Variabili aleatorie

Discrete $\quad X : (\Omega, \mathbb{P}) \longrightarrow \mathbb{R}$

universo

$$\text{Imm } X = \{ x_1, x_2, x_3, \dots \}$$

$$\text{Legge di } X = \{ p_1, p_2, p_3, \dots \} \qquad p_i = p(x_i) := \mathbb{P}\{ X = x_i \}$$
$$= \mathbb{P}\{ \omega \in \Omega : X(\omega) = x_i \}$$

Funzione di ripartizione $\quad F : \mathbb{R} \to [0,1]$

$$F(t) := \mathbb{P}\{ X \leq t \} =$$
$$= \mathbb{P}\{ \omega \in \Omega : X(\omega) \leq t \}$$

Def Media (o speranza) di X v.a. discreta è

$$E[X] := \sum_i x_i \, p_i \quad \left(= \sum_i x_i \, \mathbb{P}\{ X = x_i \} \right)$$

"expectation"

Varianza di X v.a. discreta è

$$\text{var}(X) := \sum_i (x_i - E[X])^2 p_i = \sum_i x_i^2 p_i - (E[X])^2$$

$$\Big(= \sum_i \left(x_i^2 - 2x_i E[X] + E[X]^2 \right) p_i =$$

$$= \sum_i x_i^2 p_i - 2E[X] \underbrace{\sum_i x_i p_i}_{= E[X]} + E[X]^2 \underbrace{\sum_i p_i}_{= 1} \Big)$$

Scarto quadratico medio (o deviazione standard) di X v.a.

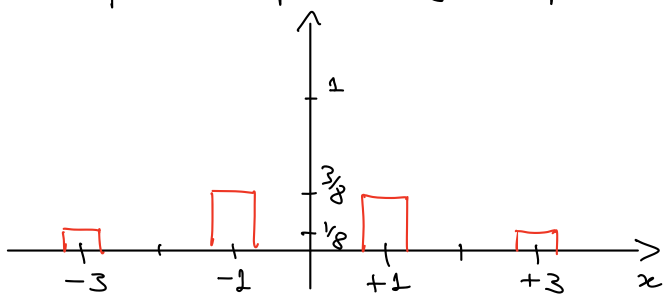
$$\sigma(X) := \sqrt{\text{var}(X)}$$

OSS $\quad$ Momento k-esimo di X v.a. discreta, $k \in \mathbb{N}$,

$$E[X^k] := \sum_i p_i \, x_i^k$$

Ad es, $\mathrm{var}(X) = E[X^2] - (E[X])^2$.

ES $\quad X$ con legge $\quad p(-3) = p(+3) = \frac{1}{8}, \quad p(-1) = p(1) = \frac{3}{8}$.



$$E[X] = \big((-3)\cdot p(-3) + (-1)\cdot p(-1) + 1\cdot p(1) + 3\cdot p(3)\big) =$$
$$= -\frac{3}{8} - \frac{3}{8} + \frac{3}{8} + \frac{3}{8} = 0$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = E[X^2] - 0 =$$
$$= \big((-3)^2\cdot p(-3) + (-1)^2\cdot p(-1) + 1^2\cdot p(1) + 3^2\cdot p(3)\big) =$$
$$= 9\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 1\cdot\frac{3}{8} + 9\cdot\frac{1}{8} = \frac{24}{8} = 3$$

$$\sigma(X) = \sqrt{3}$$

$\longleftrightarrow$

Con densità $\quad X : (\underbrace{\Omega}_{\text{universo}}, \mathbb{P}) \longrightarrow \mathbb{R}$, $\quad \mathrm{Imm}\, X = (a,b) \subseteq \mathbb{R}$

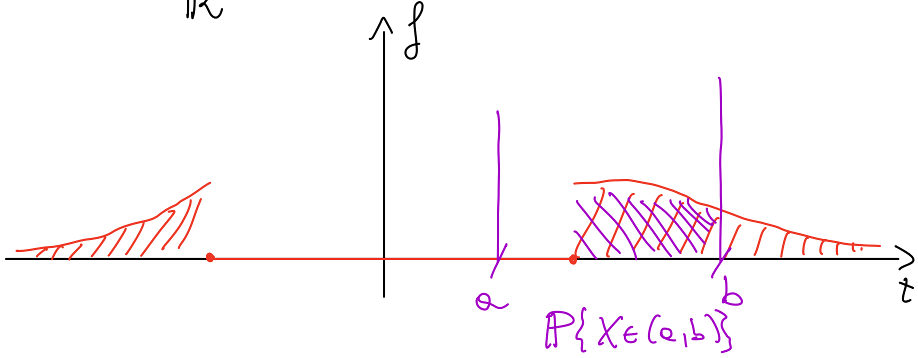
$a \in \mathbb{R} \cup \{-\infty\}$

$b \in \mathbb{R} \cup \{+\infty\}$

Densità è $f : \mathbb{R} \to [0,+\infty)$ tale che:

(i) f è integrabile $\quad (\Leftarrow f$ è continua tranne che al più in un numero finito di punti$)$

(ii) $\displaystyle\int_{\mathbb{R}} f(t)\,dt = 1 \quad$ (= Area "sotto" il grafico di f)



Legge di X v.a. con densità f è

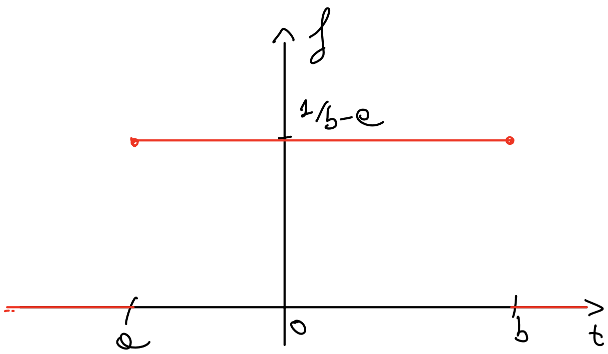
$\forall a,b \in \mathbb{R}^*$, $\quad \mathbb{P}(X \in (a,b)) = \int_a^b f(t)\,dt$ (= Area "sotto" il grafico di f tra $t=a$ e $t=b$).

$\mathbb{P}(X \in (a,b)) = \mathbb{P}\{\omega \in \Omega : X(\omega) \in (a,b)\}$

<u>OSS</u> $\forall x \in \mathbb{R}$, $\mathbb{P}\{X = x\} = 0$ se X è v.a. con densità.

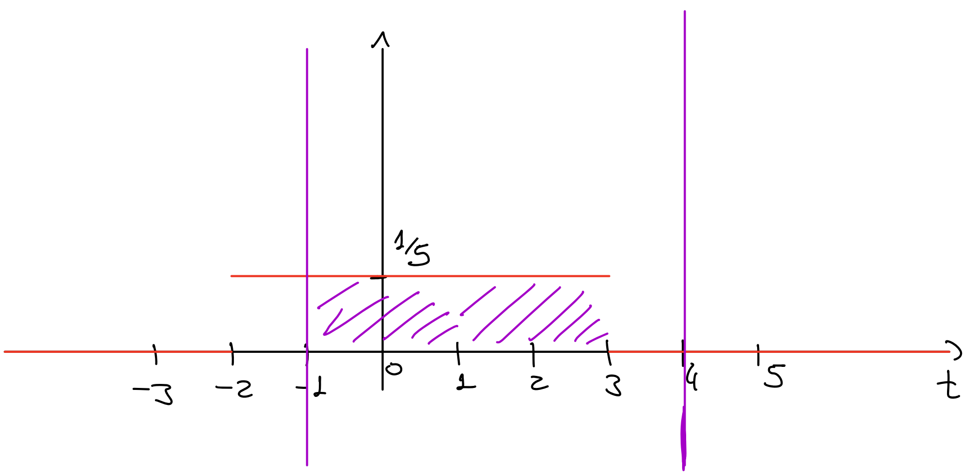
<u>ES</u> • X con densità uniforme.

$$f(t) = \begin{cases} \frac{1}{b-a} & , t \in [a,b] \\ 0 & , t \notin [a,b] \end{cases}$$



$a = -2$, $b = 3$, $\quad \frac{1}{b-a} = \frac{1}{5}$.

$$\underbrace{\mathbb{P}\{X \in (-1,4)\}} = \int_{-1}^{4} f(t)\,dt = \frac{4}{5}$$



<u>Def</u> Funzione di ripartizione $F : \mathbb{R} \to [0,1]$, $F(t) = \mathbb{P}\{X \le t\}$

Media di X v.a. con densità

$$E[X] := \int_{\mathbb{R}} t\, f(t)\,dt$$

Varianza di X v.a. con densità

$$\operatorname{var}(X) := \int_{\mathbb{R}} t^2 f(t)\,dt - (E[X])^2$$

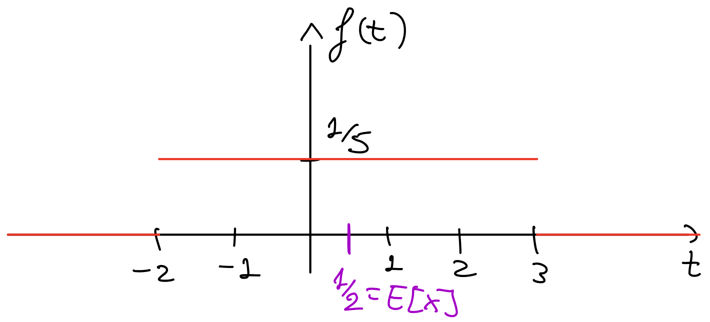
Scarto quadratico medio di X

$$\sigma(X) = \sqrt{\operatorname{var}(X)}$$

ES X v.a. con densità uniforme $f(t) = \begin{cases} 1/5, & t \in [-2,3] \\ 0, & t \notin [-2,3] \end{cases}$

$$E[X] = \int_{\mathbb{R}} t f(t)\,dt = \int_{-2}^{3} t f(t)\,dt = \int_{-2}^{3} t \cdot \frac{1}{5}\,dt =$$

$$= \frac{1}{10}\left(3^2 - (-2)^2\right) = \frac{1}{10}(9-4) = \frac{1}{2}$$



$$\operatorname{var}(X) = \int_{\mathbb{R}} t^2 f(t)\,dt - (E[X])^2 = \int_{-2}^{3} t^2 \cdot \frac{1}{5}\,dt - \left(\frac{1}{2}\right)^2 =$$

$$= \frac{1}{15}\left(3^3 - (-2)^3\right) - \frac{1}{4} = \frac{7}{3} - \frac{1}{4} = \frac{25}{12}$$

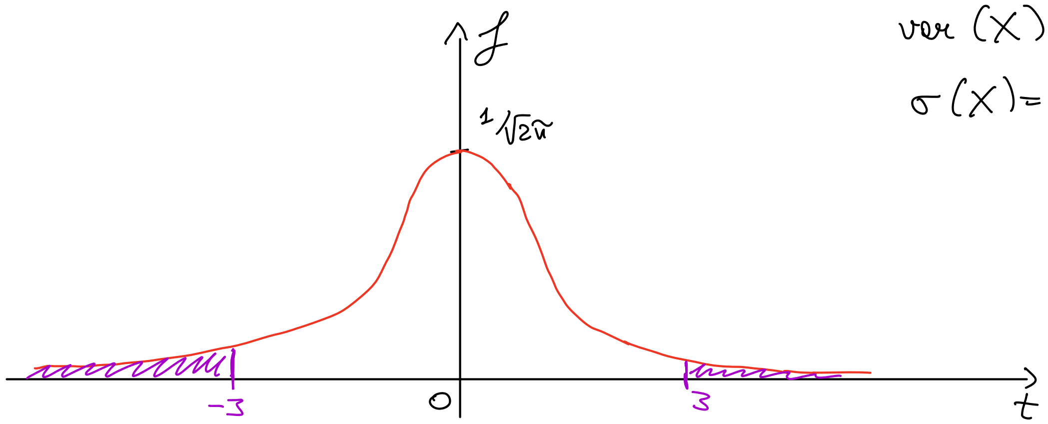
ES X v.a. con densità normale (o gaussiana) standard, $N(0,1)$

$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

$E[X] = 0$

$\operatorname{var}(X) = 1$

$\sigma(X) = 1$

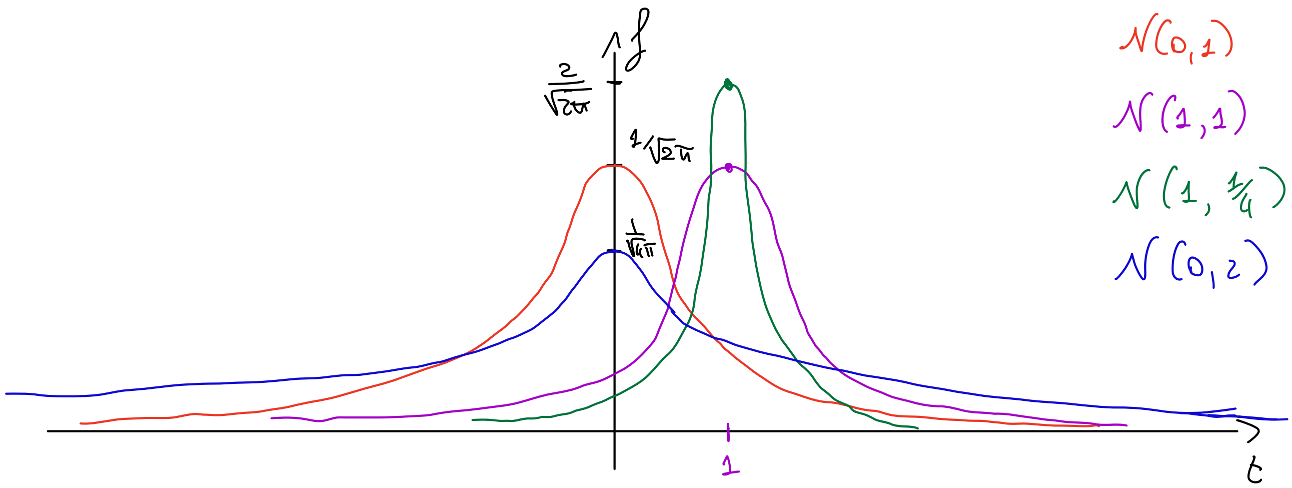


$$\underbrace{\mathbb{P}\{\,|X - E[X]| > 3\sigma(X)\}}\ \underset{\substack{\uparrow\\ E[X]=0\\ \sigma(X)=1}}{=}\ \mathbb{P}\{X < -3\} + \mathbb{P}\{X > +3\} \sim 0{,}003$$

- X v.a. con densità normale (o gaussiana) con media m e varianza σ^2

$$\mathcal{N}(m, \sigma^2)$$

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(t-m)^2}{2\sigma^2}}$$



Siano X_1 e X_2 v.a con densità normale, $X_1 \sim \mathcal{N}(m_1, \sigma_1^2)$

$$X_2 \sim \mathcal{N}(m_2, \sigma_2^2)$$

Se X_1 e X_2 sono indipendenti allora

$$X_1 + X_2 \sim \mathcal{N}(m_1 + m_2,\ \sigma_1^2 + \sigma_2^2)$$

$$E[X_1 + X_2] = E[X_1] + E[X_2], \quad \mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2)$$