2.2 Stability

Let $\dot{\underline{x}} = F(\underline{x})$ be an ordinary differential equation in $\mathbb{R}^n$ with flow $\phi_t(\cdot)$.

Definition 2.2. A point $\underline{x}$ is *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\underline{x}, \underline{y}) < \delta$ implies $d(\phi_t(\underline{x}), \phi_t(\underline{y})) < \varepsilon$ for all $t \geq 0$.

Remark 2.6. Show that it is necessary to introduce also the notion of orbital stability.

Definition 2.3. A point $\underline{x}$ is *Lyapunov asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that $d(\underline{x}, \underline{y}) < \delta$ implies

$$d(\phi_t(\underline{x}), \phi_t(\underline{y})) \xrightarrow[t\to+\infty]{} 0\,.$$

We call *domain of asymptotic stability of* $\underline{x}$ the set $D(\underline{x})$ of points $\underline{y}$ for which $d(\phi_t(\underline{x}), \phi_t(\underline{y})) \to 0$ as $t \to +\infty$. If $D(\underline{x}) = \mathbb{R}^n$ we say that $\underline{x}$ is *globally Lyapunov asymptotically stable.*

Remark 2.7. If in Definition 2.3 we drop the request that the point $\underline{x}$ is Lyapunov stable, then $\underline{x}$ is called *quasi-asymptotically stable.* In this case there exists a neighbourhood $B_\delta(\underline{x})$ so that $d(\phi_t(\underline{x}), \phi_t(\underline{y})) \to 0$ as $t \to +\infty$ for all $\underline{y} \in B_\delta(\underline{x})$, but the orbits of these points may go arbitrarily far from that of $\underline{x}$ before convergence.

It is particularly important to study the stability of a fixed point $\underline{x}_0$ for which $\phi_t(\underline{x}_0) = \underline{x}_0$ for all t in Definitions 2.2 and 2.3.

Example 2.1 (see [Gl94]). Let us consider the following differential equation in $\mathbb{R}^2$

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2+y^2}} \\ \dot{y} = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2+y^2}} \end{cases}$$

Using polar coordinates (ρ, θ) as shown in Section 2.4 (see (2.8)) with $x = \rho\cos\theta$, $y = \rho\sin\theta$, we are reduced to the equation

$$\begin{cases} \dot{\rho} = \rho(1 - \rho^2) \\ \dot{\theta} = 1 - \cos\theta \end{cases}$$

It is now easy to determine the phase portrait of the equation and deduce that $(x_0, y_0) = (1, 0)$ is a quasi-asymptotically fixed point, but it is not Lyapunov stable.

One first tool to study the stability of a fixed point is to look at the linearisation of the vector field in the point.

Definition 2.4. A fixed point $\underline{x}_0$ of a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *hyperbolic* if all the eigenvalues of the Jacobian matrix $JF(\underline{x}_0)$ have real part different from zero.

Theorem 2.8 (Hartman-Grobman)**.** *Let $\underline{x}_0$ be a hyperbolic fixed point of a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. Then there exists a neighbourhood $U(\underline{x}_0)$ and a homeomorphism $h : U(\underline{x}_0) \to \mathbb{R}^n$ which sends orbits of the differential equation $\dot{\underline{x}} = F(\underline{x})$ into orbits of the linear differential equation $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$ without changing their direction of time parametrisation[2]. In particular the homeomorphism h leaves invariant the stability properties of the fixed point $\underline{y}_0 = \underline{0}$.*

The proof can be found in Appendix C.

Theorem 2.8 implies that we can characterise a hyperbolic fixed point $\underline{x}_0$ by looking at the linear system $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$. In particular the qualitative behaviour of the orbits in a neighbourhood of $\underline{x}_0$ coincides with that of the orbits in a neighbourhood of $\underline{y}_0 = \underline{0}$. However, in general, the regularity of h in Theorem 2.8 does not increase by increasing the regularity of a general F. Hence, the "shape" of the orbits may change under the action of h.

The situation is easier in dimension two. If $\underline{x}_0 \in \mathbb{R}^2$ is a hyperbolic fixed point, then $JF(\underline{x}_0)$ is in one of the cases 1-4 excluding case 2 with vanishing trace. If we are not in case 3, the fixed point $\underline{x}_0$ can be characterised like $\underline{y}_0 = \underline{0}$ for $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$. Hence we can talk about stable and unstable nodes, stable and unstable foci, and saddles. See [Gl94, Section 5.2].

We now briefly discuss the problem of the regularity of h for $F \in C^\omega$.

Lyapunov functions

Given a real C^1 function $V(\underline{x})$, we introduce the notation $\dot{V}(\underline{x})$ for its derivative along a vector field F. Namely

$$\dot{V}(\underline{x}) := \langle \nabla V(\underline{x}) \,,\, F(\underline{x}) \rangle \tag{2.1}$$

Notice that $\dot{V}(\underline{x}) = \frac{d}{dt} V(\phi_t(\underline{x}))|_{t=0}$.

[2] A formal statement is that, if ϕ_t is the flow of the original system $\dot{\underline{x}} = F(\underline{x})$ and ψ_t is the flow of the linear system $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$, then for all $\underline{x} \in U(\underline{x}_0)$ we have $h(\phi_t(\underline{x})) = \psi_t(h(\underline{x}))$ for all $t \in \mathbb{R}$ such that $\phi_t(\underline{x}) \in U(\underline{x}_0)$.

Definition 2.5. Let $\underline{x}_0$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. A C^1 real function $V : U \to \mathbb{R}$ defined in a neighbourhood U of $\underline{x}_0$ is called a *Lyapunov function for* $\underline{x}_0$ if:

(i) $V(\underline{x}) > V(\underline{x}_0)$ for all $\underline{x} \in U \setminus \{\underline{x}_0\}$;

(ii) $\dot{V}(\underline{x}) \leq 0$ for all $\underline{x} \in U$.

If the function $V : U \to \mathbb{R}$ satisfies (i) and

(ii)' $\dot{V}(\underline{x}) < 0$ for all $\underline{x} \in U \setminus \{\underline{x}_0\}$,

it is called a *strict Lyapunov function for* $\underline{x}_0$.

Theorem 2.9 (First Lyapunov stability theorem)**.** *Let $\underline{x}_0 \in \mathbb{R}^n$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. If there exists a Lyapunov function for $\underline{x}_0$, then $\underline{x}_0$ is Lyapunov stable.*

Proof. Let $V : U \to \mathbb{R}$ be the Lyapunov function for $\underline{x}_0$. Given $\varepsilon > 0$ such that $B_\varepsilon(\underline{x}_0) \subset U$, we let

$$m := \min_{\partial B_\varepsilon(\underline{x}_0)} V \quad \text{and} \quad S_m := \{\underline{x} \in B_\varepsilon(\underline{x}_0) \,:\, V(\underline{x}) < m\}$$

By definition $V(\underline{x}_0) < m$, hence $\underline{x}_0 \in S_m$. Moreover by continuity there exists $\delta > 0$ such that $B_\delta(\underline{x}_0) \subset S_m$. We now show that if $\underline{y} \in B_\delta(\underline{x}_0)$ then $\phi_t(\underline{y}) \in B_\varepsilon(\underline{x}_0)$ for all $t \geq 0$.

Condition (ii) in Definition 2.5 implies that $V(\phi_t(\underline{y})) \leq V(\underline{y}) < m$ for all $t \geq 0$. We conclude that if there exists $t_0 > 0$ such that $\phi_{t_0}(\underline{y}) \notin B_\varepsilon(\underline{x}_0)$, then by continuity of the flow there exists $t_1 \in (0, t_0)$ such that $\phi_{t_1}(\underline{y}) \in \partial B_\varepsilon(\underline{x}_0)$. This is a contradiction to the definition of m. □

Theorem 2.10 (Second Lyapunov stability theorem)**.** *Let $\underline{x}_0 \in \mathbb{R}^n$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. If there exists a strict Lyapunov function for $\underline{x}_0$, then $\underline{x}_0$ is Lyapunov asymptotically stable.*

Proof. Let $V : U \to \mathbb{R}$ be the strict Lyapunov function for $\underline{x}_0$. By Theorem 2.9 the fixed point $\underline{x}_0$ is Lyapunov stable. We now need to show that the domain of asymptotic stability of $\underline{x}_0$ contains a ball $B_\delta(\underline{x}_0)$.

Let us fix $\varepsilon > 0$, and let $\delta > 0$ be such that $d(\underline{x}_0, \underline{y}) < \delta$ implies $d(\underline{x}_0, \phi_t(\underline{y})) < \varepsilon$ for all $t \geq 0$. Hence $\mathcal{O}^+(\underline{y}) \subset B_\varepsilon(\underline{x}_0)$ for all $\underline{y} \in B_\delta(\underline{x}_0)$, and by Proposition 1.1 we have that $\omega(\underline{y})$ is a non-empty, compact, invariant subset of $B_\varepsilon(\underline{x}_0)$ for all $\underline{y} \in B_\delta(\underline{x}_0)$.

Let us fix $\underline{y} \in B_\delta(\underline{x}_0)$. Condition (ii)' in Definition 2.5 implies that $V(\phi_t(\underline{y}))$ is a decreasing function of t, hence there exists

$$c := \lim_{t\to+\infty} V(\phi_t(\underline{y}))$$

But $V|_{\omega(\underline{y})} \equiv c$ by continuity, in fact for all $\underline{z} \in \omega(\underline{y})$ we have

$$V(\underline{z}) = \lim_{k\to\infty} V(\phi_{t_k}(\underline{y})) = c$$

where $\{t_k\}_k$ is the diverging sequence such that $\phi_{t_k}(\underline{y}) \to z$ as $k \to \infty$. Finally, since $\omega(\underline{y})$ is invariant, we have $V(\phi_t(\underline{z})) = c$ for all t, which by (2.1) implies $\dot{V}(\underline{z}) = 0$ for all $\underline{z} \in \omega(\underline{y})$. Hence $\omega(\underline{y}) \subset \{\dot{V} \equiv 0\}$, and by condition (ii)' $\omega(\underline{y}) = \{\underline{x}_0\}$.

We have thus proved that $B_\delta(\underline{x}_0) \subset D(\underline{x}_0)$. □

Corollary 2.11 (La Salle's Invariance Principle)**.** *Let $\underline{x}_0$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. If there exists a Lyapunov function for $\underline{x}_0$ defined on a neighbourhood U of $\underline{x}_0$, then for all $\underline{y} \in U$ such that $\mathcal{O}^+(\underline{y})$ is contained in U and is bounded, we have $\omega(\underline{y}) \subseteq \{\dot{V} \equiv 0\}$.*

Example 2.2. Let us consider the system in $\mathbb{R}^2$ given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = -y^3 - x - x^3 \end{cases}$$

The point $(0,0)$ is the only fixed point and it is not hyperbolic. Looking for a Lyapunov function of the form $V(x,y) = ax^2 + bx^4 + cy^2$ one finds $\dot{V}(x,y) = 2xy(a-c) + 2x^3y(2b-c) - 2cy^4$. Hence

$$V(x,y) = 2x^2 + x^4 + 2y^2$$

is a Lyapunov function for $(0,0)$, with $\{\dot{V} \equiv 0\} = \{y = 0\}$. Hence V is not a strict Lyapunov function. By Theorem 2.9 we have that $(0,0)$ is Lyapunov stable, and applying Corollary 2.11 we also obtain that there exists $\delta > 0$ such that for all $\underline{y} \in B_\delta((0,0))$ it holds $\omega(\underline{y}) \subset \{y = 0\}$. Moreover, since $\omega(\underline{y})$ is an invariant set and the only invariant subset of $\{y = 0\}$ is $\{(0,0)\}$, we have proved that $(0,0)$ is asymptotically stable.

Theorem 2.12 (Bounding functions)**.** *Let F be a vector field in $\mathbb{R}^n$, and assume that there exist a C^1 real function $V : \mathbb{R}^n \to \mathbb{R}$, a compact set $G \subset \mathbb{R}^n$ and $k \in \mathbb{R}$ such that: (a) $G \subset V_k := \{V < k\}$; (b) there exists $\delta > 0$ such that $\dot{V}(\underline{x}) \le -\delta$ for all $\underline{x} \in \mathbb{R}^n \setminus G$. Then for all $\underline{x} \in \mathbb{R}^n$ there exists $t_0 \ge 0$ such that $\phi_t(\underline{x}) \in V_k$ for all $t > t_0$.*

Proof. If $\underline{x} \in V_k$ we are done, since by assumption (b) $\dot{V}|_{\partial V_k} < 0$, and we can choose $t_0 = 0$. If $\underline{x} \notin V_k$ and $\phi_t(\underline{x}) \notin V_k$ for all $t > 0$

$$V(\phi_t(\underline{x})) - V(\underline{x}) = \int_0^t \frac{d}{ds} V(\phi_s(\underline{x}))\, ds = \int_0^t \dot{V}(\phi_s(\underline{x}))\, ds \le -\delta t$$

which implies $V(\phi_t(\underline{x})) < k$ for $t > \frac{V(\underline{x})-k}{\delta}$. Hence we find a contradiction, and we have thus proved that there exists $t_0 > 0$ such that $\phi_{t_0}(\underline{x}) \in V_k$, and as before this implies that $\phi_t(\underline{x}) \in V_k$ for all $t \ge t_0$. □

Example 2.3 (Lorenz equations). Let us consider the system in $\mathbb{R}^3$ given by

$$\begin{cases} \dot{x} = \sigma(-x+y) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}$$

with σ, r, b positive constants. We can apply Theorem 2.12 with

$$G = \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, rx^2 + y^2 + b(z-r)^2 < 2br^2 \right\}$$

$$V(x,y,z) = \frac{1}{2}\Big(rx^2 + \sigma y^2 + \sigma (z-2r)^2 \Big)$$

and $\delta = \sigma b r^2$.

Using the theory of Lyapunov functions we now give a proof of the asymptotic stability of *sinks*, i.e. hyperbolic fixed points of a C^1 vector field with all eigenvalues of the Jacobian matrix of the field with negative real part.

Corollary 2.13. *Let $\underline{x}_0$ be a hyperbolic fixed point of a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, and assume that all the eigenvalues of $JF(\underline{x}_0)$ have negative real part. Then $\underline{x}_0$ is asymptotically stable.*

Proof. Let's assume without loss of generality that $\underline{x}_0 = \underline{0}$, then the vector field F satisfies $F(\underline{0}) = \underline{0}$ and can be written as

$$F(\underline{x}) = JF(\underline{0})\underline{x} + G(\underline{x})$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function satisfying $G(\underline{0}) = \underline{0}$ and $JG(\underline{0}) = 0$.

Let $\lambda_1, \dots, \lambda_k$ be the, not necessarily distinct, negative real eigenvalues of $JF(\underline{0})$, and let $a_j \pm i b_j$, with $j = 1, \dots, \frac{1}{2}(n-k)$, be the, not necessarily distinct, couples of conjugate complex eigenvalues with $a_j < 0$. For simplicity we also assume that $JF(\underline{0})$ is written in Jordan normal form, therefore

$$JF(\underline{0}) = diag\Big(\Lambda_1, \dots, \Lambda_h, B_1, \dots B_m\Big)$$

where the Λ_j's are the Jordan blocks relative to the real eigenvalues, and the B_j's are the Jordan blocks relative to the complex eigenvalues.

Let us consider the following change of variables. For $\varepsilon > 0$ let $\underline{y} = (y_1, \dots, y_n)$ be defined as follows:

- if $(x_m, \dots, x_{m+s-1})$ are the components of $\underline{x}$ corresponding to a Jordan block Λ_j, we let $y_{m+\ell} := \varepsilon^{-\ell} x_{m+\ell}$ for $\ell = 0, \dots, s-1$;
- if $(x_p, \dots, x_{p+2s-1})$ are the components of $\underline{x}$ corresponding to a Jordan block B_j, we let $y_{p+2\ell} := \varepsilon^{-\ell} x_{p+2\ell}$ and $y_{p+2\ell+1} := \varepsilon^{-\ell} x_{p+2\ell+1}$, for $\ell = 0, \dots, s-1$.

Then it is a standard computation to verify that $\underline{y}$ satisfies the ODE

$$\dot{\underline{y}} = A_\varepsilon \underline{y} + \tilde{G}(\underline{y}),$$

with $\tilde{G}(\underline{0}) = \underline{0}$ and $J\tilde{G}(\underline{0}) = 0$ and

$$A_\varepsilon = diag\Big(\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_h, \tilde{B}_1, \dots \tilde{B}_m\Big),$$

where

$$\tilde{\Lambda}_j = \begin{pmatrix} \lambda_j & \varepsilon & 0 & \dots & 0 & 0 \\ 0 & \lambda_j & \varepsilon & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda_j & \varepsilon & 0 \\ 0 & \dots & \dots & 0 & \lambda_j & \varepsilon \\ 0 & \dots & \dots & \dots & 0 & \lambda_j \end{pmatrix}$$

and

$$\tilde{B}_j = \begin{pmatrix} R_j & \varepsilon I_2 & 0 & \dots & 0 & 0 \\ 0 & R_j & \varepsilon I_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & R_j & \varepsilon I_2 & 0 \\ 0 & \dots & \dots & 0 & R_j & \varepsilon I_2 \\ 0 & \dots & \dots & \dots & 0 & R_j \end{pmatrix}, \quad \text{with } R_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}$$

and I_2 the 2×2 identity matrix.

We now show that $V(\underline{y}) = \sum_{i=1}^{n} y_i^2$ is a strict Lyapunov function for $\underline{0}$. It is enough to study the derivative $\dot{V}(\underline{y}) = 2\sum_{i=1}^{n} y_i \dot{y}_i$.

If $(y_m, \dots, y_{m+s-1})$ are the components of $\underline{y}$ corresponding to a Jordan block $\tilde{\Lambda}_j$ we have

$$\sum_{\ell=0}^{s-1} y_{m+\ell}\, \dot{y}_{m+\ell} = \sum_{\ell=0}^{s-2} y_{m+\ell}(\lambda_j y_{m+\ell} + \varepsilon y_{m+\ell+1}) + \lambda_j y_{m+s-1}^2 + O(|\underline{y}|^3) =$$

$$\leq \lambda_j \sum_{\ell=0}^{s-1} y_{m+\ell}^2 + \frac{\varepsilon}{2}(y_m^2 + y_{m+s-1}^2) + \varepsilon \sum_{\ell=1}^{s-2} y_{m+\ell}^2 + O(|\underline{y}|^3) \leq$$

$$\leq (\lambda_j + \varepsilon) \sum_{\ell=0}^{s-1} y_{m+\ell}^2 + O(|\underline{y}|^3).$$

With an analogous argument, if $(y_m, \dots, y_{m+2s-1})$ are the components of $\underline{y}$ corresponding to a Jordan block $\tilde{B}_j$ we have

$$\sum_{\ell=0}^{s-1}(y_{m+2\ell}\, \dot{y}_{m+2\ell} + y_{m+2\ell+1}\, \dot{y}_{m+2\ell+1}) =$$

$$= \sum_{\ell=0}^{s-2} y_{m+2\ell}\,(a_j y_{m+2\ell} - b_j y_{m+2\ell+1} + \varepsilon y_{m+2\ell+2}) +$$

$$+ \sum_{\ell=0}^{s-2} y_{m+2\ell+1}\,(b_j y_{m+2\ell} + a_j y_{m+2\ell+1} + \varepsilon y_{m+2\ell+3}) +$$

$$+ y_{m+2s-2}\,(a_j y_{m+2s-2} - b_j y_{m+2s-1}) + y_{m+2s-1}\,(b_j y_{m+2s-2} + a_j y_{m+2s-1}) +$$

$$+ O(|\underline{y}|^3) =$$

$$= a_j \sum_{\ell=0}^{s-1}(y_{m+2\ell}^2 + y_{m+2\ell+1}^2) + \varepsilon \sum_{\ell=0}^{s-2} \Big(y_{m+2\ell}\, y_{m+2\ell+2} + y_{m+2\ell+1}\, y_{m+2\ell+3}\Big) +$$

$$+ O(|\underline{y}|^3) \leq$$

$$\leq (a_j + \varepsilon) \sum_{\ell=0}^{s-1}(y_{m+2\ell}^2 + y_{m+2\ell+1}^2) + O(|\underline{y}|^3).$$

If we fix $\varepsilon > 0$ such that $(\lambda_j + \varepsilon) < 0$ and $(a_j + \varepsilon) < 0$ for all eigenvalues of $JF(\underline{0})$, letting $\mu \in \mathbb{R}^-$ satisfy $(\lambda_j + \varepsilon) \le \mu < 0$ and $(a_j + \varepsilon) \le \mu < 0$ for all j, we have proved that

$$\dot{V}(\underline{y}) \le 2\mu|\underline{y}|^2 + O(|\underline{y}|^3).$$

We need to show that there exists $\delta > 0$ such that $\dot{V}(\underline{y}) < 0$ for all $\underline{y} \in B_\delta(\underline{0})$ and $\underline{y} \ne \underline{0}$. By definition of $O(\cdot)$ functions, there exist $c > 0$ and $\tilde{\delta} > 0$ such that

$$O(|\underline{y}|^3) \le c|\underline{y}|^3\,, \quad \forall\, \underline{y} \in B_{\tilde{\delta}}(\underline{0}).$$

If we choose $\delta = \min\{-\frac{2\mu}{c}, \tilde{\delta}\}$ it follows

$$\dot{V}(\underline{y}) \le 2\mu|\underline{y}|^2 + c|\underline{y}|^3 = |\underline{y}|^2(2\mu + c|\underline{y}|) < 0, \quad \forall\, \underline{y} \in B_\delta(\underline{0}) \setminus \{\underline{0}\},$$

and the proof is finished. $\square$