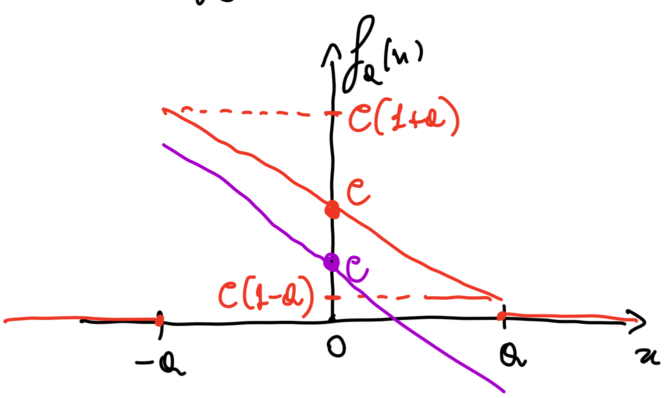
Esercizio 1

Sia $a>0$ e consideriamo la funzione

$$f_a(x) = \begin{cases} c(1-x), & -a < x < a \\ 0, & \text{altrimenti} \end{cases}$$

(a) Per quali valori di $a \in (0,+\infty)$ e di $c \in \mathbb{R}$, la f_a è la densità di una v.a.?

f_a è densità $\iff$ $\underline{f_a \geq 0}$, $\underline{f_a \text{ int.}}$, $\int_{-\infty}^{+\infty} f_a(x)\,dx = 1$



$$\left.\begin{array}{l} f_a \geq 0 \Rightarrow f_a(0) = c \geq 0 \\ c \neq 0 \text{ alt. } f_a \equiv 0 \end{array}\right] \Rightarrow \boxed{c>0}$$

$f_a'(x) = -c \quad \forall x \in (-a,a)$, garantisce che

$$f_a(a) \geq 0 \Rightarrow f_a(x) \geq 0 \quad \forall x \in (-a,a)$$

$$c(1-a) \geq 0 \Rightarrow \boxed{a \leq 1}$$

$$1 = \int_{-\infty}^{+\infty} f_a(x)\,dx = \int_{-a}^{a} c(1-x)\,dx = \left(cx - \frac{1}{2}cx^2\right)\Big|_{-a}^{a} =$$

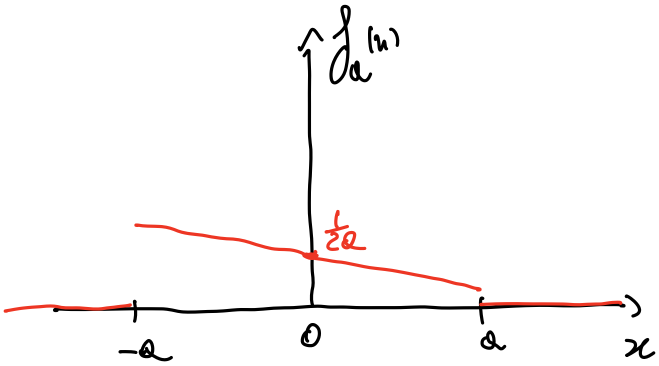
$$= \left(ca - \frac{1}{2}ca^2\right) - \left(-ca - \frac{1}{2}ca^2\right) = 2ca$$

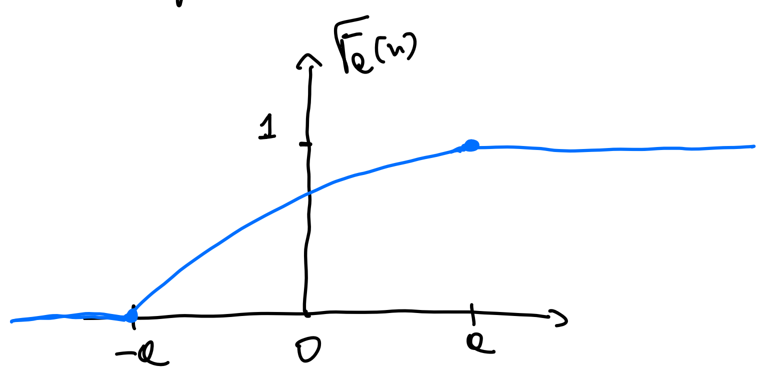
$$\Rightarrow \boxed{c = \frac{1}{2a} > 0}$$

f_a è densità $\forall\, a \in (0,1]$ e $c = \frac{1}{2a}$.

$$f_a(x) = \begin{cases} \frac{1}{2a}(1-x), & -a < x < a \\ 0, & \text{alt.} \end{cases}$$

(b) Ricavare la funz. di ripartizione $F_a(x)$





$$F_a(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_a(t)\,dt \qquad \left(F_a'(x) = f_a(x) \quad \forall x \notin \{-a, a\} \right)$$

$\forall\, x \in (-a, a)$
$$F_a(x) = F_a(-a) + \int_{-a}^{x} f_a(t)\,dt =$$
$$= 0 + \int_{-a}^{x} \frac{1}{2a}(1-t)\,dt =$$
$$= \frac{1}{2a}\left(t - \frac{1}{2}t^2\right)\Big|_{-a}^{x} = \frac{1}{2a}x\left(1 - \frac{1}{2}x\right) - \frac{1}{2a}\left(-a - \frac{1}{2}a^2\right)$$
$$= \frac{x(2-x)}{4a} + \frac{2+a}{4}$$

$$F_a(-a) = \frac{-a(2+a)}{4a} + \frac{2+a}{4} = 0$$

$$F_a(a) = \frac{a(2-a)}{4a} + \frac{2+a}{4} = \frac{a(2-a) + a(2+a)}{4a} = 1$$

(c) Studiamo i momenti $E[X^k]$.

Per quali $k \in \mathbb{N}$, $E[|X|^k] < +\infty$? Per ogni $k \in \mathbb{N}$ perché $|f_a(x)| \leq \frac{1}{2a}(1+a)$ $\forall x$, e $f_a \neq 0$ su un intervallo limitato $[-a, a]$.

$$E[|X|^k] = \int_{-\infty}^{+\infty} |x|^k f_a(x)\,dx = \int_{-a}^{a} |x|^k f_a(x)\,dx \leq \frac{a^k(1+a)}{2a} \cdot 2a \qquad \forall k \in \mathbb{N}$$

$$E[X^k] = \int_{-\infty}^{+\infty} x^k f_a(x)\,dx = \int_{-a}^{a} x^k \frac{1}{2a}(1-x)\,dx =$$

$$= \frac{1}{2a}\left(\frac{1}{k+1}x^{k+1} - \frac{1}{k+2}x^{k+2}\right)\Bigg|_{-a}^{a} =$$

$$= \frac{1}{2a}\left[\frac{a^{k+1} - (-a)^{k+1}}{k+1} - \frac{a^{k+2} - (-a)^{k+2}}{k+2}\right]$$

<u>$k=1$</u>
$$E[X] = \frac{1}{2a}\left[\frac{a^2 - a^2}{2} - \frac{a^3 - (-a^3)}{3}\right] = -\frac{a^2}{3}$$

$$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2 =$$

$$= \frac{a^2}{3} - \frac{a^4}{9} = \frac{a^2}{3}\left(1 - \frac{a^2}{3}\right).$$

(d) Calcolare $\mathbb{P}(X^2 > l)$ al variare di $l \in \mathbb{R}$.

Se $Y = X^2$, $\mathbb{P}(X^2 > l) = 1 - \mathbb{P}(X^2 \leq l) =$

$= 1 - F_Y(l)$.

Calcolare $F_Y(y)$ $\begin{cases} \text{diretta}, & \mathbb{P}(X^2 \leq y) \\ \text{indiretta}, & \text{calcolare } f_Y \text{ e poi } F_Y. \end{cases}$

È conveniente calcolare f_Y se $Y = h(X)$ con h funzione invertibile e con inversa derivabile su $\mathrm{Imm}\, X$.

In questo caso, $Y = X^2$, $h(x) = x^2$ su $[-a, a]$ e h non è invertibile su $[-a, a]$.

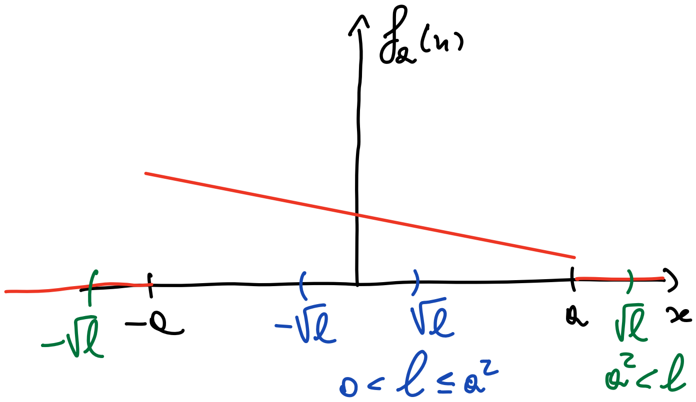
$$f_Y(y) = \begin{cases} f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right|, & \forall y \in h(\mathrm{Imm}\, X) \\ 0, & \text{alt.} \end{cases}$$

Calcoliamo direttamente $\mathbb{P}(X^2 > l)$.

$$\begin{cases} \mathbb{P}(X^2 > l) \Leftrightarrow \mathbb{P}(X > \sqrt{l}) \cup \mathbb{P}(X < -\sqrt{l}), & l \geq 0 \end{cases}$$

$$\mathbb{P}(X^2 > \ell) = 1 \qquad \forall \ell \leq 0$$

Se $\ell > 0$ $\qquad \mathbb{P}(X > \sqrt{\ell})$, $\mathbb{P}(X < -\sqrt{\ell})$



$\underline{0 < \ell \leq a^2}$

$$\mathbb{P}(X > \sqrt{\ell}) = \int_{\sqrt{\ell}}^{+\infty} f_a(x)\,dx = \int_{\sqrt{\ell}}^{a} \frac{1}{2a}(1-x)\,dx =$$

$$= \frac{1}{2a}\left(x - \frac{1}{2}x^2\right)\Big|_{\sqrt{\ell}}^{a} = \frac{a - \frac{1}{2}a^2}{2a} - \frac{1}{2a}\left(\sqrt{\ell} - \frac{1}{2}\ell\right)$$

$$\mathbb{P}(X < -\sqrt{\ell}) = \int_{-\infty}^{-\sqrt{\ell}} f_a(x)\,dx = \int_{-a}^{-\sqrt{\ell}} \frac{1}{2a}(1-x)\,dx =$$

$$= \frac{1}{2a}\left(x - \frac{1}{2}x^2\right)\Big|_{-a}^{-\sqrt{\ell}} = \frac{1}{2a}\left(-\sqrt{\ell} - \frac{1}{2}\ell\right) - \frac{1}{2a}\left(-a - \frac{1}{2}a^2\right)$$

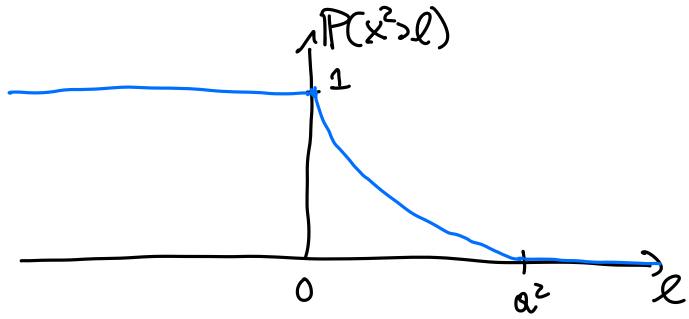
$$\mathbb{P}(X^2 > \ell) = 1 - \frac{\sqrt{\ell}}{a}$$

$\underline{\ell > a^2}$

$$\mathbb{P}(X > \sqrt{\ell}) = \int_{\sqrt{\ell}}^{+\infty} f(x)\,dx = \int_{\sqrt{\ell}}^{+\infty} 0\,dx = 0$$

$$\mathbb{P}(X < -\sqrt{\ell}) = 0$$

$$\mathbb{P}(X^2 > \ell) = \begin{cases} 1, & \ell \leq 0 \\ 1 - \frac{\sqrt{\ell}}{a}, & \ell \in (0, a^2) \\ 0, & \ell \geq a^2 \end{cases}$$



(e) Stimatori puntuali per $a \in (0,1]$.

$X_1, \dots, X_n$ campione statistico di v.a. indipendenti e con legge data da $F_a(x)$.

- Metodo dei momenti.

Stimiamo a imponendo che $E[X^k] = \frac{1}{n} \sum_{i=1}^{n} x_i^k$.

$k=1$, $E[X] = \bar{x}$

$$-\frac{a^2}{3} = \bar{x} \iff \tilde{a} = \sqrt{-3\bar{x}} \quad \text{ha senso se}$$

$$-3\bar{x} \in (0,1] \iff \bar{x} \in [-1/3, 0)$$

Questo ha senso $\tilde{a}(X_1, \dots, X_n) = \sqrt{-3\bar{X}}$ se $\bar{X} \in [-1/3, 0)$

$k=2$, $E[X^2] = \frac{1}{n} \sum_{i=1}^{n} x_i^2$

$$\frac{a^2}{3} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \iff \tilde{a} = \sqrt{\frac{3}{n}\sum_{i=1}^{n} x_i^2} \in (0,1]$$

Se $\bar{X} \notin [-\tfrac{1}{3}, 0)$, usiamo

$$\tilde{a}(X_1,\dots,X_n) = \sqrt{\frac{3}{n}\sum_{i=1}^{n} X_i^2} \quad \text{se} \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 \in \left(0, \tfrac{1}{3}\right]$$

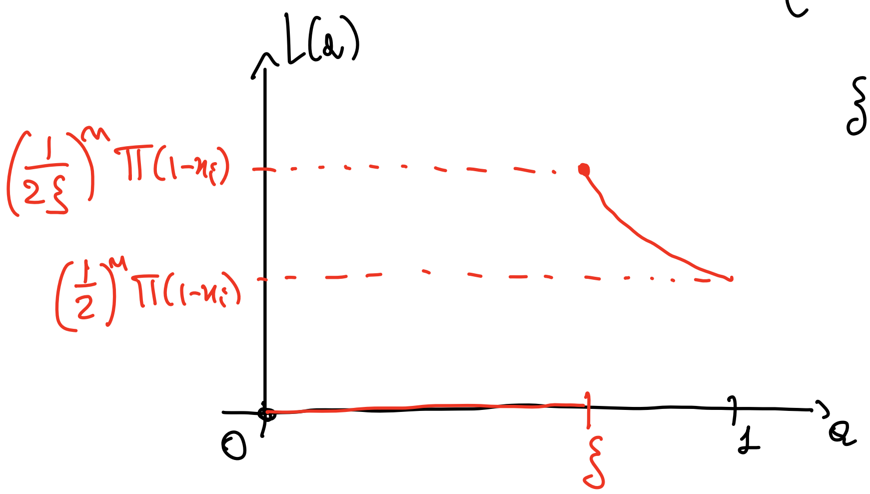
– Metodo di massima verosimiglianza

$$(0,1] \ni a \longmapsto L(a; x_1,\dots,x_n) = \prod_{i=1}^{n} f_a(x_i) = \begin{cases} \left(\frac{1}{2a}\right)^n \prod_{i=1}^{n} (1-x_i), & \text{se } x_i \in (-a,a)\ \forall i \\ 0, & \text{se } \exists\, x_i \notin (-a,a) \end{cases}$$

$$-a < x_i < a \ \forall i \iff a > x_i \text{ e } a > -x_i \ \forall i$$

$$\iff a > |x_i| \ \forall i \iff a > \max\{|x_1|,\dots,|x_n|\}$$

$$(0,1] \ni a \longmapsto L(a; x_1,\dots,x_n) = \begin{cases} \left(\frac{1}{2a}\right)^n \prod_{i=1}^{n} (1-x_i), & a > \max\{|x_1|,\dots,|x_n|\} \\ 0, & \text{alt.} \end{cases}$$



$$\xi = \max\{|x_1|,\dots,|x_n|\} < 1$$

$$\hat{a} = \max\{|x_1|,\dots,|x_n|\}$$

$$\hat{a}(X_1,\dots,X_n) = \max\{|X_1|,\dots,|X_n|\}$$

Se $\max\{|x_1|,\dots,|x_n|\} \geq 1$, il metodo fallisce.

Esercizio 2 Sia dato un campione di 80 barre di lunghezze $x_1,\dots,x_{80}$, con $\bar{x} = 10.6$ cm e $\bar{\sigma} = 1.3$ cm.

(a) Stimare la prob. che una barra abbia lunghezza > 12 cm.

$$X_1,\dots,X_{80}, \quad X_k \sim N(\mu,\sigma^2) \quad \mu,\sigma \text{ ignote.}$$

$$\mathbb{P}(X_k > 12) = \mathbb{P}(\sigma Z + \mu > 12)$$

$$X_k = \sigma Z + \mu, \quad Z \sim N(0,1)$$

Poniamo $\mu = \bar{X}(\omega) = \bar{x}$, $\sigma^2 = \begin{cases} S^2(\omega) = \bar{\sigma}^2 \\ \frac{n-1}{n} S^2(\omega) = \frac{n-1}{n}\bar{\sigma}^2 \end{cases}$

$$\mathbb{P}(X_k > 12) \sim \mathbb{P}\left(Z > \frac{12-\bar{x}}{\bar{\sigma}}\right) = \mathbb{P}\left(Z > \frac{12-10.6}{1.3}\right) =$$

$$= 1 - \Phi\left(\frac{12-10.6}{1.3}\right) \sim 0.141$$

(b) Stimare la prob. che su 200 barre prodotte dalla ditta, al più 25 siano lunghe più di 12 cm.

$Y_1,\dots,Y_{200}$, $Y_k \sim B(1,p)$ con $Y_k = 1 \iff$ barra più lunga di 12 cm

$p \sim 0.141$ $\qquad$ $Y_k = 0 \iff$ alt.

$$\mathbb{P}(Y_1 + \dots + Y_{200} \le 25) = \mathbb{P}\left(\frac{Y_1 + \dots + Y_{200} - 200\, E[Y_k]}{\sqrt{200\, Var(Y_k)}} \le \frac{25 - 200\, E[Y_k]}{\sqrt{200\, Var(Y_k)}} \right)$$

$$\approx \mathbb{P}\left(Z \le \frac{25 - 200 \cdot 0.141}{\sqrt{200 \cdot 0.141 \cdot (1 - 0.141)}} \right) =$$

$$= \Phi\left(\frac{25 - 200 \cdot 0.141}{\sqrt{200 \cdot 0.141 \cdot 0.859}} \right) \sim 0.258$$

(c) Costruire un intervallo di fiducia bilatero per la lunghezza delle barre al 90% di fiducia.
Prec. relativa ?

$X_1, \dots, X_{80}$, $X_k \sim N(\mu, \sigma^2)$ μ, σ^2 ignote

$$I = \left[\bar{x} - \frac{\bar{\sigma}}{\sqrt{n}} \tau_{(1-\alpha/2,\, n-1)},\ \bar{x} + \frac{\bar{\sigma}}{\sqrt{n}} \tau_{(1-\alpha/2,\, n-1)} \right]$$

$$\text{Prec. stima} = \frac{\bar{\sigma}}{\sqrt{n}} \tau_{(1-\alpha/2,\, n-1)} = \frac{1.3}{\sqrt{80}} \tau_{(0.95, 79)} \sim 0.242$$

$$1-\alpha = 0.9 \iff \alpha = 0.1 \iff 1 - \alpha/2 = 0.95$$

$$I = [10.358,\ 10.842]$$

$$\text{Prec. relativa} = \frac{\text{Prec. stima}}{|\bar{x}|} \sim \frac{0.242}{10.6} \sim 0.023$$
$$\sim 2.3 \cdot 10^{-2}$$

Se volessi prec. relativa $\le 10^{-2}$

$$\Longleftrightarrow \quad \frac{\frac{\bar\sigma}{\sqrt{n}}\, t_{(1-\alpha/2,\, n-1)}}{|\bar x|} \le 10^{-2} \quad \Longleftrightarrow$$

$$t_{(1-\alpha/2,\,79)} \le \frac{\sqrt{80}}{1.3} \cdot 10.6 \cdot 10^{-2}$$

$$\Longleftrightarrow \quad 1-\frac{\alpha}{2} \le F_{79}\left(\frac{\sqrt{80}}{1.3}\, 10.6 \cdot 10^{-2}\right)$$

$$\Longleftrightarrow \quad \alpha \ge 2\left[1 - F_{79}(\quad)\right]$$

$$\Longleftrightarrow \quad 1-\alpha \le 2F_{79}\left(\frac{\sqrt{80}}{1.3}\, 10.6 \cdot 10^{-2}\right) - 1$$

$$\sim 0.532$$

(e) Testare l'ipotesi che la lunghezza sia ≤ 10 cm.

$$\bar x = 10.6 \quad , \quad \bar\sigma = 1.3$$

Test unilatero per la media di un campione gaussiano con varianza ignota.

$$H_0)\ \mu \le 10 \qquad H_1)\ \mu > 10$$

$$C = \left\{ \frac{\sqrt{n}}{S}(\bar X - \mu_0) > t_{(1-\alpha,\, n-1)} \right\} \quad \text{livello } \alpha$$

p-value $\quad \bar\alpha = 1 - F_{n-1}\left(\frac{\sqrt{n}}{\bar\sigma}(\bar x - \mu_0)\right)$

$$= 1 - F_{79}\left(\frac{\sqrt{80}}{1.3}(10.6 - 10)\right)$$

$\sim 4 \cdot 10^{-5}$ da scartare.

Ipotesi $\mu \leq \mu_0$ plausibile , $\bar{\alpha} \geq 0.3$

$$1 - F_{79}\left(\frac{\sqrt{80}}{1.3} (10.6 - \mu_0) \right) \geq 0.3$$

$$\iff F_{79}\left(\frac{\sqrt{80}}{1.3} (10.6 - \mu_0) \right) \leq 0.7$$

$$\iff 10.6 - \mu_0 \leq \frac{1.3}{\sqrt{80}} \tau_{(0.7, 79)}$$

$$\iff \mu_0 \geq 10.6 - \frac{1.3}{\sqrt{80}} \tau_{(0.7, 79)} \sim 10.523$$