Sia X v.a. a valori reali

Prop (i) Sia $h : \mathbb{R} \to \mathbb{R}$ che sia ben definita su $V(X)$, l'insieme su cui assume valori X, allora

$$E[h(X)] = \begin{cases} \sum_{x_i} h(x_i)\, p_X(x_i) & \text{(discreta)} \\ \int_{-\infty}^{+\infty} h(x) f_X(x)\, dx & \text{(densità)} \end{cases}$$

se esiste finito.

<u>oss</u> $E[X^n] = E[h(X)]$ con $h(x) = x^n$

(ii) $E[aX + b] = a E[X] + b \quad \forall\, a, b \in \mathbb{R}$

(iii) Se Y è v.a. a valori reali allora

$$E[X + Y] = E[X] + E[Y]$$

(iv) $E[X] \geq 0$ se $X \geq 0$, in particolare

$$E[X] \geq E[Y] \text{ se } X \geq Y.$$

<u>dim</u> (i) $E[h(X)] = E[Y] \qquad Y = h(X)$

Se X ha valori $\{x_1, x_2, \dots\}$, Y ha valori

$$\{h(x_1), h(x_2), \dots\} = \{y_1, y_2, \dots\}$$

$$E[Y] = \sum_{y_j} y_j\, p_Y(y_j) = \sum_{y_j} y_j \Big(\sum_{\substack{x_i \text{ t.c.} \\ h(x_i) = y_j}} p_X(x_i) \Big)$$

$$= \sum_{y_j} \left(\sum_{\substack{x_i \text{ t.c.} \\ h(x_i)=y_j}} y_j \, P_X(x_i) \right) = \sum_{y_j} \sum_{\substack{x_i \text{ t.c.} \\ h(x_i)=y_j}} h(x_i) P_X(x_i)$$

$$= \sum_{x_i} h(x_i) P_X(x_i)$$

(ii) $h(x) = ax+b \quad , a, b \in \mathbb{R}$

$$E[aX+b] = E[h(X)] \quad \text{e applico (i).}$$

Applicando (iii)

$$E[aX+b] = E[aX] + E[b] =$$

$$= E[aX] + b \underset{\substack{\uparrow \\ (i)}}{=} a\,E[X] + b$$

(iii) $E[X+Y] = E[Z] \quad , Z = X+Y$

$$E[Z] = \sum_{z_k} z_k \, P_Z(z_k) = \sum_{z_k} z_k \left(\sum_{\substack{(x_i, y_j) \\ \text{t.c. } x_i + y_j = z_k}} P_{X,Y}(x_i, y_j) \right) =$$

$$= \sum_{z_k} \left(\sum_{\substack{(x_i, y_j) \\ x_i + y_j = z_k}} \overset{\substack{z_k \\ \shortparallel}}{(x_i + y_j)} P_{X,Y}(x_i, y_j) \right) =$$

$$= \sum_{x_i, y_j} (x_i + y_j) P_{X,Y}(x_i, y_j) = \sum_{x_i, y_j} x_i P_{X,Y}(x_i, y_j) +$$

$$+\sum_{x_i, y_j} y_j \, p_{X,Y}(x_i, y_j) =$$

$$= \sum_{x_i} x_i \underbrace{\sum_{y_j} p_{X,Y}(x_i, y_j)}_{p_X(x_i)} + \sum_{y_j} y_j \underbrace{\sum_{x_i} p_{X,Y}(x_i, y_j)}_{p_Y(y_j)} =$$

$$= E[X] + E[Y]$$

(iv) $\quad X \geq 0 \Rightarrow E[X] \geq 0$

$$X \geq 0 \Rightarrow f_X\big|_{(-\infty,0)} \equiv 0$$

$$\Rightarrow \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^{+\infty} x f_X(x)\,dx \geq 0$$

$\square$

Esempi

- $X \sim B(n,p) \qquad E[X] = ?$

$$E[X] = \sum_{h=0}^{n} h \, p_X(h) = \sum_{h=0}^{n} \binom{n}{h} p^h (1-p)^{n-h} h$$

X è la somma di n v.a. indipendenti $B(1,p)$

$$X = Y_1 + Y_2 + \dots + Y_n \quad , \; Y_k \sim B(1,p) \; \forall k$$

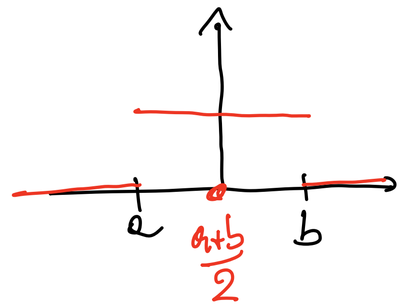
$Y_1, \dots, Y_n$ indipendenti

$$E[X] = n\,E[Y_1] = np$$

$$E[Y_1] = (0)\,p_{Y_1}(0) + (+1)\,p_{Y_1}(+1) = p$$

- X v.a. uniforme su $[a,b]$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{altr.} \end{cases}$$



$$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx =$$

$$= \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)\, dx = \int_a^b x^n \frac{1}{b-a}\, dx =$$

$$= \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \qquad \forall\, n \geq 1$$

$$E[e^X] = \int_{-\infty}^{+\infty} e^x f_X(x)\, dx = \int_a^b e^x \frac{1}{b-a}\, dx =$$

$h(x) = e^x$

$$= \frac{e^b - e^a}{b-a}$$

• X v.a. unif. su $[-1,1]$, $Y = X^2$

$$E[X^2] = E[Y]$$

$$\int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_{-\infty}^{+\infty} y\, f_Y(y)\,dy$$

Y assume valori in $[0,1]$

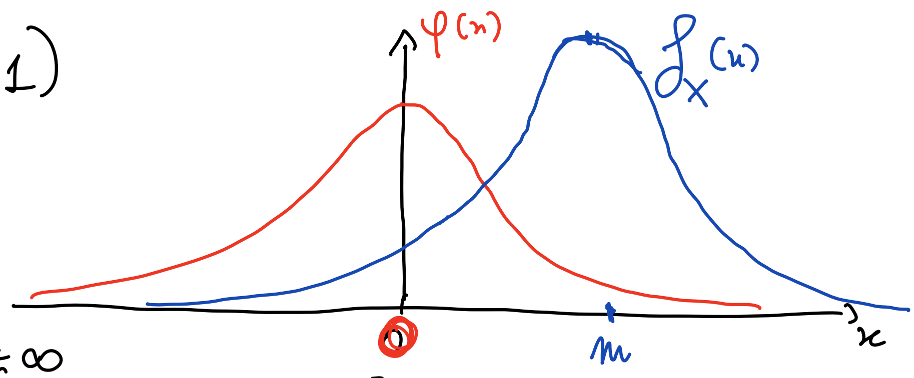
$$= \int_{-1}^{+1} x^2 \,\frac{1}{2}\,dx = \frac{x^3}{6}\Big|_{-1}^{+1} = \frac{1}{3}$$

• $X \sim N(m, \sigma^2)$, $m \in \mathbb{R}$, $\sigma > 0$.

$$E[X] = \int_{-\infty}^{+\infty} x \,\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$$

$$= E[\sigma Z + m] = \sigma \overbrace{E[Z]}^{=0} + m = m$$

$Z \sim N(0,1)$



$$E[Z] = \int_{-\infty}^{+\infty} x \,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\,dx = 0$$

$E[Z^n] \in \mathbb{R} \quad \forall n \geq 1$, $E[Z^n] = 0$ n dispari

$\boxed{\text{CASO } n=2}$

Def Si chiama VARIANZA di una v.a X a valori reali il numero $Var(X)$, se esiste finito, dato da

$$Var(X) := E[(X - E[X])^2]$$

OSS $Var(X) = 0 \iff X = \text{costante}$

Def Si chiama SCARTO QUADRATICO MEDIO (o DEVIAZIONE STANDARD) di X il numero $\sigma(X) := \sqrt{Var(X)}$

Prop (i) $Var(X) = E[X^2] - (E[X])^2$

dim $Var(X) = E[(X - E[X])^2] =$

$$= E[X^2 + (E[X])^2 - 2E[X]X] =$$

$$= E[X^2] + (E[X])^2 - 2E[E[X]X] =$$

$$= E[X^2] + (E[X])^2 - 2E[X] \cdot E[X] =$$

$$= E[X^2] - (E[X])^2$$

(ii) $Var(aX + b) = a^2 Var(X) \quad \forall a, b \in \mathbb{R}$

dim $\mathrm{Var}(aX+b) = E[(aX+b)^2] - (E[aX+b])^2 =$
$= E[a^2X^2 + 2abX + b^2] - a^2(E[X])^2 - 2abE[X]$
$- b^2 = a^2 E[X^2] + 2ab\,E[X] + b^2 - a^2(E[X])^2$
$- 2ab\,E[X] - b^2 = a^2\{E[X^2] - (E[X])^2\} =$
$= a^2 \mathrm{Var}(X).$

$$\mathrm{Var}(X+Y) \overset{?}{=} E[(X+Y)^2] - (E[X+Y])^2 =$$
$$= E[X^2] + E[Y^2] + 2E[X\cdot Y] +$$
$$- (E[X])^2 - (E[Y])^2 - 2E[X]E[Y] =$$
$$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\underbrace{(E[X\cdot Y] - E[X]E[Y])}_{\mathrm{Cov}(X,Y)}$$

Def Date due v.a. X, Y a valori reali, si chiama COVARIANZA di X e Y il numero
$$\mathrm{Cov}(X,Y) := E[(X-E[X])(Y-E[Y])]$$

OSS $\mathrm{Cov}(X,Y) = E[X\cdot Y] - E[X]E[Y]$

OSS Coefficiente di correlazione
$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma(X)\sigma(Y)}$$

Prop (i) $\mathrm{Cov}(aX+bZ+c, Y) =$

$= a\,\mathrm{Cov}(X,Y) + b\,\mathrm{Cov}(Z,Y) \quad \forall a,b,c \in \mathbb{R}$
$\forall X, Y, Z$ v.a.

(ii) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

(iii) $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

(iv) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$

Prop Se X e Y sono v.a indipendenti allora

$\mathrm{Cov}(X,Y) = 0$ (X e Y sono incorrelate)

Il viceversa è falso.

Cor Sono fatti equivalenti:

(i) X e Y sono incorrelate

(ii) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

(iii) $E[X \cdot Y] = E[X] \cdot E[Y]$

Teorema (i) Disuguaglianza di Markov

Sia Y v.a. a valori non-negativi, allora $\forall a > 0$ si ha

$$\mathbb{P}(Y > a) \leq \frac{E[Y]}{a}$$

(ii) Disuguaglianza di Chebyshev

Sia X v.a. a valori reali, allora $\forall d > 0$ si ha

$$\mathbb{P}(|X - E[X]| > d) \leq \frac{Var(X)}{d^2}$$

<u>Oss</u> $d = n \cdot \sigma(X)$

$$\mathbb{P}(|X - E[X]| > n\,\sigma(X)) \leq \frac{1}{n^2}$$

<u>dim</u> (i) Y v.a. a valori in $[0, +\infty)$

$$E[Y] = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy = \int_{0}^{+\infty} y\, f_Y(y)\, dy =$$

$$= \int_{\{Y > a\}} y\, f_Y(y)\, dy + \int_{\{Y \leq a\}} y\, f_Y(y)\, dy \geq$$

$$\geq \int_{\{Y > a\}} y\, f_Y(y)\, dy > \int_{\{Y > a\}} a\, f_Y(y)\, dy = a \cdot \mathbb{P}(Y > a)$$

(ii) X v.a. a valori reali, $d > 0$

$Y := (X - E[X])^2$ per (i)

$$\mathbb{P}(Y > d^2) \leq \frac{E[Y]}{d^2}$$

$$\mathbb{P}\left((X - E[X])^2 > d^2\right) = \mathbb{P}\left(|X - E[X]| > d\right) \leq$$

$$\leq \frac{E[(X - E[X])^2]}{d^2} = \frac{Var(X)}{d^2}$$

$\square$

Esempi

- $X \sim B(n, p)$

$X = Y_1 + \ldots + Y_n$, $Y_k \sim B(1, p)$ indipendenti

$$\Rightarrow Var(X) = Var(Y_1) + \ldots + Var(Y_n) =$$

$$= n\, Var(Y_k) = n\, p(1-p)$$

$$Var(Y_k) = E[Y_k^2] - (E[Y_k])^2 = p(1-p)$$

$$= 0^2 \cdot p_{Y_k}(0) + 1^2\, p_{Y_k}(1) - p^2 =$$

$$= p - p^2 = p(1-p)$$

- $X \sim N(m, \sigma^2)$, $m \in \mathbb{R}$, $\sigma > 0$

$\qquad E[X] \qquad \sigma(X) = \sqrt{Var(X)}$

$X = \sigma Z + m$, $\quad Z \sim N(0, 1)$

$$Var(X) = Var(\sigma Z + m) = \sigma^2 Var(Z)$$

$$Var(Z) = 1 \Rightarrow Var(X) = \sigma^2$$

$X \sim N(m_1, \sigma_1^2)$ $\quad Y \sim N(m_2, \sigma_2^2)$ indip.

$$X + Y \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$$

ES $\quad Var(Z) = 1$.