\mathcal{B}}\, |g|_{s,\beta} \leq C\, \|f\|_{C^1}\, \|g\|_{C^1}\, \theta^t\,,$$

where we have used that, by definition, $\mathcal{B}$ is contained in the dual of C_s^β. □

We now give a sketch of the main steps of the proof of Theorem 6.13. For all the details, we refer to [Li04].

Proof of Theorem 6.13. The idea is to overcome the problems due to the existence of a central direction for the action of the time-t maps of the flow $\Phi = \{\varphi_t\}$ in the tangent space to $T^1\Sigma$, by using the spectral properties of the infinitesimal generator of the group $\mathcal{L}_t$. First, one needs to prove the following lemma. We fix a $\beta \in (0,1)$ as in the statement of Theorem 6.10.

Lemma 6.14. *The operators $\mathcal{L}_t$ extend to a group of bounded operators on $\mathcal{B}$ and $\mathcal{B}_w$, and they form a strongly continuous group. In addition, for all $f \in \mathcal{B}_w$*

$$|\mathcal{L}_t f|_w \leq |f|_w\,,$$

and for each $\beta' \in (0,\beta)$ there exists a constant $C_1 > 0$ such that for all $f \in \mathcal{B}$

$$\|\mathcal{L}_t f\|_{\mathcal{B}} \leq \|f\|_{\mathcal{B}}\,, \quad \|\mathcal{L}_t f\|_{\mathcal{B}} \leq 3e^{-\sigma\beta' t}\, \|f\|_{\mathcal{B}} + C_1\, |f|_w\,.$$

Then, the operators $\mathcal{L}_t$ are equi-continuous on $\mathcal{B}$ with spectrum $spec(\mathcal{L}_t|_{\mathcal{B}}) \subseteq B(0,1) = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$. In addition, one can consider the *infinitesimal generator*

$$L(f) := \lim_{t \to 0} \frac{\mathcal{L}_t f - f}{t}$$

which has domain $D(L)$ containing the smooth functions, on which it coincides with the vector field defining the flow. By Lemma 6.14, the spectrum of L, $spec(L)$, is contained in the half plane

$\{\Re(\zeta) \leq 0\}$, and 0 is an eigenvalue corresponding to constant functions in $D(L)$. Then, we can consider the *resolvent*

$$\{\Re(\zeta) > 0\} \ni \zeta \mapsto R(\zeta)f := (\zeta Id - L)^{-1} f = \int_0^{+\infty} e^{-\zeta t}\, \mathcal{L}_t f\, dt\,. \tag{6.6}$$

Lemma 6.15. *For each $\zeta \in \{\Re(\zeta) > 0\}$, $R(\zeta)$ is compact as an operator from $\mathcal{B}$ to $\mathcal{B}_w$. In addition, for $f \in D(L^2)$ it holds*

$$R(\zeta)f - \zeta^{-1} f + \zeta^{-2} L(f) = \zeta^{-2} R(\zeta) L^2(f)\,, \tag{6.7}$$

and for $f \in D(L^2) \cap C^0$, using $\zeta = \rho + i\eta$, $\rho, \eta \in \mathbb{R}$, one can write

$$\mathcal{L}_t f = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(\rho+i\eta)t}\, R(\rho + i\eta) f\, d\eta\,, \quad \text{with } \rho = \Re(\zeta) > 0 \text{ fixed.} \tag{6.8}$$

To obtain the estimate (6.5), we need to shift the contour of integration in (6.8) to the half plane $\{\Re(\zeta) < 0\}$. And, to this aim, we need a better control of the spectrum of L.

Lemma 6.16. *For each $\zeta \in \mathbb{C}$ with $\rho = \Re(\zeta) > 0$, and for all $f \in C^1(T^1\Sigma)$, it holds*

$$\|R(\zeta)\|_{\mathcal{B}} \leq \rho^{-1}\,, \quad \|R(\zeta)^j f\|_{\mathcal{B}} \leq \frac{3}{(\rho + \sigma\beta')^j}\, \|f\|_{\mathcal{B}} + C_1\, \rho^{-j}\, |f|_w\,,$$

where σ is as in Definition 6.11, and β' and C_1 are as in Lemma 6.14.

In addition, there exist $\bar\rho > 1$, $\bar\eta > 0$, and $\nu \in (0,1)$, such that for all $\zeta = \rho + i\eta$ with $\rho \in [\bar\rho^{-1}, \bar\rho]$ and $|\eta| \geq \bar\eta$, the spectral radius of $R(\zeta)|_{\mathcal{B}}$ is bounded by $\nu\, \rho^{-1}$.

Proof. The first inequality follows from (6.6) and Lemma 6.14, since

$$\|R(\zeta)f\|_{\mathcal{B}} \leq \int_0^{+\infty} e^{-\rho t}\, \|\mathcal{L}_t f\|_{\mathcal{B}}\, dt \leq \rho^{-1}\, \|f\|_{\mathcal{B}}\,, \quad \forall\, f \in C^1\,.$$

For the second inequality, we use the formula

$$R(\zeta)^j f = \frac{1}{(j-1)!} \int_0^{+\infty} t^{j-1}\, e^{-\zeta t}\, \mathcal{L}_t f\, dt\,, \quad \forall\, j \in \mathbb{N}\,.$$

Again from Lemma 6.14, it follows,

$$\begin{aligned} \|R(\zeta)^j f\|_{\mathcal{B}} &\leq \frac{1}{(j-1)!} \int_0^{+\infty} t^{j-1}\, e^{-\rho t}\, \|\mathcal{L}_t f\|_{\mathcal{B}}\, dt \leq \\ &\leq \frac{1}{(j-1)!} \int_0^{+\infty} t^{j-1}\, e^{-\rho t} \left(3e^{-\sigma\beta' t}\, \|f\|_{\mathcal{B}} + C_1\, |f|_w\right) dt = \\ &= \frac{3}{(\rho + \sigma\beta')^j}\, \|f\|_{\mathcal{B}} + C_1\, \rho^{-j}\, |f|_w\,. \end{aligned}$$

The proof of the last statement can be found in [Li04, Sections 5-6]. $\square$

Applying Theorem A.1, the inequalities in Lemma 6.16 show that, for all $\zeta = \rho + i\eta$ with $\rho > 0$, we have $essspec(R(\zeta)|_{\mathcal{B}}) \subseteq B(0, (\rho + \sigma\beta')^{-1})$. Since, by (6.6), $L = \zeta - R(\zeta)^{-1}$ for all $\zeta \in \{\Re(\zeta) > 0\}$, setting $A_\zeta(w) := \zeta - w^{-1}$, we find that

$$essspec(L|_{\mathcal{B}}) \subseteq \bigcap_{\zeta = \rho + i\eta, \rho > 0} A_\zeta\Big(B(0, (\rho + \sigma\beta')^{-1})\Big) = \big\{\ell \in \mathbb{C} \,:\, \Re(\ell) \leq -\sigma\beta'\big\}\,.$$

Actually, more can be obtained by using the last statement in Lemma 6.16. Namely, there exists $r_0 > 0$ such that

$$spec(L|_{\mathcal{B}}) \cap \{\ell \in \mathbb{C} \,:\, \Re(\ell) > -r_0\} = \{0\}\,. \tag{6.9}$$

We need the last estimate on the resolvent.

Lemma 6.17. *There exists $r \in (0, r_0)$ for which there exists $C_2 > 0$ such that for all $\rho \in [-r, 0]$ and $\eta \in \mathbb{R}$, we have*

$$\|R(\rho + i\eta)\|_{\mathcal{B}} \le C_2\,(1 + \sqrt{|\eta|})\,.$$

Now, we are ready to obtain the thesis of the theorem for $f \in D(L^2) \cap C^0$. By (6.9), we can move the line of integration in (6.8) to the left of $\{\Re(\zeta) = 0\}$, and write

$$\mathcal{L}_t f = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(-r+i\eta)t}\, R(-r+i\eta) f\, d\eta = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(-r+i\eta)t}\left(R(-r+i\eta) - (-r+i\eta)^{-1}\right) f\, d\eta\,,$$

where we have used the Residue Theorem to add the last term which has vanishing integral. By (6.7) and Lemma 6.17, it follows

$$\begin{aligned}\|\mathcal{L}_t f\|_{\mathcal{B}} &\le \frac{1}{2\pi}\, e^{-rt}\int_{-\infty}^{+\infty} \frac{1}{r^2+\eta^2}\left(\|L(f)\|_{\mathcal{B}} + \|R(r+i\eta)\|_{\mathcal{B}}\,\|L^2(f)\|_{\mathcal{B}}\right) d\eta \le \\ &\le \frac{1}{2\pi}\, e^{-rt}\int_{-\infty}^{+\infty} \frac{1}{r^2+\eta^2}\left(\|L(f)\|_{\mathcal{B}} + C_2\,(1+\sqrt{|\eta|})\,\|L^2(f)\|_{\mathcal{B}}\right) d\eta \le \\ &\le C_3\left(\|L(f)\|_{\mathcal{B}} + \|L^2(f)\|_{\mathcal{B}}\right) e^{-rt}\,.\end{aligned}$$

Finally, we consider any $f \in C^1$, with $\int_{T^1\Sigma} f\, d\mu = 0$, and prove the statement. Let $\xi : \mathbb{R} \to [0, +\infty)$ be a fixed C^∞ function with support in $(0, 1)$ and $\int_{\mathbb{R}} \xi(t)dt = 1$. Then, for each $\varepsilon > 0$ let $\xi_\varepsilon(t) := \varepsilon^{-1}\,\xi(\varepsilon^{-1}t)$, and set

$$f_\varepsilon := \int_0^\varepsilon \xi_\varepsilon(t)\,(\mathcal{L}_t f)\, dt\,.$$

Then, $f_\varepsilon \in D(L^n) \cap C^1$ for all $n \in \mathbb{N}$, and

$$\|f_\varepsilon - f\|_{\mathcal{B}} \le \int_0^\varepsilon \xi_\varepsilon(t)\,\|\mathcal{L}_t f - f\|_{C^\beta}\, dt \le \varepsilon^\beta\, \|f\|_{C^1} \sup_{t\in[0,1]} |\varphi_{-t}|_{C^1}\,.$$

Moreover,

$$\|L^2 f_\varepsilon\|_{\mathcal{B}} \le \int_0^\varepsilon |\xi_\varepsilon''(t)|\,\|\mathcal{L}_t f\|_{\mathcal{B}}\, dt \le \varepsilon^{-2}\,\|\xi''\|_1\,\|f\|_{C^1}\,.$$

Hence, for all $t > 0$ and all $\varepsilon > 0$, we have

$$\|\mathcal{L}_t f\|_{\mathcal{B}} \le \|\mathcal{L}_t f_\varepsilon\|_{\mathcal{B}} + \|f_\varepsilon - f\|_{\mathcal{B}} \le \left(C_4\,\varepsilon^{-2}\,e^{-rt} + C_5\,\varepsilon^\beta\right)\|f\|_{C^1}\,.$$

Choosing $\varepsilon = e^{-r(\beta+2)^{-1}t}$, we find that for all $f \in C^1$, with $\int_{T^1\Sigma} f\, d\mu = 0$, and all $t > 0$

$$\|\mathcal{L}_t f\|_{\mathcal{B}} \le C\|f\|_{C^1}\,\vartheta^t$$

with $\vartheta = e^{-r\beta(\beta+2)^{-1}}$. □

7. Zeta functions counting periodic orbits

Given a set M and a map $T : M \to M$ for which $Fix(T^n)$, the set of fixed points of the iterate T^n, is finite for all $n \in \mathbb{N}$, one can build the *Zeta function*

$$Z^T(w) := \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \#Fix(T^n)\,, \tag{7.1}$$

a complex function to control the exponential rate of increase of $\#Fix(T^n)$ by the convergence radius of $Z(w)$. This idea was introduced in [AM65], where it was proved that a C^1-dense set of diffeomorphisms of a smooth compact manifold satisfies

$$h(T) := \limsup_{n\to+\infty} \frac{1}{n} \log\left(\#Fix(T^n)\right) < +\infty\,,$$

hence $Z(w)$ has a positive radius of convergence.

We now show the properties of the Zeta function for a simple example, a subshift of finite type. Let $\mathcal{A}$ be a finite alphabet of N elements, for example $\mathcal{A} = \{1, \dots, N\}$, and let $\Omega = \mathcal{A}^{\mathbb{Z}}$ denote the set of bi-infinite words with symbols from $\mathcal{A}$, that is

$$\Omega := \{\omega = (\omega_i)_{i\in\mathbb{Z}} \,:\, \omega_i \in \mathcal{A},\, \forall\, i \in \mathbb{Z}\}\,.$$

We then consider a $N \times N$ matrix M with coefficients $m_{hk} \in \{0, 1\}$ for $h, k \in \mathcal{A}$, which we call the *transition matrix*, and restrict our attention to the infinite words in $\Omega_M \subseteq \Omega$ which are allowed by M, that is

$$\Omega_M := \left\{\omega \in \Omega \,:\, m_{\omega_i\,\omega_{i+1}} = 1,\, \forall\, i \in \mathbb{Z}\right\}\,.$$

The *subshift of finite type* is the dynamical system defined by the discrete-time action on Ω_M of the *shift map*

$$\sigma : \Omega_M \to \Omega_M\,, \qquad (\sigma(\omega))_i = \omega_{i+1} \quad \forall\, i \in \mathbb{Z}\,. \tag{7.2}$$

When endowed with the product topology over the discrete set $\mathcal{A}$, Ω_m is a compact space and the shift map is continuous.

We now state two preliminary results. The first is a corollary of the well-known *Perron-Frobenius Theorem* on matrices with non-negative coefficients.

Definition 7.1. A $N \times N$ matrix M with non-negative coefficients is called *primitive* if there exists $m \in \mathbb{N}$ for which all entries of M^m are positive.

Lemma 7.2. *If M is primitive, then M has a unique real positive maximal eigenvalue λ_M.*

Lemma 7.3. *For the fixed points of the iterates of the shift map, it is true that*

$$\#Fix(\sigma^n) = trace(M^n)\,.$$

Hence, if M is primitive,

$$h(\sigma) = \lim_{n\to+\infty} \frac{1}{n} \log\left(\#Fix(\sigma^n)\right) = \log \lambda_M\,,$$

where λ_M is the maximal eigenvalue of M.

Lemma 7.3 implies that the Zeta function $Z^\sigma(w)$ for the subshift of finite type on Ω_M has radius of convergence λ_M^{-1} and it defines an analytic function on $\{|w| < \lambda_M^{-1}\}$. However, it is possible to obtain a meromorphic continuation for $Z^\sigma(w)$ to $\mathbb{C}$. For $|w| < \lambda_M^{-1}$, we can write

$$
\begin{aligned}
Z^\sigma(w) &= \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \#Fix(\sigma^n) = \exp \sum_{n=1}^{+\infty} \frac{w^n}{n}\, trace(M^n) = \exp \sum_{n=1}^{+\infty} \frac{1}{n}\, trace((wM)^n) \\
&= \exp\, trace\Big(\sum_{n=1}^{+\infty} \frac{1}{n} (wM)^n \Big) = \exp\, trace\Big(-\log(Id - wM) \Big) \\
&= \det \exp\, \Big(-\log(Id - wM) \Big) = \frac{1}{\det(Id - wM)} \,.
\end{aligned} \tag{7.3}
$$

This is the required continuation, which shows that $Z^\sigma(w)$ is a meromorphic function on $\mathbb{C}$ with poles of finite order at the values w_j for which w_j^{-1} is an eigenvalue of M. Hence, under the assumptions of Lemma 7.2, the smallest pole is a simple one at λ_M^{-1}.

This is the property we would like to obtain also for a general dynamical system by using a suitable transfer operator instead of M. This would give us information on the growth of the number of periodic points, for which in general we don't have an explicit relation like that in Lemma 7.3. Before coming back to the geodesic flow on negatively curved surfaces, let's make a last remark on the Zeta function (7.1).

Let's write $w = e^{-\alpha h}$ with $\alpha \in \mathbb{C}$ and $h = h(\sigma)$. Indicating with "ppo" the set of primitive periodic orbits and by $\ell(\cdot)$ the period of such an orbit, we obtain[8] with some abuse of notation

$$
\begin{aligned}
Z^\sigma(\alpha) &= \exp \sum_{n=1}^{+\infty} \frac{1}{n} e^{-\alpha n h} \#Fix(\sigma^n) = \exp \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{\omega \in Fix(\sigma^n)} e^{-\alpha n h} \\
&= \exp \sum_{\gamma\, \in\, \mathrm{ppo}} \sum_{m=1}^{+\infty} \frac{1}{m} e^{-\alpha m \ell(\gamma) h} = \exp \sum_{\gamma\, \in\, \mathrm{ppo}} -\log\Big(1 - e^{-\alpha \ell(\gamma) h}\Big) \\
&= \prod_{\gamma\, \in\, \mathrm{ppo}} \Big(1 - e^{-\alpha \ell(\gamma) h}\Big)^{-1},
\end{aligned} \tag{7.4}
$$

which is convergent for $\Re(\alpha) > 1$.

The last expression has the advantage of being well defined also for continuous-time dynamical systems, under the assumption that primitive periodic orbits are at most a countable infinity.

Let $\Phi = \{\varphi_t\}$ be the geodesic flow on a smooth closed connected orientable negatively curved Riemannian surface Σ. Given a primitive periodic orbit (ppo) γ for the flow, and denoting by $\ell(\gamma)$ its length or period, we consider the counting function

$$
(0, +\infty) \ni T \mapsto \pi(T) := \#\, \{\gamma \in ppo\, :\, \ell(\gamma) \le T\}\,. \tag{7.5}
$$

By Theorems 3.4 and 6.6, the flow Φ is Anosov and mixing with respect to the volume form on $T^1\Sigma$. As a corollary we then obtain

Proposition 7.4 ([Bo72])**.** *Let Σ be a smooth closed connected orientable Riemannian surface with negative curvature. Then, the geodesic flow has countably many primitive periodic orbits and,*

$$
h(\Phi) := \lim_{T \to +\infty} \frac{1}{T} \log \pi(T)
$$

exists, is finite, and coincides with the topological entropy of the flow.

The following result gives a more precise estimate of the asymptotic behavior of $\pi(T)$.

[8] This is the analogous computation of that giving the Euler product expression for the Riemann Zeta function.

Theorem 7.5 ([PP83, PS88]). *Let $\Phi = \{\varphi_t\}$ be the geodesic flow on a smooth closed connected orientable negatively cruved Riemannian surface Σ. Then, letting $h = h(\Phi)$, there exists $c < h$ such that*

$$\pi(T) = \int_2^{e^{hT}} \frac{1}{\log \tau}\, d\tau + O(e^{cT})\,.$$

The idea of the proof is to obtain "good" continuation properties for the Zeta function of the flow Φ, and to do that one can use a flow under a function with base map a subshift of finite type. The properties of the zeta functions are obtained by relating them to generalised complex transfer operators associated to the base map. This procedure is part of the *thermodynamic formalism* approach to dynamical systems.

In these notes we give the proof of the simpler result

$$\pi(T) \sim \frac{e^{hT}}{hT}\,. \tag{7.6}$$

For the proof of Theorem 7.5 we refer the reader to [PS88] and the survey in [Ma04].

7.1. **Thermodynamic formalism for subshifts of finite type.** Let $\mathcal{A}$ be a finite alphabet with N symbols, M a $N \times N$ transition matrix, and $\Omega_M^+ := (\mathcal{A}^{\mathbb{N}_0})_M$ the set of one-sided infinite sequences with symbols from $\mathcal{A}$ and admissible with respect to M. Let the shift map σ act on Ω_M^+ as in (7.2). This defines a *one-sided subshift of finite type.*

For $\vartheta \in (0,1)$, consider the following metric on Ω_M^+. For $\omega, \tilde\omega \in \Omega_M^+$, let

$$N(\omega, \tilde\omega) := \inf \{ i \in \mathbb{N}_0 \,:\, \omega_i \neq \tilde\omega_i \}\ ,$$

with $N(\omega,\tilde\omega) = +\infty$ if $\omega_i = \tilde\omega_i$ for all $i \in \mathbb{N}_0$, and define

$$d_\vartheta : \Omega_M^+ \times \Omega_M^+ \to [0, +\infty)\,, \quad d_\vartheta(\omega, \tilde\omega) = \vartheta^{N(\omega,\tilde\omega)}\,.$$

The metric d_ϑ induces on Ω_M^+ the product topology, hence $(\Omega_M^+, d_\vartheta)$ is a compact metric space.

Let $\mathcal{B}_\vartheta$ be the space of Lipschitz functions on Ω_M^+ endowed with the norm

$$\|u\|_\vartheta := \|u\|_\infty + |u|_\vartheta\,, \quad |u|_\vartheta := \sup_{\omega \neq \tilde\omega} \frac{|u(\omega) - u(\tilde\omega)|}{d_\vartheta(\omega,\tilde\omega)}\,.$$

Fixed $u \in \mathcal{B}_\vartheta$, we define the *Ruelle transfer operator* $\mathcal{L}_u$ to be the linear operator

$$\mathcal{L}_u : \mathcal{B}_\vartheta \to \mathcal{B}_\vartheta\,, \quad (\mathcal{L}_u f)(\omega) = \sum_{\sigma(\tilde\omega) = \omega} e^{u(\tilde\omega)}\, f(\tilde\omega)\,. \tag{7.7}$$

The function u is called the *potential* of $\mathcal{L}_u$. By varying the choice of the potential, one may reduce $\mathcal{L}_u$ to the transfer operator defined in the previous section.

We now recall classical results for the Ruelle transfer operators. A classical reference is [PP90]. First, it is easy to see that $\mathcal{L}_u$ is bounded. If u is real and $\mathcal{L}_u 1 = 1$, we say that u is *normalised.*

Definition 7.6. Two functions $u, v \in \mathcal{B}_\vartheta$ are said to be *cohomologous*, and we write $v \sim u$, if there exists a continuous function g such that $v = u + g \circ \sigma - g$.

Lemma 7.7. *If $u \in \mathcal{B}_\vartheta$ is real valued and $\mathcal{L}_u$ has a real positive eigenfunction f_β with eigenvalue $\beta \in (0, +\infty)$, then $v = u - \log \beta - \log f_\beta \circ \sigma + \log f_\beta \sim u$ is normalised.*

We now state the main result for the transfer operators $\mathcal{L}_u$ associated to a subshift of finite type. The proof of the parts (a) and (c), known as the *Ruelle-Perron-Frobenius Theorem*, follows by arguing as in the proof of Theorem A.2. For the other parts we refer the reader to [PP90].

Theorem 7.8 ([PP90]). *Let M be a primitive transition matrix. Given a potential $u \in \mathcal{B}_\vartheta$, we have:*

(a) *If u is real valued and normalised, then:*
  (i) *$\mathcal{L}_u$ has 1 as simple maximal eigenvalue with eigenfunction $1 \in \mathcal{B}_\vartheta$;*
  (ii) *$\mathcal{L}_u$ has a spectral gap;*
  (iii) *There exists a unique probability measure m such that $\mathcal{L}_u^* m = m$, where $\mathcal{L}_u^*$ is the dual operator, and m is σ-invariant;*
  (iv) *For all $f \in C^0(\Omega_M^+)$, it holds $\mathcal{L}_u^n f \to \int_{\Omega_M^+} f\, dm$ uniformly as $n \to +\infty$.*

(b) *If u is complex valued with $\Re(u)$ normalised, then:*
  (i) *The spectral radius of $\mathcal{L}_u$ is bounded above by 1;*
  (ii) *If $\mathcal{L}_u$ has an eigenvalue β of modulus 1 then it is simple and unique, and $\mathcal{L}_u = \beta\, \mathcal{M}\mathcal{L}_{\Re(u)}\mathcal{M}^{-1}$ where $\mathcal{M}$ is a multiplication operator. In addition, $\mathcal{L}_u$ has a spectral gap;*
  (iii) *If $\mathcal{L}_u$ has no eigenvalues of modulus 1, then its spectral radius is strictly smaller than 1.*

(c) *If u is real valued and non-normalised, then:*
  (i) *$\mathcal{L}_u$ has a simple maximal positive eigenvalue β with real positive eigenfunction $f_\beta \in \mathcal{B}_\vartheta$;*
  (ii) *$\mathcal{L}_u$ has a spectral gap;*
  (iii) *There exists a unique probability measure m such that $\mathcal{L}_u^* m = \beta m$, where $\mathcal{L}_u^*$ is the dual operator, and $f_\beta m$ is σ-invariant;*
  (iv) *For all $f \in C^0(\Omega_M^+)$, it holds $\beta^{-n}\mathcal{L}_u^n f \to f_\beta \int_{\Omega_M^+} f\, dm$ uniformly as $n \to +\infty$.*

(d) *Let $D \subset \mathcal{B}_\vartheta$ denotes the set of non-necessarily real functions u such that $\mathcal{L}_u$ has a simple maximal eigenvalue $\beta(u)$ with spectral gap (clearly real valued functions are in D). Then:*
  (i) *D is open;*
  (ii) *the* pressure *$P(u) := \log \beta(u)$, fixed to be real for real valued potentials, is defined mod $2\pi i$ and is an analytic map;*
  (iii) *if $u \in D$ and $u \sim v + c + 2\pi i \tilde{v}$ for some integer valued function $\tilde{v}$, then $P(u) = P(v) + c$ (mod $2\pi i$).*

Having generalised the transfer operator, we can consider the analogous generalisation for the Zeta function (7.1). Instead of simply counting the fixed points of σ^n, we can weigh them by using a potential.

Given $u \in \mathcal{B}_\vartheta$, we define the *dynamical Zeta function* of σ by

$$Z_u^\sigma(w) := \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \sum_{\omega \in Fix(\sigma^n)} \prod_{j=0}^{n-1} e^{u(\sigma^j(\omega))}\,. \tag{7.8}$$

Clearly $Z_0^\sigma(w) = Z^\sigma(w)$.

Proposition 7.9. *If $u \in \mathcal{B}_\vartheta$ is real, then $Z_u^\sigma(w)$ has radius of convergence $e^{-P(u)} = \beta(u)^{-1}$. In particular, $P(0) = h(\sigma)$.*

Proof. If $u(\omega) = u(\omega_0, \omega_1)$, namely u only depends on the first two symbols of ω, then we can repeat the argument in (7.3) with the matrix $M_u = e^u M$, for which $(m_u)_{hk} = e^{u(h,k)} m_{hk}$ for all $h, k \in \mathcal{A}$, to obtain

$$Z_u^\sigma(w) = \frac{1}{\det(Id - wM_u)}\,.$$

At the same time, the operator $\mathcal{L}_u$ is given by

$$(\mathcal{L}_u f)(\omega) = \sum_{h \in \mathcal{A}} m_{h\omega_0}\, e^{u(h\omega)}\, f(h\omega) = \sum_{h \in \mathcal{A}} m_{h\omega_0}\, e^{u(h,\omega_0)}\, f(h\omega)\,.$$

Then, the real positive eigenfunction f_β, relative to the maximal eigenvalue β given in Theorem 7.8-(c-i), is of the form $f_\beta(\omega) = f_\beta(\omega_0)$, hence is represented by a row vector $F_\beta = (F_h)_{h \in \mathcal{A}}$ which satisfies $F_\beta M_u = \beta F_\beta$. Therefore, $\beta(u)$ is the maximal eigenvalue of the primitive matrix M_u, and the result follows in this case.

For any real $u \in \mathcal{B}_\vartheta$, for all $\varepsilon > 0$ there exists $\tilde{u} \in \mathcal{B}_\vartheta$ depending on a finite number of symbols such that $\|u - \tilde{u}\| < \varepsilon$. The result follows by using that the pressure function is Lipschitz continuous with constant 1. $\square$

In the following, we are interested in the convergence of a dynamical zeta function Z_u^σ at $w = 1$, which for a real valued u is equivalent to $P(u) < 0$.

The next result is a consequence of the analytic properties of the pressure.

Theorem 7.10 ([PP90])**.** *Let $u \in \mathcal{B}_\vartheta$ with $\Re(u)$ normalised, hence $P(\Re(u)) = 0$. Then:*

(i) If the spectral radius of $\mathcal{L}_u$ is strictly smaller than 1, then there exists $\varepsilon > 0$ such that for all $v \in \mathcal{B}_\vartheta$ with $\|u - v\|_\vartheta < \varepsilon$, the dynamical zeta function Z_v^σ converges absolutely at $w = 1$.

(ii) If the spectral radius of $\mathcal{L}_u$ is equal to 1, then then there exists $\varepsilon > 0$ such that for all $v \in \mathcal{B}_\vartheta$ with $\|u - v\|_\vartheta < \varepsilon$, the series

$$\bar{Z}_v^\sigma(w) := \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \sum_{\omega \in Fix(\sigma^n)} \Big(\prod_{j=0}^{n-1} e^{v(\sigma^j(\omega))} - e^{n\,P(v)} \Big)$$

converges absolutely at $w = 1$. In addition, the expression

$$v \mapsto Z_v^\sigma(1) := \frac{\bar{Z}_v^\sigma(1)}{1 - e^{P(v)}}$$

gives an extension of $Z_u^\sigma(1)$ to a neighbourhood of u, which is non-zero and analytic provided that $e^{P(u)} \neq 1$, that is 1 is not an eigenvalue of $\mathcal{L}_u$.

7.2. **Proof of** (7.6)**.** Let $Z^\Phi(\alpha)$ be the Zeta function of the geodesic flow Φ

$$Z^\Phi(\alpha) = \prod_{\gamma \,\in\, \text{ppo}} \Big(1 - e^{-\alpha h \ell(\gamma)}\Big)^{-1} \tag{7.9}$$

where $h = h(\Phi)$ is as in Proposition 7.4 and is convergent for $\Re(\alpha) > 1$.

Using the next result we reduce the problem to the study of the dynamical zeta function of a subshift of finite type.

Theorem 7.11 ([Bo73])**.** *There exists a subshift of finite type (Ω_M, σ) on a finite alphabet $\mathcal{A}$ with primitive transition matrix M, and a positive roof function $\tau \in \mathcal{B}_\vartheta$ for some $\vartheta \in (0,1)$, such that the vertical flow $V = \{V_t\}$ defined on $(\Omega_M)_\tau$ with base map σ has topological entropy $h(V) = h(\Phi)$ and its zeta function $Z^V(\alpha)$ is such that $Z^\Phi(\alpha)/Z^V(\alpha)$ is non-zero and analytic in the region $\Re(\alpha) > 1 - \varepsilon$ for some $\varepsilon > 0$.*

We are thus interested in studying $Z^V(\alpha)$. First, we note that there is a one-to-one correspondence between the periodic orbits for V and for σ. If γ' is a periodic orbit for V then there exists exactly one $\gamma = \{\omega, \sigma(\omega), \ldots, \sigma^{n-1}(\omega)\}$ periodic for σ with period n such that

$$\ell(\gamma') = \sum_{j=0}^{n-1} \tau(\sigma^j(\omega))\,.$$

Hence, we can repeat the computations in (7.4) backward to obtain for $h = h(V) = h(\Phi)$

$$
\begin{aligned}
Z^V(\alpha) &= \prod_{\gamma' \,\in\, \mathrm{ppo(V)}} \left(1 - e^{-\alpha h \ell(\gamma')}\right)^{-1} = \exp \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{\omega \in Fix(\sigma^n)} e^{-\alpha h \sum_{j=0}^{n-1} \tau(\sigma^j(\omega))} \\
&= \exp \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{\omega \in Fix(\sigma^n)} \prod_{j=0}^{n-1} e^{-\alpha h \tau(\sigma^j(\omega))} = Z^{\sigma}_{-\alpha h \tau}(1)\,.
\end{aligned}
$$

Hence, we are reduced to study the convergence properties of the dynamical Zeta function (7.8) of the subshift of finite type with potential $u_\alpha(\cdot) := -\alpha h \tau(\cdot) \in \mathcal{B}_\vartheta$. Moreover, without loss of generality, we can consider the one-sided subshift σ on Ω_M^+. Thus, we are ready to apply the results from Section 7.1.

Lemma 7.12. *Let $G \in C((\Omega_M^+)_\tau)$ and $g(\omega) := \int_0^{\tau(\omega)} G(\omega, s)\, ds$. Then, there exists a constant $c = c(G)$ such that $P(g - c\tau) = 0$. In addition, $c(0) = h(V)$, hence $P(u_1) = P(-h\tau) = 0$, that is u_1 is normalised.*

Thanks to Lemma 7.12, we write $\alpha = 1 + i\eta$ and apply Theorem 7.10 to $u_{1+i\eta}$ for all $\eta \in \mathbb{R}$. The first question is whether 1 may be an eigenvalue for $\mathcal{L}_\eta := \mathcal{L}_{u_{1+i\eta}}$.

Proposition 7.13. *For all $\eta \in \mathbb{R} \setminus \{0\}$, 1 is not an eigenvalue of $\mathcal{L}_\eta$.*

Proof. We use that the vertical flow $V = \{V_t\}$ on $(\Omega_M^+)_\tau$ is mixing. In particular, it is *weakly mixing*, that is the Koopman operator on L^2 has no eigenvalue of modulus one different from the maximal eigenvalue 1.

Let $\eta \in \mathbb{R} \setminus \{0\}$, and assume that $\mathcal{L}_\eta$ has 1 as an eigenvalue. That is, the potential $u_{1+i\eta}(\cdot) = -h\tau(\cdot) - i\eta h \tau(\cdot)$ is such that $\Re(u_{1+i\eta}) = u_1$ is normalised, since $P(-h\tau) = 0$, and $\mathcal{L}_\eta$ has a simple maximal eigenvalue at $1 = e^{P(u_1)}$.

Let $f_1 \in \mathcal{B}_\vartheta$ be the eigenfunction of $\mathcal{L}_\eta$ with eigenvalue 1, then

$$
\begin{aligned}
|f_1(\omega)| = |(\mathcal{L}_{u_{1+i\eta}} f_1)(\omega)| &= \Big| \sum_{\sigma(\tilde\omega) = \omega, \tilde\omega \in \Omega_M^+} e^{-h\tau(\tilde\omega)}\, e^{-i\eta h \tau(\tilde\omega)}\, f_1(\tilde\omega) \Big| \\
&\le \sum_{\sigma(\tilde\omega) = \omega, \tilde\omega \in \Omega_M^+} e^{-h\tau(\tilde\omega)}\, |f_1(\tilde\omega)| = (\mathcal{L}_{u_1} |f_1|)(\omega)\,.
\end{aligned}
$$

Let now m be the unique probability measure for which $\mathcal{L}_{u_1}^* m = m$. Then,

$$
\int_{\Omega_M^+} |f_1|\, dm = \int_{\Omega_M^+} (\mathcal{L}_{u_1} |f_1|)\, dm\,.
$$

It follows that $|f_1|$ is a multiple of the eigenfunction of $\mathcal{L}_{u_1}$ for the eigenvalue 1, hence $|f_1|$ is m-a.e. constant.

In addition, recalling that $\mathcal{L}_{u_1} 1 = 1$, we write

$$
f_1(\omega) = (\mathcal{L}_{u_{1+i\eta}} f_1)(\omega) = \left(\mathcal{L}_{u_1}\left(e^{-i\eta h\tau} f_1\right)\right)(\omega) = \sum_{\sigma(\tilde\omega) = \omega, \tilde\omega \in \Omega_M^+} e^{-h\tau(\tilde\omega)}\, e^{-i\eta h\tau(\tilde\omega)}\, f_1(\tilde\omega)\,.
$$

Hence, the rightmost term is a convex combination of complex numbers with modulus $|f_1|$ which is a complex number with modulus $|f_1|$. Therefore,

$$
\forall\, \omega \in \Omega_M^+\,, \quad e^{-i\eta h\tau(\omega)}\, f_1(\omega) = f_1(\sigma(\omega))\,.
$$

This gives a contradiction with the vertical flow being weakly mixing. Indeed, setting for all $(\omega, s) \in (\Omega_M^+)_\tau$, $F(\omega, s) := f_1(\omega) e^{-i\eta hs}$, we have

$$F(\omega, \tau(\omega)) = f_1(\omega) e^{-i\eta h\tau(\omega)} = f_1(\sigma(\omega)) = F(\sigma(\omega), 0)\,,$$

hence F is well defined as a continuous function on $(\Omega_M^+)_\tau$ and in L^2. Moreover,

$$F(V_t(\omega, s)) = F(\omega, s+t) = f_1(\omega) e^{-i\eta h(s+t)} = e^{-i\eta ht} F(\omega, s)\,, \quad \forall\, t \in \mathbb{R}\,.$$

The mixing property of V implies that $\eta ht = 0$, hence $\eta = 0$. This is a contradiction. □

Applying Theorem 7.10 to $Z^V(\alpha) = Z^\sigma_{-\alpha h\tau}(1)$, Proposition 7.13 implies the existence of an analytic continuation of $Z^V(\alpha)$ to a neighborhood of $\{\Re(\alpha) \geq 1\}$ with a simple pole at $\alpha = 1$. Theorem 7.11 then implies that

Corollary 7.14. *The Zeta function $Z^\Phi(\alpha)$ of the geodesic flow Φ defined in* (7.9) *converges for $\Re(\alpha) > 1$, has a simple pole at $\alpha = 1$, and has a non-zero analytic extension to a neighborhood of $\Re(\alpha) = 1$.*

By the properties in Corollary 7.14, we can write

$$\begin{aligned}
\frac{(Z^\Phi)'(\alpha)}{Z^\Phi(\alpha)} &= \frac{d}{d\alpha} \log Z^\Phi(\alpha) = -\sum_{\gamma\,\in\,\text{ppo}} \sum_{m=1}^{+\infty} h\ell(\gamma)\, e^{-\alpha m\ell(\gamma)h} \\
&= -\int_0^{+\infty} e^{-\alpha ht}\, d\Big(\sum_{m=1}^{+\infty} \sum_{\gamma\,\in\,\text{ppo},\ m\ell(\gamma)\,\leq\, t} h\ell(\gamma)\Big) \\
&= \frac{1}{1-\alpha} + W(\alpha)\,,
\end{aligned}$$

where $W(\alpha)$ is analytic in $\Re(\alpha) \geq 1$. A classical Ikehara-Wiener Tauberian Theorem (see [PP90, Theorem 6.7]) implies that

$$\psi(T) := \sum_{m=1}^{+\infty} \sum_{\gamma\,\in\,\text{ppo},\ m\ell(\gamma)\,\leq\, T} h\ell(\gamma) \sim e^{hT}\,, \quad \text{as } T \to +\infty.$$

In addition, recalling the definition of the counting function $\pi(T)$ in (7.5),

$$\psi(T) = \sum_{\gamma\,\in\,\text{ppo},\ \ell(\gamma)\,\leq\, T} \Big\lfloor \frac{T}{\ell(\gamma)} \Big\rfloor\, h\ell(\gamma) \leq hT\,\pi(T)\,,$$

hence,

$$\liminf_{T\to+\infty} \frac{hT\,\pi(T)}{e^{hT}} \geq 1\,. \tag{7.10}$$

In the other direction, let $\xi \in (0, 1)$. Then,

$$\begin{aligned}
\pi(T) &= \pi(\xi T) + \sum_{\xi T \leq \ell(\gamma) \leq T} 1 \leq \pi(\xi T) + \sum_{\ell(\gamma) \leq T} \frac{\ell(\gamma)}{\xi T} \\
&\leq \pi(\xi T) + \frac{\psi(T)}{\xi hT}\,.
\end{aligned}$$

Since $\pi(\xi T) = o(e^{hT})$, we have

$$\limsup_{T\to+\infty} \frac{hT\,\pi(T)}{e^{hT}} \leq \frac{1}{\xi}\,. \tag{7.11}$$

From (7.10) and (7.11) for all $\xi \in (0, 1)$, we obtain (7.6).

Appendix A. Exponential decay of correlations for discrete-time dynamical systems

In this appendix we apply the transfer operator method to prove exponential decay of correlations for smooth expanding and hyperbolic maps. For more information we refer to [DKL21]. We recall the following result, known as *Hennion's Theorem* (see e.g. [DKL21, Theorem 1.1]) and similar to other spectral results, such as Ionescu-Tulcea-Marinescu's Theorem.

Theorem A.1. *Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{B}_w, |\cdot|_w)$ two Banach spaces with $\mathcal{B}$ compactly embedded in $\mathcal{B}_w$, and let $\mathcal{L} : \mathcal{B}_w \to \mathcal{B}_w$ be a linear continuous operator for which $\mathcal{L}(\mathcal{B}) \subseteq \mathcal{B}$. If there exist constants $C_H > 0$ and $M > \kappa > 0$ for which, for all $f \in \mathcal{B}$ and $j \in \mathbb{N}$, the following hold*

$$|\mathcal{L}^j f|_w \leq C_H\, M^j\, |f|_w\,, \tag{A.1}$$

$$\|\mathcal{L}^j f\|_{\mathcal{B}} \leq C_H\, \kappa^j\, \|f\|_{\mathcal{B}} + C_H\, M^j\, |f|_w\,; \tag{A.2}$$

then, using the notation $B(0,r) = \{\zeta \in \mathbb{C} \,:\, |\zeta| \leq r\}$, the spectrum and the essential spectrum of $\mathcal{L}$ satisfy

$$spec(\mathcal{L}|_{\mathcal{B}}) \subseteq B(0, M)\,, \quad essspec(\mathcal{L}|_{\mathcal{B}}) \subseteq B(0, \kappa)\,.$$

The first result we prove shows how to apply the spectral properties of the transfer operator for one-dimensional smooth expanding maps.

Theorem A.2. *Let $T : S^1 \to S^1$ be C^2 and expanding, that is there exists $\lambda > 1$ such that $|T'(x)| \geq \lambda$ for all $x \in S^1$. Then, there exists a unique T-invariant probability measure μ which is ergodic, mixing, and absolutely continuous with respect to the Lebesgue measure with density $h(x)$. Moreover, $h \in W^{1,2}(S^1)$ and T has exponential decay of correlations on $W^{1,2}(S^1)$ and $L^2(S^1)$.*

Proof. We begin by recalling the standard continuous embedding for the Sobolev and Lebesgue spaces of S^1 with Lebesgue measure,

$$W^{1,2}(S^1) \subset_c L^2(S^1) \subset L^1(S^1)\,, \quad W^{1,1}(S^1) \subset C^0(S^1)\,, \tag{A.3}$$

where $\subset_c$ denotes a compact embedding[9]. Moreover, we will need that for all $x \in S^1$ the set of preimages

$$\bigcup_{k\in\mathbb{N}} T^{-k}(x) = \Big\{ y \in S^1 \,:\, \exists\, k \in \mathbb{N} \text{ for which } T^k(y) = x \Big\}$$

is dense in S^1. This follows from the expanding property of T, indeed for all $\varepsilon > 0$ and all $y \in S^1$ there exists $k \in \mathbb{N}$ for which $S^1 \subset T^k(y - \varepsilon, y + \varepsilon)$.

Let now introduce the transfer operator $\mathcal{L}$ associated to T, defined as the linear operator on L^1 which is the dual to the Koopman operator $Ug = g \circ T$ on L^∞ with respect to the Lebesgue measure. That is,

$$\int_{S^1} f\,(Ug)\,dx = \int_{S^1} (\mathcal{L}f)\,g\,dx\,, \quad \forall\, f \in L^1,\, g \in L^\infty\,. \tag{A.4}$$

All $x \in S^1$ have a finite number of preimages, that is

$$N := \max_{x\in S^1} \Big(\#\, \big\{ y \in S^1 \,:\, T(y) = x \big\} \Big) \leq \|T'\|_\infty < \infty\,, \quad \forall\, x \in S^1\,. \tag{A.5}$$

Hence $\mathcal{L}f$ is well defined by

$$\mathcal{L} : L^1 \to L^1\,, \quad \mathcal{L}f(x) := \sum_{T(y)=x} \frac{1}{|T'(y)|}\, f(y)\,. \tag{A.6}$$

[9]From now on, we drop S^1 from the notation of the Sobolev and Lebesgue spaces.

Step 1. The transfer operator $\mathcal{L}$ is positive and continuous on L^1, with $\|\mathcal{L}\|_1 = 1$. In particular, the spectrum satisfies spec$(\mathcal{L}|_{L^1}) \subseteq B(0,1)$, where $B(0,1) = \{\zeta \in \mathbb{C} : |\zeta| \le 1\}$.

It is immediate that $f \ge 0$ implies $\mathcal{L}f \ge 0$, hence $\mathcal{L}$ is positive. Then, by (A.4),

$$\int_{S^1} \mathcal{L}f \, dx = \int_{S^1} f \, dx \,, \quad \forall\, f \in L^1 \,,$$

from which it follows that $\mathcal{L}f \in L^1$ and $\|\mathcal{L}f\|_1 \le \|f\|_1$. In addition, $\|\mathcal{L}f\|_1 = \|f\|_1$ for all $f \ge 0$. Hence, $\|\mathcal{L}\|_1 = 1$.

Step 2. There exists $h \in W^{1,2}$ such that $\mathcal{L}h = h$. In particular, $d\mu(x) = h(x)dx$ is a T-invariant probability measure which is equivalent to the Lebesgue measure, that is $h(x) > 0$ for all $x \in S^1$.

We claim that $f \in W^{1,2}$ implies $\mathcal{L}f \in W^{1,2}$. This follows from (A.6). First, we write

$$\begin{aligned} \|\mathcal{L}f\|_2^2 &= \int_{S^1} (\mathcal{L}f)\,(\mathcal{L}f)\, dx = \int_{S^1} f\,(U\mathcal{L}f)\, dx \le \|f\|_2 \Big(\int_{S^1} (\mathcal{L}f)^2 \circ T \, dx \Big)^{\frac{1}{2}} = \\ &= \|f\|_2 \Big(\int_{S^1} (\mathcal{L}f)^2\, (\mathcal{L}1) dx \Big)^{\frac{1}{2}} \le \|f\|_2 \, \|\mathcal{L}1\|_\infty \, \|\mathcal{L}f\|_2 \le \frac{N}{\lambda} \, \|f\|_2 \, \|\mathcal{L}f\|_2 \,, \end{aligned} \tag{A.7}$$

where 1 denotes the constant function and we have used (A.5). Then,

$$(\mathcal{L}f)'(x) = \sum_{T(y)=x} \operatorname{sgn}(T') \Big(\frac{1}{|T'(y)|^2} \, f'(y) - \frac{T''(y)}{|T'(y)|^3} \, f(y) \Big) \tag{A.8}$$

where $\operatorname{sgn}(T')$ is ± 1 according to whether T' is always positive or negative, from which

$$|(\mathcal{L}f)'| \le \Big| \mathcal{L}\Big(\frac{f'}{|T'|} \Big) \Big| + \Big| \mathcal{L}\Big(\frac{T''\, f}{|T'|^2} \Big) \Big| \,. \tag{A.9}$$

It follows

$$\|(\mathcal{L}f)'\|_2 \le \frac{N}{\lambda} \, \||T'|^{-1} \, f'\|_2 + \frac{N}{\lambda} \, \|T'' \, |T'|^{-2} \, f\|_2 \le \frac{N}{\lambda^2} \, \|f'\|_2 + \frac{\|T''\|_\infty \, N}{\lambda^3} \, \|f\|_2 \,.$$

However, it turns out that for $f \in W^{1,2}$ we can bound $\|\mathcal{L}^j f\|_{W^{1,2}}$ uniformly for all $j \ge 1$. First, we consider the norm $\|\mathcal{L}^j f\|_{W^{1,1}}$. From Step 1, we have

$$\|\mathcal{L}^j f\|_1 \le \|f\|_1 \,, \quad \forall\, j \in \mathbb{N} \,.$$

Moreover, from (A.9), we obtain

$$\|(\mathcal{L}f)'\|_1 \le \||T'|^{-1}\|_\infty \, \|f'\|_1 + \|T'' \, |T'|^{-2}\|_\infty \, \|f\|_1 \le \frac{1}{\lambda} \, \|f'\|_1 + \frac{\|T''\|_\infty}{\lambda^2} \, \|f\|_1 \,.$$

Iterating, we get

$$\begin{aligned} \|(\mathcal{L}^j f)'\|_1 \le & \frac{1}{\lambda} \, \|(\mathcal{L}^{j-1} f)'\|_1 + \frac{\|T''\|_\infty}{\lambda^2} \, \|f\|_1 \le \frac{1}{\lambda^2} \, \|(\mathcal{L}^{j-2} f)'\|_1 + \|T''\|_\infty \Big(\frac{1}{\lambda^3} + \frac{1}{\lambda^2} \Big) \|f\|_1 \le \cdots \\ & \cdots \le \frac{1}{\lambda^j} \, \|f'\|_1 + \frac{\|T''\|_\infty}{\lambda^2} \Big(\sum_{i=0}^{j-1} \lambda^{-i} \Big) \|f\|_2 \le \frac{1}{\lambda^j} \, \|f'\|_1 + \frac{\|T''\|_\infty}{\lambda(\lambda-1)} \, \|f\|_1 \,. \end{aligned}$$

Finally,

$$\|\mathcal{L}^j f\|_{W^{1,1}} \le \frac{1}{\lambda^j} \, \|f'\|_1 + \Big(1 + \frac{\|T''\|_\infty}{\lambda(\lambda-1)} \Big) \|f\|_1 \,, \quad \forall\, j \in \mathbb{N} \,. \tag{A.10}$$

Now, using (A.3), we obtain

$$\|\mathcal{L}^j f\|_\infty \le \|\mathcal{L}^j f\|_{W^{1,1}} \le \Big(2 + \frac{\|T''\|_\infty}{\lambda(\lambda-1)} \Big) \|f\|_{W^{1,1}} \,, \quad \forall\, j \in \mathbb{N} \,.$$

Hence, we can repeat the argument in (A.7) for $\mathcal{L}^j f$ to obtain that

$$\exists\, C_1 = 1 + \frac{\|T''\|_\infty}{\lambda(\lambda-1)} > 0 \text{ such that } \|\mathcal{L}^j f\|_2 \le C_1\, \|f\|_2\,, \quad \forall\, f \in L^2\,, \forall\, j \in \mathbb{N}\,. \tag{A.11}$$

To obtain the estimate for $\|(\mathcal{L}^j f)'\|_2$, we first note that

$$(\mathcal{L}^j f)(x) = \sum_{T^j(y)=x} \frac{1}{|(T^j)'(y)|}\, f(y)\,.$$

Hence, similarly to (A.8), we write

$$\begin{aligned}(\mathcal{L}^j f)'(x) &= \sum_{T^j(y)=x} \operatorname{sgn}(T')\left(\frac{1}{|(T^j)'(y)|^2}\, f'(y) - \frac{(T^j)''(y)}{|(T^j)'(y)|^3}\, f(y)\right) = \\ &= \operatorname{sgn}(T')\left(\mathcal{L}^j\Big(\frac{f'}{|(T^j)'|}\Big)(x) - \mathcal{L}^j\Big(\frac{(T^j)''\, f}{|(T^j)'|^2}\Big)(x)\right).\end{aligned}$$

For all $j \in \mathbb{N}$, we have the estimate $\||(T^j)'|^{-1}\|_\infty \le \lambda^{-j}$, and we can write

$$(T^j)''(x) = \sum_{i=0}^{j-1} \frac{T''(T^i(x))}{T'(T^i(x))}\, (T^i)'(x)\, (T^j)'(x)\,,$$

from which, using that $T \in C^2$, for all $j \in \mathbb{N}$

$$\frac{|(T^j)''(x)|}{|(T^j)'(x)|^2} \le \Big\|\frac{T''}{T'}\Big\|_\infty \sum_{i=0}^{j-1} \Big|\frac{(T^i)'(x)}{(T^j)'(x)}\Big| \le \Big\|\frac{T''}{T'}\Big\|_\infty \sum_{i=0}^{j-1} \lambda^{i-j} \le \frac{1}{\lambda-1}\Big\|\frac{T''}{T'}\Big\|_\infty =: C_2\,.$$

Thus, we get that

$$\exists\, C_2 > 0 \text{ such that } \|(\mathcal{L}^j f)'\|_2 \le C_1\, \lambda^{-j}\, \|f'\|_2 + C_1\, C_2\, \|f\|_2\,, \quad \forall\, f \in W^{1,2}\,, \forall\, j \in \mathbb{N}\,. \tag{A.12}$$

Now, since $1 \in W^{1,2}$, for all $n \in \mathbb{N}$,

$$g_n := \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{L}^j 1 \in W^{1,2}$$

and $\|g_n\|_2 \le C_1$ since $\|\mathcal{L}^j 1\|_2 \le C_1$ for all j. By the compactness of the embedding of $W^{1,2}$ in L^2, there exists a subsequence $\{n_k\}_k$ and $h \in L^2$ such that

$$\lim_{k\to+\infty} \|g_{n_k} - h\|_1 = 0\,.$$

First, we show that $\mathcal{L}h = h$. It is a consequence of the continuity of $\mathcal{L}$ on L^1 and of the following limits in L^1

$$\mathcal{L}h = \lim_{k\to\infty} \mathcal{L}g_{n_k} = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \mathcal{L}^j 1 = \lim_{k\to\infty} \left(g_{n_k} + \frac{\mathcal{L}^{n_k}1 - 1}{n_k}\right) = h\,.$$

From (A.4) we obtain that $d\mu(x) = h(x)dx$ is a T-invariant probability measure. One gets

$$\int_{S^1} (Ug)\, d\mu = \int_{S^1} (Ug)\, h\, dx = \int_{S^1} g\, (\mathcal{L}h)\, dx = \int_{S^1} g\, h\, dx = \int_{S^1} g\, d\mu\,, \quad \forall\, g \in L^\infty\,.$$

Moreover,

$$\mu(S^1) = \int_{S^1} h\, dx = \lim_{k\to\infty} \int_{S^1} g_{n_k}\, dx = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \int_{S^1} \mathcal{L}^j 1\, dx = 1\,.$$

We now prove that $h \in W^{1,2}$. Since $g_n \in W^{1,2}$ and $g_{n_k} \to h$ in L^1, for all $\psi \in C^\infty$ we have

$$\int_{S^1} g'_{n_k}\, \psi \, dx = \int_{S^1} g_{n_k}\, \psi' \, dx \xrightarrow[k\to+\infty]{} \int_{S^1} h\, \psi' \, dx \, .$$

Moreover, by (A.12),

$$\|g'_{n_k}\|_2 \le \frac{1}{n_k} \sum_{j=0}^{n_k-1} \|(\mathcal{L}^j 1)'\|_2 \le C_1\, C_2 \, , \quad \forall\, k \in \mathbb{N} \, ,$$

therefore, up to a subsequence there exists a function $h' \in L^2$ such that $g'_{n_k} \rightharpoonup h'$ for $k \to \infty$ in L^2. Then, for all $\psi \in C^\infty$

$$\int_{S^1} g'_{n_k}\, \psi \, dx \xrightarrow[k\to+\infty]{} \int_{S^1} h'\, \psi \, dx \, ,$$

and together with the previous value of the limit, we get that h' is the weak derivative of h in L^2. Therefore, $h \in W^{1,2}$.

It remains to prove that $h > 0$. First, since $\mathcal{L}$ is a positive operator, we have $g_n \ge 0$ for all $n \in \mathbb{N}$, hence $h \ge 0$. By contradiction, let $x_0 \in S^1$ for which $h(x_0) = 0$. Then,

$$0 = h(x_0) = (\mathcal{L}^k h)(x_0) = \sum_{T^k(y)=x_0} \frac{1}{|(T^n)'(y)|}\, h(y) \quad \forall\, k \in \mathbb{N} \quad \Rightarrow \quad h(y) = 0 \quad \forall\, y \in \bigcup_{k\in\mathbb{N}} T^{-k}(x_0) \, .$$

But the last set is dense, and since $h \in W^{1,2}$ hence is continuous, this implies $h \equiv 0$. This is absurd since $\mu(S^1) = 1$.

Step 3. The spectrum of $\mathcal{L}$ on $W^{1,2}$ satisfies spec$(\mathcal{L}|_{W^{1,2}}) \cap \{|\zeta| = 1\} = \{1\}$, and 1 is a simple eigenvalue.

Let $f \in W^{1,2} \subset W^{1,1}$, $f \not\equiv 0$, satisfy $\mathcal{L} f = e^{i\ell} f$ for $\ell \in \mathbb{R}$. We need to show that $\ell = 0$ and $f \equiv h$. Since $h > 0$, we let $g := f/h$. We have

$$e^{i\ell}\, g = e^{i\ell}\, \frac{f}{h} = \frac{\mathcal{L} f}{h} \quad \Leftrightarrow \quad g(x) = e^{-i\ell}\, \frac{1}{h(x)} \sum_{T(y)=x} \frac{1}{|T'(y)|}\, g(y)\, h(y) \, , \quad \forall\, x \in S^1 \, .$$

Since g is continuous, because f and h are, we find $x_0 \in S^1$ such that $|g(x_0)| = \|g\|_\infty$. Using the previous equality and $\mathcal{L} h = h$, we obtain

$$\|g\|_\infty = |g(x_0)| \le \frac{1}{h(x_0)} \Big(\sum_{T(y)=x_0} \frac{1}{|T'(y)|}\, h(y) \Big) \sup_{T(y)=x_0} |g(y)| = \sup_{T(y)=x_0} |g(y)| \, .$$

As above, using that the set of preimages of every point is dense, we find that $|g(x)| = \|g\|_\infty$ for all $x \in S^1$, hence $g(x) = c\, e^{i\ell(x)}$ for some $c \in \mathbb{R}$ and a continuous $\ell : S^1 \to \mathbb{R}$. Then, for all $x \in S^1$,

$$e^{i\ell(x)} = e^{-i\ell}\, \frac{1}{h(x)} \sum_{T(y)=x} \frac{1}{|T'(y)|}\, e^{i\ell(y)}\, h(y) = e^{-i\ell} \sum_{T(y)=x} \frac{h(y)}{|T'(y)|\, h(T(y))}\, e^{i\ell(y)} \, .$$

The right hand side is a convex combination of points on S^1, since it has to be a point on S^1, we have that there exists $y \in T^{-1}(x)$ for which

$$e^{i\ell(x)} = e^{-i\ell}\, e^{i\ell(y)} \quad \Leftrightarrow \quad \ell(y) - \ell(T(y)) = \ell \, , \quad \forall\, y \in S^1 \, .$$

Using that the probability measure μ is T-invariant, we obtain

$$\ell = \int_{S^1} \Big(\ell(y) - \ell(T(y)) \Big)\, d\mu = 0 \, .$$

Hence, $g(x) \equiv c$ and $f(x) = c\, h(x)$.

Step 4. The map T has exponential decay of correlations on $W^{1,2}$ and L^2.

By Definition 6.9, we have to show that there exist constants $C > 0$ and $\vartheta \in (0,1)$ such that

$$|C_T(f,g,n)| \le C\, \|f\|_{W^{1,2}}\, \|g\|_2\, \vartheta^n\,, \quad \forall\, n \in \mathbb{N},\ f \in W^{1,2}\,,\ g \in L^2\,.$$

In Step 2, we have proved that $\mathcal{L}(W^{1,2}) \subset W^{1,2}$ and that we can apply Theorem A.1 to $\mathcal{L}$ with $\mathcal{B}_w = L^2$ and $\mathcal{B} = W^{1,2}$. In particular, (A.11)-(A.12) imply (A.1)-(A.2) with

$$C_H = \max\{C_1, 1 + C_1\, C_2\}\,, \quad M = 1\,, \quad \kappa = \lambda^{-1} \in (0,1)\,.$$

Therefore,

$$essspec(\mathcal{L}|_{W^{1,2}}) \subseteq B(0, \lambda^{-1})\,,$$

and applying the spectral theorem, we find that there exist linear operators $K, R : W^{1,2} \to W^{1,2}$, with K compact and R continuous, such that $RK = KR = 0$, and for all $j \in \mathbb{N}$,

$$\mathcal{L}^j f = K^j f + R^j f \quad \text{with} \quad \|R^j f\|_{W^{1,2}} \le \lambda^{-j}\, \|f\|_{W^{1,2}}\,, \quad \forall\, f \in W^{1,2}\,.$$

In particular, K corresponds to the projections on the eigenspaces of the eigenvalues of $\mathcal{L}|_{W^{1,2}}$ in $\{|\zeta| > \lambda^{-1}\}$. By Step 3, 1 is a simple eigenvalue and it's the only one on $\{|\zeta| = 1\}$, hence, setting

$$\vartheta := \max\left\{|\rho|\, :\, \rho \text{ is an eigenvalue of } \mathcal{L} \text{ with } \lambda^{-1} < |\rho| < 1\right\}\,,$$

we can write $K = K_1 + K_\vartheta$, where K_1 is the projection on h and K_ϑ is the projection associated to the other eigenvalues in $\{|\zeta| > \lambda^{-1}\}$. Then, $K_1 K_\vartheta = K_\vartheta K_1 = 0$, and writing $\tilde{R} = K_\vartheta + R$ we obtain $K_1 \tilde{R} = \tilde{R} K_1 = 0$, and for all $j \in \mathbb{N}$,

$$\mathcal{L}^j f = K_1^j f + \tilde{R}^j f \quad \text{with} \quad \|\tilde{R}^j f\|_{W^{1,2}} \le \vartheta^j\, \|f\|_{W^{1,2}}\,, \quad \forall\, f \in W^{1,2}\,. \tag{A.13}$$

We now describe K_1. There exists a functional $k_1 : W^{1,2} \to \mathbb{R}$ such that, for all $f \in W^{1,2}$, $K_1 f = k_1(f)\, h$. For all $f \in W^{1,2}$, we have

$$\int_{S^1} f\, dx = \int_{S^1} \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{L}^j f\, dx = \int_{S^1} \Big(\frac{1}{n} \sum_{j=0}^{n-1} K_1^j f + \frac{1}{n} \sum_{j=0}^{n-1} \tilde{R}^j f \Big)\, dx =$$

$$= \int_{S^1} K_1 f\, dx + \int_{S^1} \frac{1}{n} \sum_{j=0}^{n-1} \tilde{R}^j f\, dx \le \int_{S^1} K_1 f\, dx + \frac{1}{n} \sum_{j=0}^{n-1} \|\tilde{R}^j f\|_{W^{1,2}} = k_1(f) + o(1) \quad \text{as } n \to +\infty.$$

Hence,

$$k_1(f) = \int_{S^1} f\, dx\,, \quad \forall\, f \in W^{1,2}\,.$$

Finally, we recall that $f, h \in W^{1,2}$ implies $f \cdot h \in W^{1,2}$ and there exists a constant $c > 0$ such that

$$\|f \cdot h\|_{W^{1,2}} \le c\, \|f\|_{W^{1,2}}\, \|h\|_{W^{1,2}}\,. \tag{A.14}$$

Then, for all $f \in W^{1,2}$ and $g \in L^2$ we write

$$\int_{S^1} f\, (U^n g)\, d\mu = \int_{S^1} f\, h\, (U^n g)\, dx = \int_{S^1} (\mathcal{L}^n (fh))\, g\, dx =$$

$$= \int_{S^1} (K_1^n (fh))\, g\, dx + \int_{S^1} (\tilde{R}^n (fh))\, g\, dx = \int_{S^1} k_1(fh)\, h\, g\, dx + \int_{S^1} (\tilde{R}^n (fh))\, g\, dx =$$

$$= \Big(\int_{S^1} f\, d\mu \Big) \Big(\int_{S^1} g\, d\mu \Big) + \int_{S^1} (\tilde{R}^n (fh))\, g\, dx\,.$$

Since,

$$\Big| \int_{S^1} (\tilde{R}^n (fh))\, g\, dx \Big| \le \|\tilde{R}^n (fh)\|_{W^{1,2}}\, \|g\|_2 \le \|\tilde{f} h\|_{W^{1,2}}\, \|g\|_2 \le c\, \|h\|_{W^{1,2}}\, \vartheta^n\, \|f\|_{W^{1,2}}\, \|g\|_2\,,$$

where we have used (A.14), we set $C = c\,\|h\|_{W^{1,2}}$ to obtain

$$|C_T(f,g,n)| = \Big| \int_{S^1} f\,(U^n g)\,d\mu - \Big(\int_{S^1} f\,d\mu\Big)\Big(\int_{S^1} g\,d\mu\Big)\Big| \le C\,\|f\|_{W^{1,2}}\,\|g\|_2\,\vartheta^n\,.$$

Step 5. The map T is mixing, hence ergodic, with respect to the probability measure μ. By Proposition 6.4, it is enough to show that

$$\lim_{n\to\infty} C_T(f,g,n) = 0\,, \quad \forall\, f,g \in L^2\,.$$

We know that the previous limit holds with exponential decay if $f \in W^{1,2}$. Since $W^{1,2}$ is dense in L^2, for all $f \in L^2$ and for all $\varepsilon > 0$ there exists $f_0 \in W^{1,2}$ such that $\|fh - f_0\|_2 < \varepsilon$. Hence,

$$\Big| \int_{S^1} f\,d\mu - \int_{S^1} f_0\,dx \Big| \le \int_{S^1} |fh - f_0|\,dx < \varepsilon\,,$$

and, for all $g \in L^2$ and all $n \in \mathbb{N}$,

$$\Big| \int_{S^1} f\,(U^n g)\,d\mu - \int_{S^1} (\mathcal{L}^n f_0)\,g\,dx \Big| \le \int_{S^1} |\mathcal{L}^n(fh) - \mathcal{L}^n f_0|\,g\,dx \le \|\mathcal{L}^n(fh) - \mathcal{L}^n f_0\|_2\,\|g\|_2 < \varepsilon\,C_1\,\|g\|_2\,,$$

where we have used (A.11). Therefore, for all $n \in \mathbb{N}$, we write

$$\begin{aligned} |C_T(f,g,n)| &\le \Big| \int_{S^1} f\,(U^n g)\,d\mu - \int_{S^1} (\mathcal{L}^n f_0)\,g\,dx \Big| + \Big| \int_{S^1} f\,d\mu - \int_{S^1} f_0\,dx \Big|\,\|g\|_2 + \\ &\quad + \Big| \int_{S^1} (\mathcal{L}^n f_0)\,g\,dx - \Big(\int_{S^1} f_0\,dx\Big)\Big(\int_{S^1} g\,d\mu\Big)\Big| < \\ &\quad < (1 + C_1)\,\varepsilon\,\|g\|_2 + C\,\|f_0\|_{W^{1,2}}\,\|g\|_2\,\vartheta^n\,, \end{aligned}$$

where $C > 0$ and $\vartheta \in (0,1)$ are as in Step 4. This proves that for all $f,g \in L^2$

$$\lim_{n\to\infty} |C_T(f,g,n)| < (1 + C_1)\,\varepsilon\,\|g\|_2\,, \quad \forall\, \varepsilon > 0\,,$$

hence the result of this last step and the theorem are proved. □

In order to introduce the difficulties one needs to face when studying Anosov flows, here we sketch the proof of the exponential decay of correlations via the transfer operator method for the simpler case of a hyperbolic discrete dynamical system. Namely, we consider maps for which the tangent space has a splitting in stable and unstable directions. The existence of the central direction for Anosov flows will make the argument more difficult as explained in Section 6.1.

Here, we consider the simplest possible example of a hyperbolic discrete system, the *toral automorphisms*.

Theorem A.3. *Let $A \in SL(2,\mathbb{Z})$ be symmetric with eigenvalues $\lambda > 1$ and $\lambda^{-1} \in (0,1)$, and orthogonal eigenvectors v^u and v^s, respectively. Let $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ be the toral automorphism defined by*

$$\mathbb{T}^2 \ni (x,y) \longmapsto T_A(x,y) := A \begin{pmatrix} x \\ y \end{pmatrix} \; (\textit{mod}\ \mathbb{Z}^2)\,.$$

Then, the Lebesgue measure m on $\mathbb{T}^2$ is a T_A-invariant probability measure, and T_A is ergodic and mixing with respect to m. Moreover, T_A has exponential decay of correlations for couples of observables in $C^\infty(\mathbb{T}^2)$.

Proof. Since $A \in SL(2,\mathbb{Z})$, the map T_A is continuous, invertible, and preserves the area. Hence, the Lebesgue measure m is T_A-invariant. By Proposition 6.4, the mixing property of T_A can be proved by checking that

$$\lim_{n\to\infty} C_{T_A}(f,g,n) = 0\,, \quad \forall\, f,g \in L^2(\mathbb{T}^2,\mathbb{C})$$

directly. For all $f \in L^2(\mathbb{T}^2, \mathbb{C})$, we can use the Fourier series expansion and write

$$f(x,y) = \sum_{(n,m)\in\mathbb{Z}^2} \hat{f}_{n,m}\, e^{2\pi i(nx+my)} .$$

Then, it is enough to consider the case $f(x,y) = e^{2\pi i(nx+my)}$ and $g(x,y) = e^{2\pi i(n'x+m'y)}$ for some $n, n', m, m' \in \mathbb{Z}$. We have

$$C_{T_A}(f,g,k) = \int_{\mathbb{T}^2} \bar{f}\,(U^k g)\, dx\, dy = \int_{\mathbb{T}^2} e^{-2\pi i(nx+my)}\, e^{2\pi i(<(n',m'),T_A^k(x,y)>}\, dx\, dy =$$

$$= \int_{\mathbb{T}^2} e^{-2\pi i(nx+my)}\, e^{2\pi i(<(n',m'),A^k(x,y)>}\, dx\, dy = \int_{\mathbb{T}^2} e^{-2\pi i(nx+my)}\, e^{2\pi i(<A^k(n',m'),(x,y)>}\, dx\, dy =$$

$$= \int_{\mathbb{T}^2} e^{2\pi i(<A^k(n',m')-(n,m),(x,y)>}\, dx\, dy ,$$

where we have used that $T_A^k(x,y) = A^k(x,y) \pmod{\mathbb{Z}^2}$ and that A is symmetric. By the properties of A, for all (n',m') there exist $c_u, c_s \in \mathbb{R}$ such that $(n,m) = c_u\, v^u + c_s\, v^s$ as a vector in $\mathbb{R}^2$. Hence,

$$A^k(n',m') \sim \lambda^k\, c_u\, v^u , \quad \text{as } k \to +\infty.$$

In particular, this implies that for all $n, n', m, m' \in \mathbb{Z}$ there exists $\bar{k} \in \mathbb{N}$ such that $A^k(n',m') \neq (n,m)$ for all $k \geq \bar{k}$. In conclusion,

$$C_{T_A}(f,g,k) = \int_{\mathbb{T}^2} \bar{f}\,(U^k g)\, dx\, dy = 0 , \quad \forall\, k \geq \bar{k} ,$$

and this proves that T_A is mixing with respect to m.

We now use the transfer operator method to show exponential decay of correlations. We follow [DKL21, Section 3.4]. Since $\det(A) = 1$, defining the transfer operator $\mathcal{L}$ as in (A.4), we have that for all $f \in L^1$ and $g \in L^\infty$,

$$\int_{\mathbb{T}^2} f\,(Ug)\, dx\, dy = \int_{\mathbb{T}^2} (\mathcal{L}f)\, g\, dx\, dy \quad \Rightarrow \quad \mathcal{L}f = f \circ T_A^{-1} .$$

Arguing as in the proof of Theorem A.2, we expect that $\mathcal{L}$ regularizes observables in the expanding direction spanned by v^u at each point $(x,y) \in \mathbb{T}^2$, but the contrary happens in the contracting one. Therefore, we need to look at norms which capture high order derivability in the v^u direction and high order nice distributional behavior in the v^s direction.

Let[10] $f \in C^\infty$. We use the following notations

$$\partial_u f := < v^u, \nabla f > \quad \text{and} \quad \partial_s f := < v^s, \nabla f > .$$

Fix a $\delta > 0$ and consider the space of real functions in $C_0^\infty([-\delta,\delta])$. For all $q \in \mathbb{N}_0$, let $\psi^{(q)}$ denote the q-th derivative for $\psi \in C_0^\infty([-\delta,\delta])$. Then, we set

$$\|\psi\|_{C^q} := \max_{0\leq q'\leq q} \|\psi^{(q')}\|_\infty .$$

Let's now introduce the Banach spaces we will use to apply Theorem A.1. For all $p, q \in \mathbb{N}_0$ and $f \in C^\infty$, consider the norm

$$\|f\|_{p,q} := \sup_{(x,y)\in\mathbb{T}^2}\ \sup_{0\leq p'\leq p} \Big(\sup_{\psi\in C_0^\infty([-\delta,\delta]),\|\psi\|_{C^q}\leq 1} \int_{-\delta}^{\delta} \partial_u^{p'} f((x,y)+tv^s)\, \psi(t)\, dt \Big) . \tag{A.15}$$

Then, we define $\mathcal{B}^{p,q}$ to be the Banach space obtained by completing C^∞ with respect to $\|\cdot\|_{p,q}$. The spaces $\mathcal{B}^{p,q}$ have the properties stated in the next lemma. For the proof we refer to [DKL21, Lemmas 3.8-3.9-3.10].

[10] When not specified we refer to real functions defined on $\mathbb{T}^2$.

Lemma A.4. *For all $p, q \in \mathbb{N}_0$ the spaces $\mathcal{B}^{p,q}$ satisfy:*

(i) If $p > 0$, then $\mathcal{B}^{p,q}$ is continuously embedded in $(C^q)^*$, the dual of C^q.

(ii) The operator ∂_u is bounded as an operator from $\mathcal{B}^{p+1,q}$ to $\mathcal{B}^{p,q}$, with kernel given by the constant functions.

(iii) If $p > 0$, the space $\mathcal{B}^{p,q}$ embeds compactly in $\mathcal{B}^{p-1,q+1}$.

For our aims, it is enough to fix $p = 1$ and $q = 0$.

Step 1. If in Theorem A.1 we choose $(\mathcal{B}_w, |\cdot|_w) = (\mathcal{B}^{0,1}, \|\cdot\|_{0,1})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}}) = (\mathcal{B}^{1,0}, \|\cdot\|_{1,0})$, then (A.1) *and* (A.2) *hold.*

Let $f \in C^\infty$. First, we want to estimate $\|\mathcal{L}^j f\|_{0,1}$ for all $j \in \mathbb{N}$. For all $\psi \in C^\infty([-\delta, \delta])$, since T^{-1} stretches the lines in the v^s direction by a factor λ, we have

$$\begin{aligned}\int_{-\delta}^{\delta} \mathcal{L}^j f((x,y) + tv^s)\, \psi(t)\, dt &= \int_{-\delta}^{\delta} f(T^{-j}(x,y) + t\lambda^j v^s)\, \psi(t)\, dt \\ &= \lambda^{-j} \int_{-\delta\lambda^j}^{\delta\lambda^j} f(T^{-j}(x,y) + tv^s)\, \psi(\lambda^{-j} t)\, dt\,,\end{aligned}$$

which can be reduced to the sum of $c_1 \cdot \lambda^j$ terms, for a fixed constant $c_1 > 0$, of the form

$$\int_{t_i-\delta}^{t_i+\delta} f((x,y) + tv^s)\, \psi(\lambda^{-j} t)\, \psi_i(t)\, dt\,,$$

with $\{\psi_i\}_i$ a C^∞ partition of unity with ψ_i having supports of size δ and $\|\psi_i\|_{C^1} \le c_2$. Then, we find

$$\Big| \int_{-\delta}^{\delta} \mathcal{L}^j f((x,y) + tv^s)\, \psi(t)\, dt \Big| \le c_1\, \|f\|_{0,1}\, \|\psi\|_{C^1}\,.$$

Hence,

$$\|\mathcal{L}^j f\|_{0,1} \le c_1\, \|f\|_{0,1}\,, \quad \forall\, j \in \mathbb{N}\,.$$

This proves (A.1) with $M = 1$.

Let's now consider $\|\mathcal{L}^j f\|_{1,0}$. For all $\psi \in C^\infty([-\delta, \delta])$, we estimate the integrals

$$\int_{-\delta}^{\delta} \partial_u^{p'} (\mathcal{L}^j f)((x,y) + tv^s)\, \psi(t)\, dt\,, \quad \text{for } p' = 0, 1.$$

When $p' = 0$, we follow the same computations as above, but at the end we use, for all i, the estimate

$$\Big| \int_{t_i-\delta}^{t_i+\delta} f((x,y) + tv^s)\, \psi(\lambda^{-j} t)\, \psi_i(t)\, dt \Big| \le c_2\, \|\psi\|_\infty\, \|f\|_{0,1}\,.$$

Let's now set $p' = 1$. We recall that, being A symmetric, $\nabla(f \circ T^{-j}) = A^{-j}\, \nabla f$, hence

$$\partial_u(\mathcal{L}^j f) = < v^u, \nabla(f \circ T^{-j}) > = < A^{-j} v^u, (\nabla f) \circ T^{-j} > = \lambda^{-j}\, (\partial_u f) \circ T^{-j}\,. \tag{A.16}$$

Therefore,

$$\begin{aligned}\Big| \int_{-\delta}^{\delta} \partial_u(\mathcal{L}^j f)((x,y) + tv^s)\, \psi(t)\, dt \Big| &= \lambda^{-j} \Big| \int_{-\delta}^{\delta} \mathcal{L}^j (\partial_u f)((x,y) + tv^s)\, \psi(t)\, dt \Big| \le \\ &\le c_1\, \lambda^{-j}\, \|\partial_u f\|_{0,0}\, \|\psi\|_\infty \le c_1\, \lambda^{-j}\, \|f\|_{1,0}\, \|\psi\|_\infty\,.\end{aligned}$$

We have thus proved

$$\|\mathcal{L}^j f\|_{1,0} \le c_1\, \lambda^{-j}\, \|f\|_{1,0} + c_2\, \|f\|_{0,1}\,, \quad \forall\, j \in \mathbb{N}\,.$$

This proves (A.2) with $M = 1$ and $\kappa = \lambda^{-1} \in (0,1)$.

Step 2. The spectrum of $\mathcal{L}$ satisfies spec$(\mathcal{L}|_{\mathcal{B}^{1,0}}) \cap \{|\zeta| > \lambda^{-1}\} = \{1\}$.

By Step 1, we have $spec(\mathcal{L}|_{\mathcal{B}^{0,q}}) \subseteq B(0,1)$ for all $q \in \mathbb{N}_0$. Let $f \in \mathcal{B}^{1,0}$, $f \neq 0$ satisfy $\mathcal{L}f = \ell f$ for some $\ell \in \left\{|\zeta| > \lambda^{-1}\right\}$. Then,

$$\ell\, \partial_u f = \partial_u(\mathcal{L}f) = \lambda^{-1}\, \mathcal{L}(\partial_u f)$$

as shown in (A.16). Hence, $\partial_u f \in \mathcal{B}^{0,0}$ by Lemma A.4-(ii) and it is an eigenfunction of $\mathcal{L}$ with eigenvalue $\ell\lambda \in \{|\zeta| > 1\}$. Since $spec(\mathcal{L}|_{\mathcal{B}^{0,0}}) \subseteq B(0,1)$, it follows that $\partial_u f = 0$. Therefore, by Lemma A.4-(ii), f is constant. This implies $\ell = 1$.

Step 3. The map T_A has exponential decay of correlations for couples of observables in $C^\infty(\mathbb{T}^2)$.
We can repeat the argument in Theorem A.2 - Step 4, to get

$$|C_{T_A}(f, g, n)| \leq \max\{c_1, c_2\}\, \|f\|_{1,0}\, \|g\|_\infty\, \lambda^{-n}\,.$$

This concludes the proof. $\square$

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