Funzione generatrice dei momenti

Def Data X v.a. a valori reali, si chiama FUNZIONE GENERATRICE DEI MOMENTI DI X

$$t \longmapsto \varphi_X(t) = E[e^{tX}]$$

$$\varphi_X(t) = \begin{cases} \sum_{x_i} e^{tx_i} p_X(x_i) & (X \text{ discreta}) \\ \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx & (X \text{ con densità}) \end{cases}$$

Si chiama dominio di φ_X l'insieme dei valori t per cui $\varphi_X(t) < +\infty$.

OSS

- $t = 0 \in$ dominio di φ_X

$$\varphi_X(0) = 1$$

- Ad esempio se X ha densità $f(x) = \frac{1}{\pi(1+x^2)}$ per ogni $x \in \mathbb{R}$, allora

$$\varphi_X(t) = \int_{-\infty}^{+\infty} \frac{e^{tx}}{\pi(1+x^2)}\,dx = +\infty \quad \forall\, t \neq 0$$

- Se X ha densità $f(x) = \begin{cases} \frac{c}{x^4}, & x \geq 1 \\ 0, & \text{alt.} \end{cases}$

$$\varphi_X(t) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx = \int_1^{+\infty} c \cdot \frac{e^{tx}}{x^4}\,dx < +\infty$$

$\Leftrightarrow \quad t \in (-\infty, 0]$.

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)\,dx = \int_1^{+\infty} c \cdot x^{n-4}\,dx < +\infty$$

$$4 - n > 1 \Leftrightarrow n < 3$$

X ha momenti finiti solo per $n = 1, 2$.

<u>Teorema</u> Siano X e Y v.a. a valori reali, e φ_X e φ_Y le rispettive funz. generatrici dei momenti. Se esiste un intervallo (a,b) con $a<b$, t.c. φ_X e φ_Y sono definite in (a,b) e $\varphi_X(t) = \varphi_Y(t)$ $\forall\, t \in (a,b)$, allora X e Y sono equidistribuite.

<u>Teorema</u> Sia X v.a. a valori reali. Se esiste $\varepsilon > 0$ t.c. φ_X è definita su $(-\varepsilon, \varepsilon)$, allora X ammette momento finito $E[X^n]$ per ogni $n \geq 1$. Inoltre

$$E[X^n] = \frac{d^n \varphi_X}{dt^n}(0) \quad \forall\, n \geq 1.$$

<u>Prop</u> (i) Sia X v.a. e siano $a, b \in \mathbb{R}$, allora

$$\varphi_{aX+b}(t) = e^{bt} \varphi_X(at)$$

<u>dim</u>

$$\varphi_{aX+b}(t) = E[e^{(aX+b)t}] = E[e^{bt} \cdot e^{(at)X}] =$$
$$= e^{bt} E[e^{(at)X}] = e^{bt} \varphi_X(at) \quad \square$$

(ii) Siano X, Y indipendenti allora

$$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$$

<u>dim</u>

$$\varphi_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX} e^{tY}] \overset{X, Y \text{ indip.}}{=}$$
$$= E[e^{tX}] E[e^{tY}] = \varphi_X(t)\varphi_Y(t) \quad \square$$

<u>Applicazioni</u>

- $X \sim B(n, p)$, $\varphi_X(t) = ?$

$X = Y_1 + \dots + Y_n$, $\{Y_k\}$ indip. con $Y_k \sim B(1, p)$ $\forall k$

$$\varphi_{Y_k}(t) = \sum_{h=0}^{1} e^{t \cdot h} p_{Y_k}(h) = e^{0 \cdot t}(1-p) + e^{1 \cdot t} \cdot p =$$
$$= e^t p + 1 - p \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \varphi_X(t) = \varphi_{Y_1}(t) \cdots \varphi_{Y_n}(t) = (1-p+e^t p)^n \quad \forall t \in \mathbb{R}.$$

- X v.a. geometrica di par. p. $\quad p_X(h) = (1-p)^{h-1} p \quad \forall h \in \mathbb{N}$

$$\varphi_X(t) = \sum_{h=1}^{+\infty} e^{th}\, p_X(h) = \sum_{h=1}^{+\infty} e^{th} (1-p)^{h-1} p =$$

$$= \frac{p}{1-p} \sum_{h=1}^{+\infty} \left((1-p)e^t\right)^h < +\infty$$

$$\Longleftrightarrow \quad (1-p)e^t < 1 \quad \Longleftrightarrow \quad t < \log \frac{1}{1-p}$$

$\varphi_X(t)$ ha dominio $\left(-\infty, \log \frac{1}{1-p}\right) \ni 0$

$$\Rightarrow E[X^n] < +\infty \quad \forall n \geq 1$$

- (esercizio) $\quad X \sim \text{Poisson}(\lambda), \quad Y \sim \text{Poisson}(\mu)$, indip.

$$\Rightarrow X + Y \sim \text{Poisson}(\lambda+\mu).$$

Calcolo φ_{X+Y} e mostro che ha la forma della funz. gen. dei momenti di Poisson$(\lambda+\mu)$.

- X v.a. con densità uniforme su $[a,b]$, $\quad f_X(x) = \begin{cases} \frac{1}{b-a}, & [a,b] \\ 0, & \text{alt.} \end{cases}$

$$\varphi_X(t) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx = \int_a^b e^{tx} \cdot \frac{1}{b-a}\,dx \overset{t \neq 0}{=}$$

$$= \frac{1}{t} e^{tx} \cdot \frac{1}{b-a} \Big|_a^b = \frac{e^{bt} - e^{at}}{t(b-a)}$$

$$\varphi_X(t) = \begin{cases} \dfrac{e^{bt} - e^{at}}{t(b-a)}, & \forall\, t \in \mathbb{R} \setminus \{0\} \\ 1, & t = 0 \end{cases}$$

$$E[X] = \frac{d}{dt}\varphi_X(0) = \lim_{t \to 0} \frac{\varphi_X(t) - \varphi_X(0)}{t} =$$

$$= \lim_{t \to 0} \frac{\dfrac{e^{bt} - e^{at}}{t(b-a)} - 1}{t} = \lim_{t \to 0} \frac{e^{bt} - e^{at} - t(b-a)}{t^2(b-a)} =$$

$$= \lim_{t \to 0} \frac{\left[\cancel{1} + \cancel{bt} + \frac{1}{2}b^2t^2 + o(t^2)\right] - \left[\cancel{1} + \cancel{at} + \frac{1}{2}a^2t^2 + o(t^2)\right] - \cancel{t(b-a)}}{t^2(b-a)} =$$

$$= \frac{1}{2}\,\frac{b^2 - a^2}{b-a} = \frac{1}{2}(a+b)$$

$$E[X^2] = \frac{d^2}{dt^2}\varphi_X(0) = \int_a^b x^2 \frac{1}{b-a}\, dx = \frac{b^3 - a^3}{3(b-a)}$$

- $X \sim N(m, \sigma^2)$, $\quad m = E[X] \in \mathbb{R}$

  $\sigma = \sigma(X) = \sqrt{Var(X)} > 0$

$\varphi_X(t) = ?$ Se $Z \sim N(0,1)$, $X = \sigma Z + m$

$$\varphi_X(t) = e^{mt}\,\varphi_Z(\sigma t)$$

$$\boxed{\varphi_Z(t)} = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\left(tx - \frac{x^2}{2}\right)}\, dx =$$

$$tx - \frac{x^2}{2} = \frac{1}{2}\left(-t^2 + 2tx - x^2\right) + \frac{1}{2}t^2 = -\frac{1}{2}(t-x)^2 + \frac{1}{2}t^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{t^2/2} e^{-\frac{1}{2}(t-x)^2} dx \underset{y=t-x}{=} e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy$$

$$\boxed{= e^{t^2/2}} \qquad \forall t \in \mathbb{R}$$

$$X \sim N(m, \sigma^2) \Rightarrow \boxed{\varphi_X(t) =} e^{mt} \varphi_Z(\sigma t) = e^{mt} e^{\frac{\sigma^2 t^2}{2}}$$

$$\boxed{= e^{mt + \frac{\sigma^2 t^2}{2}}} \qquad \forall t \in \mathbb{R}$$

$X_1 \sim N(m_1, \sigma_1^2)$, $X_2 \sim N(m_2, \sigma_2^2)$ indip.

$$\Rightarrow X_1 + X_2 \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$$

dim $X_1 + X_2$ ha funz. generatrice dei momenti

$$\varphi_{X_1+X_2}(t) \underset{\text{Prop}}{=} \varphi_{X_1}(t)\, \varphi_{X_2}(t) \underset{\square}{=} e^{m_1 t + \frac{\sigma_1^2 t^2}{2}} e^{m_2 t + \frac{\sigma_2^2 t^2}{2}} =$$

$$= e^{(m_1 + m_2)t + \frac{(\sigma_1^2 + \sigma_2^2) t^2}{2}} \underset{\square}{=} \varphi_Y(t) \qquad \forall t \in \mathbb{R}$$

se $Y \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$

Quindi per il Teorema, $X_1 + X_2$ e Y sono equidistribuite. Allora $X_1 + X_2 \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$ $\square$

ESERCIZIO $\quad X \sim B(n,p) \quad , \varphi_X(t) = (1-p+pe^t)^n$

$$Y = \frac{X - E[X]}{\sqrt{Var(X)}} \quad , \quad E[Y] = \frac{1}{\sqrt{Var(X)}} E[X - E[X]] = 0$$

$$Var(Y) = \frac{Var(X)}{(\sqrt{Var(X)})^2} = 1$$

$$Y = \frac{X - np}{\sqrt{np(1-p)}} = \frac{1}{\sqrt{np(1-p)}} X - \sqrt{\frac{np}{1-p}}$$

$$\varphi_Y(t) = e^{-\sqrt{n}\sqrt{\frac{p}{1-p}}\,t}\, \varphi_X\left(\frac{t}{\sqrt{np(1-p)}}\right) =$$

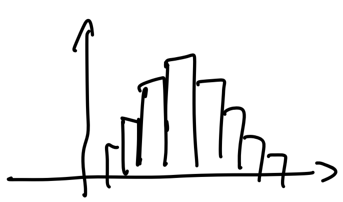
$$= e^{-\sqrt{n}\sqrt{\frac{p}{1-p}}\,t} \left(1-p+p\,e^{t/\sqrt{np(1-p)}}\right)^n$$

$\forall t \neq 0$

$$\lim_{n\to+\infty} \varphi_Y(t) = e^{t^2/2}$$

Sia $\{X_k\}$ una successione di v.a. indipendenti ed equidistribuite.

$h_1 \dots h_{100}$, $X_1 \dots X_{100}$ e $X_i(\omega) = h_i$.



$\bar{h}$

$Var(h)$

Teoremi limite

Sia $\{X_k\}$ succ. di v.a. a valori reali indipendenti ed equidistribuite.

OSS $\dfrac{X_1+\dots+X_n}{n}$ $\boxed{\cancel{\dfrac{X+X+\dots+X}{n}=X}}$

$$E\left[\frac{X_1+\dots+X_n}{n}\right]=\frac{1}{n}\sum_{k=1}^{n}E[X_k]=E[X_k]$$

$$Var\left(\frac{X_1+\dots+X_n}{n}\right)=\frac{1}{n^2}\sum_{k=1}^{n}Var(X_k)=\frac{Var(X_k)}{n}\xrightarrow[n\to+\infty]{}0$$

Teorema (Legge debole dei grandi numeri)

Sia $\mu=E[X_k]$ $\forall k\geq 1$, e supponiamo che $E[X_k^2]<+\infty$, $\forall k\geq 1$. Allora

$$\forall \varepsilon>0 \quad \lim_{n\to+\infty}\mathbb{P}\left(\left|\frac{X_1+X_2+\dots+X_n}{n}-\mu\right|>\varepsilon\right)=0$$

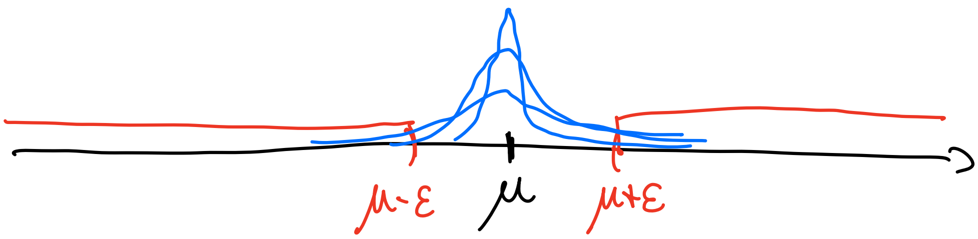
Def Se $\{Y_k\}$ succ. di v.a. e Z v.a., si dice che Y_k converge a Z in probabilità se

$$\forall \varepsilon>0 \quad \lim_{n\to+\infty}\mathbb{P}(|Y_k-Z|>\varepsilon)=0$$

Legge debole dei grandi numeri ci dice che

$\dfrac{X_1+\dots+X_n}{n}$ converge in probabilità a μ.

$$\mathbb{P}\left(\underbrace{\left|\frac{X_1+\dots+X_n}{n}-\mu\right|>\varepsilon}\right)$$



<u>dim</u>

<u>Prop</u> Sia $\{Y_k\}$ succ. di v.a. t.c.

- $\exists\, c \in \mathbb{R}$ t.c. $E[Y_k] \xrightarrow[k\to+\infty]{} c$
- $Var(Y_k) \xrightarrow[k\to+\infty]{} 0$

Allora Y_k converge in probabilità a c.

ossia $\forall \varepsilon>0 \quad \lim_{k\to+\infty} \mathbb{P}(|Y_k-c|>\varepsilon)=0$.

<u>dim</u> (prop) $\forall \varepsilon>0$, $\mathbb{P}(|Y_k-E[Y_k]|>\varepsilon) \le \dfrac{Var(Y_k)}{\varepsilon^2}$

non basta.

$$\mathbb{P}(|Y_k-c|>\varepsilon) = \mathbb{P}((Y_k-c)^2>\varepsilon^2) \overset{\text{Markov}}{\le} \frac{E[(Y_k-c)^2]}{\varepsilon^2}$$

Ma $E[(Y_k-c)^2] = E[(Y_k - E[Y_k] + E[Y_k] - c)^2] =$

$$= E[(Y_k - E[Y_k])^2] + (E[Y_k]-c)^2 +$$

$$+ 2(E[Y_k]-c)\underbrace{E[Y_k - E[Y_k]]}_{=0} =$$

$$= Var(Y_k) + (E[Y_k]-c)^2 \xrightarrow[k\to+\infty]{} 0$$

$$\Rightarrow \forall \varepsilon > 0 \quad \mathbb{P}(|Y_k - c| > \varepsilon) \le \frac{E[(Y_k-c)^2]}{\varepsilon^2} \xrightarrow[k\to+\infty]{} 0$$

$\square$.

$$Y_k = \frac{X_1 + \dots + X_k}{k}, \quad c = \mu. \quad E[Y_k] = \mu \quad \forall k \ge 1$$

$$Var(Y_k) = \frac{Var(X_i)}{k} \xrightarrow[k\to+\infty]{} 0$$

$\overset{\text{Prop}}{\Rightarrow}$ $\frac{X_1+\dots+X_k}{k}$ converge a μ in probabilità

$\square$.

<u>OSS</u> Posso applicare la prop. a $\frac{X_1+\dots+X_n}{n}$

anche nel caso in cui

- $\{X_k\}$ succ. di v.a. con $E[X_k]=\mu$ $\forall k$, $Var(X_k) \le M$ $\forall k \ge 1$ e incorrelate.
- $\{X_k\}$ succ. di v.a. con $E\left[\frac{X_1+\dots+X_n}{n}\right] \to \mu$, $Var\left(\frac{X_1+\dots+X_k}{k}\right) \to 0$ e incorrelate.