Esercizio x casa Costruire gli stimatori puntuali (con i due metodi) per il parametro λ nel caso di $X_1, \dots, X_n$ campione statistico di v.a. di Poisson (λ).

Esempi Supponiamo che $X_1, \dots, X_n$ siano distribuite con densità

$$f_\vartheta(x) = \begin{cases} (\vartheta+1)\, x^\vartheta, & 0 < x < 1 \\ 0, & \text{alt.} \end{cases}$$

Stimiamo $\vartheta \geq 0$.

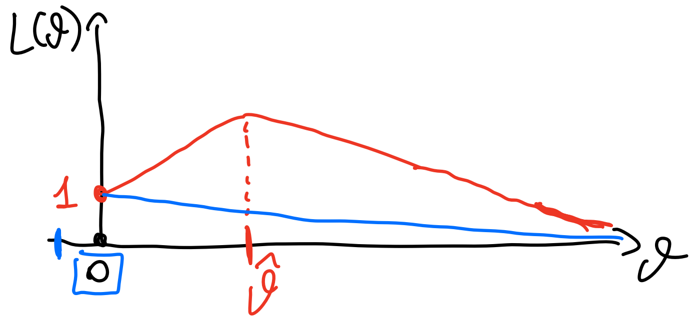
① Massima verosimiglianza

$$[0,+\infty) \ni \vartheta \longmapsto L(\vartheta; x_1, \dots, x_n) = \prod_{i=1}^n f_\vartheta(x_i) =$$

$$= \begin{cases} \prod_{i=1}^n (\vartheta+1)\, x_i^\vartheta, & \text{se } x_i \in (0,1)\ \forall i \\ 0, & \text{se } \exists\, x_i \notin (0,1) \ !! \end{cases}$$

il metodo non funziona

Supponiamo che $x_i \in (0,1)\ \forall i = 1, \dots, n$.



$$L(\vartheta) = (\vartheta+1)^n (x_1 \cdot \ldots \cdot x_n)^\vartheta$$

$$L'(\vartheta) = 0 \iff \frac{d}{d\vartheta}\left[n \log(\vartheta+1) + \vartheta \log\left(\prod_{i=1}^{n} x_i \right)\right] = 0$$

$$\iff \frac{n}{\vartheta+1} + \log\left(\prod_{i=1}^{n} x_i\right) = 0 \iff \vartheta = \frac{n}{\log\left(\frac{1}{\prod x_i}\right)} - 1$$

È una buona stima? $\vartheta \geq 0$?

$$\frac{n}{\log\left(\frac{1}{\prod x_i}\right)} - 1 \geq 0 \iff \prod_{i=1}^{n} x_i \geq e^{-n}$$

$$\hat{\vartheta} = \begin{cases} \dfrac{n}{\log\left(\frac{1}{\prod x_i}\right)} - 1 & , \text{ se } \boxed{\prod_{i=1}^{n} x_i \geq e^{-n}} \\ 0 & , \text{ se } \boxed{\prod_{i=1}^{n} x_i < e^{-n}} \end{cases}$$

2 $$E_\vartheta[X^k] = \frac{1}{n}\sum_{i=1}^{n} x_i^k$$

$\boxed{k=1}$

$$E_\vartheta[X] = \overline{x}$$

$$E_\vartheta[X] = \int_{-\infty}^{+\infty} x\, f_\vartheta(x)\, dx = \int_0^1 x \cdot (\vartheta+1) x^\vartheta \, dx =$$

$$= \frac{\vartheta+1}{\vartheta+2} x^{\vartheta+2}\Big|_0^1 = \frac{\vartheta+1}{\vartheta+2}$$

$$\frac{\vartheta+1}{\vartheta+2} = \overline{x} \iff \vartheta+1 = \overline{x}(\vartheta+2) \iff$$

$$(1-\bar{x})\vartheta = 2\bar{x}-1 \iff \vartheta = \frac{2\bar{x}-1}{1-\bar{x}}$$

$$\uparrow$$

$$x_i \in (0,1) \ \forall i$$

$$0 < \bar{x} < 1$$

È una buona stima? $\vartheta \geq 0$?

$$\frac{2\bar{x}-1}{1-\bar{x}} \geq 0 \iff \bar{x} \geq \frac{1}{2}$$

$$\tilde{\vartheta} = \frac{2\bar{x}-1}{1-\bar{x}} \quad \text{se} \quad \bar{x} \geq \frac{1}{2}$$

Se $\bar{x} < \frac{1}{2}$, consideriamo $\boxed{k=2}$.

$$E_\vartheta[X^2] = \int_{-\infty}^{+\infty} x^2 f_\vartheta(x)\,dx = \int_0^1 x^2 (\vartheta+1) x^\vartheta\,dx =$$

$$= \frac{\vartheta+1}{\vartheta+3} x^{\vartheta+3}\Big|_0^1 = \frac{\vartheta+1}{\vartheta+3}$$

$$\frac{\vartheta+1}{\vartheta+3} = \sum_{i=1}^{n} \frac{x_i^2}{n} = \overline{m}_2 \iff \vartheta+1 = \overline{m}_2(\vartheta+3)$$

$$\iff (1-\overline{m}_2)\vartheta = 3\overline{m}_2 - 1$$

$$\iff \vartheta = \frac{3\overline{m}_2-1}{1-\overline{m}_2}$$

È una buona stima se $\overline{m}_2 \geq \frac{1}{3}$.

Se $\overline{m}_2 < \frac{1}{3}$, si continua con $\boxed{k=3}$.

ESEMPI Supponiamo che le v.a. del campione statistico abbiano densità

$$f_\vartheta(x) = \begin{cases} \frac{1}{2\vartheta}, & \text{se } x \in [-\vartheta, \vartheta] \\ 0, & \text{alt.} \end{cases}$$

Stimare $\vartheta > 0$.

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$$(0, +\infty) \ni \vartheta \longmapsto L(\vartheta; x_1, \dots, x_n) = \prod_{i=1}^{n} f_\vartheta(x_i) =$$

$$= \begin{cases} \left(\frac{1}{2\vartheta}\right)^n, & \text{se } x_i \in [-\vartheta, \vartheta] \ \forall i \\ 0, & \text{se } \exists\, x_i \notin [-\vartheta, \vartheta] \end{cases}$$

$$x_i \in [-\vartheta, \vartheta] \ \forall i \iff -\vartheta \leq x_i \leq \vartheta \ \forall i \iff$$

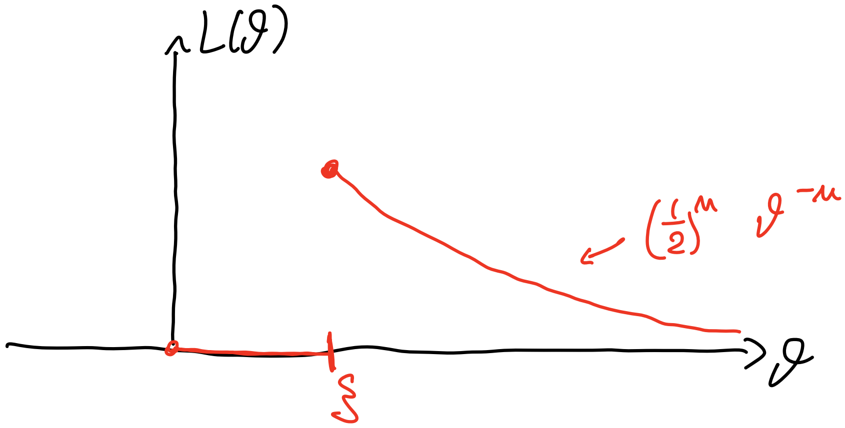
$$-\vartheta \leq x_i \ \text{ e } \ x_i \leq \vartheta \ \forall i \iff -x_i \leq \vartheta \text{ e } x_i \leq \vartheta \ \forall i$$

$$\iff \vartheta \geq \max\{x_i, -x_i\} \ \forall i \iff$$

$$\iff \vartheta \geq |x_i| \ \forall i \iff \vartheta \geq \max\{|x_1|, |x_2|, \dots |x_n|\}$$

Sia $\xi := \max\{|x_1|, \dots, |x_n|\}$

$$(0,+\infty) \ni \vartheta \longmapsto L(\vartheta; x_1,\dots,x_n) = \begin{cases} \left(\frac{1}{2\vartheta}\right)^n, & \vartheta \geq \xi \\ 0, & \vartheta < \xi \end{cases}$$



Quindi $\quad \hat{\vartheta} = \max\{|x_1|,\dots,|x_n|\}$

<u>oss</u> Abbiamo ottenuto la statistica campionaria

$$g(X_1,\dots,X_n) = \max\{|X_1|,\dots,|X_n|\}$$

come stima del parametro ϑ.

(2) $\quad E_\vartheta[X^k] = \frac{1}{n}\sum_{i=1}^{n} x_i^k \quad , \quad k=1,2,\dots$

$\boxed{k=1}$ $\quad E_\vartheta[X] = \int_{-\infty}^{+\infty} x\, f_\vartheta(x)\,dx = \int_{-\vartheta}^{\vartheta} x \cdot \frac{1}{2\vartheta}\,dx = 0$

$0 = \bar{x}$ $\qquad$ NON TROVO SOLUZIONI

$\boxed{k=2}$ $\quad E_\vartheta[X^2] = \int_{-\infty}^{+\infty} x^2 f_\vartheta(x)\,dx = \int_{-\vartheta}^{\vartheta} x^2 \cdot \frac{1}{2\vartheta}\,dx =$

$$= \frac{1}{2\vartheta} \cdot \left.\frac{x^3}{3}\right|_{-\vartheta}^{\vartheta} = \frac{\vartheta^3}{3\vartheta} = \frac{\vartheta^2}{3}$$

$$\frac{\vartheta^2}{3} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \underset{\vartheta>0}{\Longleftrightarrow} \vartheta = \left(\frac{3}{n}\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

$$\boxed{\tilde{\vartheta} = \left(\frac{3}{n}\sum_{i=1}^{n} x_i^2\right)^{1/2}}$$

<u>OSS</u> 1. $E[g(X_1,\dots,X_n)] = \vartheta$?

La statistica campionaria come stima del parametro ϑ è corretta ?

2. $E[(g(X_1,\dots,X_n)-\vartheta)^2]$ deve essere il più piccolo possibile

$$E[(g(X_1,\dots,X_n)-\vartheta)^2] = Var(g(X_1,\dots,X_n)) + \underbrace{(E[g(X_1,\dots,X_n)]-\vartheta)^2}_{\text{bias}}.$$

$\overline{X}$ media campionaria $\quad S^2$ varianza campionaria

<u>Teorema</u> Sia $X_1,\dots,X_n$ campione statistico di v.a. gaussiane di tipo $N(\mu,\sigma^2)$. Allora:

(i) $\overline{X}$ e S^2 sono indipendenti ;

(ii) $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ e

$$\frac{(n-1)\,S^2}{\sigma^2} \sim \chi^2(n-1)$$ [ha densità chi-quadro e $(n-1)$ gradi di libertà]

(iii) $$T = \frac{\sqrt{n}\,(\overline{X} - \mu)}{S} \sim T_{n-1}$$

[ha densità di Student a $(n-1)$ gradi di libertà]

OSS $$\frac{\sqrt{n}\,(\overline{X}-\mu)}{S} = \frac{X_1 + \dots + X_n - n\cdot\mu}{\sqrt{n\cdot S^2}}$$

Def Si chiama densità Gamma di parametri r e λ $(r>0,\ \lambda>0)$ la funzione

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(r)}\,\lambda^r\, x^{r-1} e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

dove $\Gamma(r) := \int_0^{+\infty} t^{r-1} e^{-t}\,dt$.

OSS $\Gamma(r+1) = r\,\Gamma(r) \quad \forall\, r>0$,

$\Gamma(n) = (n-1)! \quad \forall\, n \in \mathbb{N}$

Prop f è una buona densità

dim
- $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
- f integrabile
- $\int_{-\infty}^{+\infty} f(x)\, dx = 1$.

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_0^{+\infty} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}\, dx =$$

$$= \frac{1}{\Gamma(r)} \lambda^r \int_0^{+\infty} \frac{(\lambda x)^{r-1}}{\lambda^{r-1}} e^{-\lambda x}\, dx =$$

$$= \frac{1}{\Gamma(r)} \lambda \int_0^{+\infty} (\lambda x)^{r-1} e^{-\lambda x}\, dx \underset{t=\lambda x}{=} \frac{\lambda}{\Gamma(r)} \int_0^{+\infty} t^{r-1} e^{-t} \frac{1}{\lambda}\, dt$$

$$= \frac{1}{\Gamma(r)} \int_0^{+\infty} t^{r-1} e^{-t}\, dt = 1. \qquad \square$$

Una v.a. con densità Gamma di parametri r e λ si indica con

$$\Gamma(r, \lambda).$$

OSS $\Gamma(1, \lambda) \sim$ esponenziale (λ)

$$f(x) = \begin{cases} \frac{1}{\Gamma(1)} \lambda^1 \cdot x^{1-1} e^{-\lambda x} = \lambda e^{-\lambda x} & , x > 0 \\ 0 & , \text{alt.} \end{cases}$$

Prop • $X \sim \Gamma(r,\lambda)$ ammette tutti i momenti finiti, e $E[X^\beta] = \frac{\Gamma(r+\beta)}{\Gamma(r)\lambda^\beta}$ $\forall \beta > 0$

In particolare $E[X] = \frac{\Gamma(r+1)}{\Gamma(r)\lambda} = \frac{r}{\lambda}$

$$Var(X) = \frac{r}{\lambda^2}.$$

• $X \sim \Gamma(r,\lambda)$ ha funzione generatrice dei momenti

$$\varphi_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r \quad \forall t \in (-\infty, \lambda)$$

__dim__ $\varphi_X(t) = E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} f(x)\, dx =$

$$= \int_0^{+\infty} e^{tx} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} dx =$$

$$= \int_0^{+\infty} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{x(t-\lambda)} dx = \quad (<+\infty \Leftrightarrow t-\lambda<0)$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^{+\infty} \frac{(\lambda-t)^{r-1}}{(\lambda-t)^{r-1}} x^{r-1} e^{-(\lambda-t)x} dx =$$

$$= \frac{\lambda^r}{\Gamma(r)(\lambda-t)^{r-1}} \int_0^{+\infty} [(\lambda-t)x]^{r-1} e^{-(\lambda-t)x} dx \overset{y=(\lambda-t)x}{=}$$

$$= \frac{\lambda^r}{\Gamma(r)(\lambda-t)^{r-1}} \int_0^{+\infty} y^{r-1} e^{-y} \frac{dy}{\lambda-t} = \frac{\lambda^r}{(\lambda-t)^r} \qquad \square$$

<u>Prop</u> Se $X \sim \Gamma(r, \lambda)$ e $Y \sim \Gamma(s, \lambda)$, e sono indipendenti, allora $X+Y \sim \Gamma(r+s, \lambda)$.

<u>dim</u> X, Y indip $\Rightarrow \varphi_{X+Y}(t) = \varphi_X(t) \cdot \varphi_Y(t)$

Quindi $\varphi_{X+Y}(t) = \left(\frac{\lambda}{\lambda-t}\right)^r \cdot \left(\frac{\lambda}{\lambda-t}\right)^s = \left(\frac{\lambda}{\lambda-t}\right)^{r+s}$

$\Rightarrow X+Y$ ha distribuzione $\Gamma(r+s, \lambda)$. $\square$