2.3 Integrals of motion and invariant sets

Conservative systems and first integrals

Definition 2.6. A C^1 function $I : \mathbb{R}^n \to \mathbb{R}$ is a *first integral* for a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ if $\dot{I}(\underline{x}) = 0$ for all $\underline{x} \in \mathbb{R}^n$, with $\dot{I}(\underline{x})$ defined as in (2.1).

If $I : \mathbb{R}^n \to \mathbb{R}$ is a first integral for a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, then its level sets are invariant for the differential equation $\dot{\underline{x}} = F(\underline{x})$, so that in particular orbits of $\dot{\underline{x}} = F(\underline{x})$ lie in the level sets of I.

An important example of differential equations with a first integral are Hamiltonian systems with Hamiltonian function independent of time.

Definition 2.7. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a C^1 function and use the notation $(\underline{x}, \underline{y})$ for points in $\mathbb{R}^{2n}$, with $\underline{x}, \underline{y} \in \mathbb{R}^n$. The *Hamiltonian vector field associated to* H is $F_H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given for $i = 1, \dots, n$, by $(F_H)_i = \partial H / \partial y_i$ and $(F_H)_{(n+i)} = -\partial H / \partial x_i$, and H is called the *Hamiltonian function* of the field. The system of differential equations in $\mathbb{R}^{2n}$ with field F_H is called the *Hamiltonian system of* H.

A particular case are conservative mechanical systems with one degree of freedom, systems which describe for example the motion in $\mathbb{R}$ of a point of mass m under conservative forces. In this case the Hamiltonian function has the form

$$H : \mathbb{R}^2 \to \mathbb{R}\,, \quad H(x,y) = \frac{1}{2m}\, y^2 + W(x) \tag{2.2}$$

where $W(x) \in C^1$ is the potential energy of the system. We recall that in this case the Hamiltonian system associated to H is

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x,y) = \frac{1}{m}\, y \\ \dot{y} = -\frac{\partial H}{\partial x}(x,y) = -W'(x) \end{cases}$$

and corresponds to the second-order differential equation $m\ddot{x} = -W'(x)$.

Proposition 2.14. *A C^1 function H is a first integral for the Hamiltonian vector field F_H.*

Proof. A simple computation gives

$$\dot{H} = \langle \nabla H\,,\, F_H \rangle = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial x_i} \frac{\partial H}{\partial y_i} - \frac{\partial H}{\partial y_i} \frac{\partial H}{\partial x_i} \Big) \equiv 0.$$

□

Theorem 2.15 (Liouville theorem)**.** *A Hamiltonian system in $\mathbb{R}^{2n}$ with C^2 Hamiltonian function H preserves the $2n$-dimensional Lebesgue measure of the sets.*

Proof. For $A \subset \mathbb{R}^{2n}$ let $\phi_t(A)$ be the evolution of the set at time t, and let m be the $2n$-dimensional Lebesgue measure. Then

$$m(\phi_t(A)) = \int_{\phi_t(A)} 1\, dm = \int_A |\det(J\phi_t)|\, dm.$$

The variation equation of a differential equation shows that $J\phi_t$ satisfies the Cauchy problem

$$\begin{cases} \frac{d}{dt}\, J\phi_t(\underline{x}) = JF_H(\phi_t(\underline{x}))\, J\phi_t(\underline{x}) \\ J\phi_t(\underline{x})|_{t=0} = I \end{cases}$$

where I is the identity matrix. The solution to the previous Cauchy problem is then

$$J\phi_t(\underline{x}) = \exp\Big(\int_0^t JF_H(\phi_s(\underline{x}))\, ds \Big)\, I,$$

and using the identity $\det(\exp(M)) = \exp(\mathrm{tr}(M))$, valid for any finite square matrix M, we obtain

$$\det(J\phi_t(\underline{x})) = \exp\Big(\int_0^t \mathrm{tr}(JF_H(\phi_s(\underline{x})))\, ds \Big).$$

Then

$$m(\phi_t(A)) = \int_A \exp\left(\int_0^t \mathrm{div}(F_H)(\phi_s(\underline{x}))\, ds \right)\, dm.$$

Since

$$\mathrm{div}(F_H) = \sum_{i=1}^n \Big(\frac{\partial^2 H}{\partial x_i \partial y_i} - \frac{\partial^2 H}{\partial y_i \partial x_i} \Big) \equiv 0,$$

it follows that

$$m(\phi_t(A)) = m(A)\,, \quad \forall\, t \in \mathbb{R}$$

and the proof is finished. □

Corollary 2.16. *A Hamiltonian system in $\mathbb{R}^{2n}$ cannot have fixed points which are sinks or sources.*

Let us consider mechanical Hamiltonian systems with one degree of freedom with Hamiltonian function $H(x, y)$ as in (2.2). Applying the general theory of the previous sections and the results in this section, one can easily prove the following characterisation of the fixed points.

Proposition 2.17. *Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function written as in* (2.2). *Then the fixed points of the associated Hamiltonian system are of the form* $(x_0, 0)$ *with* $W'(x_0) = 0$.
If $W''(x_0) < 0$ *then* $(x_0, 0)$ *is a hyperbolic fixed point of saddle type, if* $W''(x_0) > 0$ *it is not hyperbolic and it is a center.*
If $W''(x_0) = 0$ *the point* $(x_0, 0)$ *is not hyperbolic and one needs to use the level sets of* $H(x, y)$ *to study the dynamics in a neighbourhood of the point.*

Example 2.4. The Hamiltonian function of a pendulum of mass m and length ℓ in a vertical gravitational field with potential energy $W(h) = mgh$ is

$$H(x, y) = \frac{1}{2m\ell^2}\, y^2 + mg\ell(1 - \cos x)\,,$$

Consider the motion of this pendulum in presence of a constant friction given by $-\mu\, y$, with $\mu \geq 0$.

Example 2.5. Study the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 - \mu\, y \end{cases}$$

with $\mu \in \mathbb{R}$.

Invariant sets

It is in general difficult to find explicit expressions for invariant sets. However, there are particular easy situations. For example, given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \ldots, F_n)$, if there exists $c \in \mathbb{R}$ such that $F_i(x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_n) = 0$ for all $x_j \in \mathbb{R}$ with $j \neq i$, then the hyperplane $\{x_i = c\}$ is an invariant set. This can be proved by the following method.

Proposition 2.18. *Let $I : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and for $c \in \mathbb{R}$ let $I_c := \{I(\underline{x}) = c\}$ be a non-empty level set of I such that $\nabla I|_{I_c} \not\equiv 0$. The level set I_c is invariant for a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ if $\dot{I}|_{I_c} \equiv 0$.*

Proof. Let $\underline{x}_0 \in I_c$ such that $\nabla I(\underline{x}_0) \neq 0$. Then there exists a local differentiable change of coordinates $\underline{y} = h(\underline{x})$ such that in a neighbourhood $U(\underline{x}_0)$ we have $I_c \cap U = \{y_n = 0\}$ and let $\underline{x}_0 = (\underline{\tilde{y}}_0, 0)$ with $\underline{\tilde{y}}_0 \in \mathbb{R}^{n-1}$. Hence, in these new coordinates $\nabla I \in \mathrm{Span}\{(0, \ldots, 0, 1)\}$ in U.

Then, from $\dot{I}|_{I_c} \equiv 0$, we have that $F_n|_U \equiv 0$. Let $\tilde{F} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be defined as $\tilde{F}(y_1, \ldots, y_{n-1}) = (F_1(y_1, \ldots, y_{n-1}, 0), \ldots, F_{n-1}(y_1, \ldots, y_{n-1}, 0))$.

Then by the local uniqueness of the solutions to the system $\dot{\underline{x}} = F(\underline{x})$, the solution with initial condition in $\underline{x}_0$ coincides in U with $(\tilde{\phi}_t(\underline{\tilde{y}}_0), 0)$, where $\tilde{\phi}_t$ is the flow of the system $\dot{\underline{\tilde{y}}} = \tilde{F}(\underline{\tilde{y}})$. Hence, the solution is in I_c. This proves the invariance of I_c. □

Example 2.6. Given the system

$$\begin{cases} \dot{x} = x^2 - y - 1 \\ \dot{y} = (x-2)y \end{cases}$$

the lines $y = 0$, $y = x + 1$ and $y = 3x - 3$ are invariant sets.

Stable and unstable manifolds

An important example of invariant sets is given by the *stable* and *unstable manifolds* of a hyperbolic fixed point.

Definition 2.8. Let $\underline{x}_0$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with flow $\phi_t(\cdot)$, and let U be a neighbourhood of $\underline{x}_0$. The *local stable manifold* $W^s_{loc}(\underline{x}_0)$ *of* $\underline{x}_0$ *in* U is the set

$$W^s_{loc}(\underline{x}_0) := \{\underline{x} \in U \ : \ \phi_t(\underline{x}) \in U \text{ for all } t \geq 0, \ \phi_t(\underline{x}) \to \underline{x}_0 \text{ as } t \to +\infty\}$$

Analogously, the *local unstable manifold* $W^u_{loc}(\underline{x}_0)$ *of* $\underline{x}_0$ *in* U is the set

$$W^u_{loc}(\underline{x}_0) := \{\underline{x} \in U \ : \ \phi_t(\underline{x}) \in U \text{ for all } t \leq 0, \ \phi_t(\underline{x}) \to \underline{x}_0 \text{ as } t \to -\infty\}$$

Theorem 2.19 (Stable and unstable manifolds)**.** *Let $\underline{x}_0$ be a fixed point of a C^k, $k \geq 1$, vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with flow $\phi_t(\cdot)$. Let's assume that $\underline{x}_0$ is hyperbolic and let $E^s(\underline{0})$ and $E^u(\underline{0})$ be the stable and unstable eigenspaces associated to the linear system $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$. Then there exists $\varepsilon > 0$ such that there exist local stable and unstable manifolds, $W^s_{loc}(\underline{x}_0)$ and $W^u_{loc}(\underline{x}_0)$, of $\underline{x}_0$ in $B_\varepsilon(\underline{x}_0)$ with the following properties:*

(i) $W^s_{loc}(\underline{x}_0)$ *and* $W^u_{loc}(\underline{x}_0)$ *are unique in* $B_\varepsilon(\underline{x}_0)$*;*

(ii) $W^s_{loc}(\underline{x}_0)$ *is forward invariant, and* $W^u_{loc}(\underline{x}_0)$ *is backward invariant;*

(iii) $W^s_{loc}(\underline{x}_0)$ *and* $W^u_{loc}(\underline{x}_0)$ *are* C^k *manifolds,* $\dim W^s_{loc}(\underline{x}_0) = \dim E^s(\underline{0})$ *and* $\dim W^u_{loc}(\underline{x}_0) = \dim E^u(\underline{0})$*;*

(iv) $W^s_{loc}(\underline{x}_0)$ *is tangential to* $\underline{x}_0 + E^s(\underline{0})$ *at* $\underline{x}_0$*, and* $W^u_{loc}(\underline{x}_0)$ *is tangential to* $\underline{x}_0 + E^u(\underline{0})$ *at* $\underline{x}_0$*.*

Proof of Theorem 2.19 in $\mathbb{R}^2$ (see [HSD]). Without loss of generality, let's assume that the fixed point is $(x_0, y_0) = (0,0)$. If $\dim E^s = 2$ or $\dim E^u = 2$, the proof is trivial since in these cases either $W^s_{loc}(0,0)$ or $W^u_{loc}(0,0)$, respectively, coincide with a ball around $(0,0)$ and the properties of the statement follow from the Hartman-Grobman Theorem 2.8.

The interesting case is when $\dim E^s = \dim E^u = 1$ and $(0,0)$ is a saddle. Up to a change of variables, we can assume that the system is written as

$$\begin{cases} \dot{x} = -\lambda\, x + f(x,y) \\ \dot{y} = \mu\, y + g(x,y) \end{cases} \tag{2.3}$$

with $\lambda, \mu > 0$, $f, g \in C^k$ with $f(0,0) = g(0,0) = 0$, and $f, g = O(x^2 + y^2)$ if $k \geq 2$, and $f, g, \partial_x f, \partial_y f, \partial_x g, \partial_y g = o(\sqrt{x^2+y^2})$ if $k = 1$. Hence, $E^s = \mathrm{Span}\{(1,0)\}$ and $E^u = \mathrm{Span}\{(0,1)\}$.

We give the proof for the local stable manifold, it follows analogously for the local unstable one. For any $\varepsilon > 0$ and $M > 1$, introduce the following notations:

$$\begin{aligned} &D_\varepsilon := \{|x| \leq \varepsilon,\ |y| \leq \varepsilon\}\,, \quad C_M := \{|x| \geq M\,|y|\}\,, \\ &S^\pm_\varepsilon := C_M \cap \{x = \pm\varepsilon\}\,, \quad C^\pm_M := C_M \cap \{x \gtrless 0\}\,. \end{aligned} \tag{2.4}$$

The proof is divided into different steps.

Step I. There exists $\varepsilon_0 > 0$ such that for all $M > 1$ we have $\dot{x}|_{D_\varepsilon \cap C^\pm_M} \lessgtr 0$. By assumption, there exists $\varepsilon > 0$ such that

$$|f(x,y)| \leq \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2+y^2}\,, \quad \forall\,(x,y) \in D_\varepsilon.$$

Then, on $D_\varepsilon \cap C^+_M$, we have

$$\begin{aligned} \dot{x} &= -\lambda\,x + f(x,y) \leq -\lambda\,x + \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2+y^2} \leq \\ &\leq -\lambda\,x + \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2\left(1+\frac{1}{M^2}\right)} \leq x\left(-\lambda + \frac{\lambda}{2}\right) = -\frac{\lambda}{2}\,x < 0. \end{aligned}$$

Similarly, on $D_\varepsilon \cap C^-_M$, we have

$$\begin{aligned} \dot{x} &= -\lambda\,x + f(x,y) \geq -\lambda\,x - \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2+y^2} \geq \\ &\geq -\lambda\,x - \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2\left(1+\frac{1}{M^2}\right)} \geq |x|\left(\lambda - \frac{\lambda}{2}\right) = \frac{\lambda}{2}\,|x| > 0. \end{aligned}$$

Step II. For any $M > 1$, there exists $\varepsilon_1 = \varepsilon_1(M) > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$ on the boundary of $D_\varepsilon \cap C_M$ the field F points towards the outside of C_M.
First of all, we consider $\varepsilon \in (0, \varepsilon_0)$ with ε_0 from Step I. Let us study the case $x, y > 0$. We have $\dot{x}|_{\partial(D_\varepsilon \cap C_M^+)} < 0$. It is then enough to prove that $\dot{y}|_{\partial(D_\varepsilon \cap C_M^+)} > 0$. Fixed $M > 1$, by assumption, there exists $\varepsilon_1 < \varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_1)$

$$|g(x,y)| \leq \frac{\mu}{2\sqrt{1+M^2}} \sqrt{x^2+y^2}\,, \quad \forall\,(x,y) \in D_\varepsilon.$$

Then, on $\partial(D_\varepsilon \cap C_M^+)$, we have

$$\dot{y} = \mu\, y + g(x,y) \geq \mu\, y - \frac{\mu}{2\sqrt{1+M^2}} \sqrt{x^2+y^2} =$$

$$= \mu\, y - \frac{\mu}{2\sqrt{1+M^2}} \sqrt{y^2(M^2+1)} = y\left(\mu - \frac{\mu}{2}\right) = \frac{\mu}{2}\, y > 0.$$

The other cases follow analogously.

Step III. For $\varepsilon \in (0, \varepsilon_1)$, on S_ε^+ there exist non-empty open intervals I_+ and I_- such that for all $(x_0, y_0) \in I_\pm$ the orbit $\phi_t(x_0, y_0)$ intersects $\partial(D_\varepsilon \cap C_M^+) \cap \{y \gtrless 0\}$.
The existence and the properties of the intervals I_+ and I_- follow from Steps I and II, and from the local uniqueness and the continuity with respect to the initial conditions of the solutions to (2.3).

Step IV. There exists $\varepsilon_2 < \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2)$ The set $S_\varepsilon^+ \setminus (I_+ \cup I_-)$ consists of a single point $(\varepsilon, \bar{y}_+(\varepsilon))$.
By the properties of the solutions to (2.3), there exist y_1, y_2 such that

$$S_\varepsilon^+ \setminus (I_+ \cup I_-) = \{(\varepsilon, y)\ : y \in [y_1, y_2]\}\,.$$

We need to show that $y_1 = y_2 = \bar{y}_+$.

Let's assume that $y_1 < y_2$. It is known that multiplying a vector field $F(x,y)$ by a non-vanishing function $h(x,y)$, the orbits of the system do not change but only their time-parametrisation is affected. Let $h(x,y) = 1/(\lambda - f(x,y)/x)$ in $D_\varepsilon \cap C_M$. Then the system in $D_\varepsilon \cap C_M$ becomes

$$\begin{cases} \dot{x} = -x \\ \dot{y} = \frac{\mu\, y + g(x,y)}{\lambda - \frac{f(x,y)}{x}} = \frac{\mu}{\lambda}\, y + \tilde{g}(x,y) \end{cases} \tag{2.5}$$

with $\tilde{g}, \partial_y \tilde{g} = o(\sqrt{x^2+y^2})$. There exists $\varepsilon_2 < \varepsilon_1$ such that for all $M > 1$ and all $\varepsilon \in (0, \varepsilon_2)$ we have

$$\left|\frac{\partial \tilde{g}}{\partial y}(x,y)\right| \leq \frac{\mu}{2\lambda\sqrt{2}} \sqrt{x^2+y^2}\,, \quad \forall\,(x,y) \in D_\varepsilon.$$

Then the solutions to (2.5) with initial condition (ε, y) are of the form $(\varepsilon\, e^{-t}, y(t))$. Hence, we can compute the vertical distance between the orbits $\phi_t(\varepsilon, y_1)$ and $\phi_t(\varepsilon, y_2)$ by computing the distance $y_2(t) - y_1(t)$ of the second components of the solutions to (2.5) with initial conditions (ε, y_1) and (ε, y_2). We have

$$\begin{aligned}\frac{d}{dt}(y_2(t)-y_1(t)) &= \frac{\mu}{\lambda}\, y_2(t) + \tilde{g}(\varepsilon\, e^{-t}, y_2(t)) - \frac{\mu}{\lambda}\, y_1(t) + \tilde{g}(\varepsilon\, e^{-t}, y_1(t)) = \\ &= \frac{\mu}{\lambda}\,(y_2(t)-y_1(t)) + \frac{\partial \tilde{g}}{\partial y}(\varepsilon\, e^{-t}, \xi(t))\,(y_2(t)-y_1(t)) \geq \\ &\geq (y_2(t)-y_1(t)) \left(\frac{\mu}{\lambda} - \frac{\mu}{2\lambda\sqrt{2}}\,\varepsilon\, e^{-t}\sqrt{1+\frac{1}{M^2}}\right) \geq \\ &\geq \frac{\mu}{2\lambda}\,(y_2(t)-y_1(t)).\end{aligned}$$

Hence, $(y_2(t) - y_1(t)) \to +\infty$, which contradicts that the orbits of the set $S_\varepsilon^+ \setminus (I_+ \cup I_-)$ are forward asymptotic to $(0,0)$. We have thus proved that $y_1 = y_2 = \bar{y}_+$.

Conclusion part I.
By Steps I-IV, for all $M > 1$ there exists $\varepsilon_2 = \varepsilon_2(M) > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$ we obtain the existence of a unique point $(\varepsilon, \bar{y}_+(\varepsilon))$ in S_ε^+ whose orbit is forward asymptotic to $(0,0)$. Therefore, fixing a $\bar{M} > 1$ the local stable manifold $W^s_{loc}(0,0)$ in $D_\varepsilon \cap C_M^+$ for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$ is given by the forward orbit of the point $(\varepsilon_2(\bar{M}), \bar{y}_+(\varepsilon_2(\bar{M})))$. An analogous argument shows the existence of the local stable manifold $W^s_{loc}(0,0)$ in $D_\varepsilon \cap C_M^-$ for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$.

This shows the uniqueness of $W^s_{loc}(0,0)$, its forward invariance, its regularity since the orbits of a system inherit the regularity of the vector field, and that its dimension is 1. It remains to prove that $W^s_{loc}(0,0)$ is tangent at $(0,0)$ to $E^s = \mathrm{Span}\{(1,0)\}$, that is to the x-axis.

Step V. Fixing a $\bar{M} > 1$, for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$ the orbit $\phi_t(\varepsilon, \bar{y}_+(\varepsilon))$ has vanishing angular coefficient as $t \to +\infty$.

Let $(x_\varepsilon(t), y_\varepsilon(t))$ denote the two components of $\phi_t(\varepsilon, \bar{y}_+(\varepsilon))$. We need to show that $y_\varepsilon(t)/x_\varepsilon(t) \to 0$ as $t \to +\infty$.

By the local uniqueness of the solutions to (2.3), for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$ the point $(\varepsilon, \bar{y}_+(\varepsilon))$ is in the orbit of $(\varepsilon', \bar{y}_+(\varepsilon'))$ for all $\varepsilon' \in (\varepsilon, \varepsilon_2(\bar{M}))$. This shows that, for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$, for all $t \geq 0$ there exists $\tilde{\varepsilon}(t) < \varepsilon$ such that $(x_\varepsilon(t), y_\varepsilon(t)) = (\tilde{\varepsilon}(t), \bar{y}_+(\tilde{\varepsilon}(t)))$. Hence, as $t \to +\infty$ we have $\tilde{\varepsilon}(t) \to 0^+$ so that $(x_\varepsilon(t), y_\varepsilon(t))$ is in $\cap_{\bar{M} \leq M \leq \tilde{M}(t)} C^+_M$ for some $\tilde{M}(t) \to +\infty$. This shows that $y_\varepsilon(t)/x_\varepsilon(t) \leq 1/\tilde{M}(t) \to 0^+$ as $t \to +\infty$.

Conclusion part II.
Step V concludes the proof of the theorem. □

Given a hyperbolic fixed point $\underline{x}_0$, one can introduce a notion of *global* stable and unstable manifolds. However, these sets in general have weaker properties than the local counterparts.

Definition 2.9. Let $\underline{x}_0$ be a hyperbolic fixed point of a C^k, $k \geq 1$, vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with flow $\phi_t(\cdot)$. The *global stable* and *unstable manifolds* of $\underline{x}_0$ are defined as

$$W^s(\underline{x}_0) := \bigcup_{t \leq 0} \phi_t(W^s_{loc}(\underline{x}_0)), \qquad W^u(\underline{x}_0) := \bigcup_{t \geq 0} \phi_t(W^u_{loc}(\underline{x}_0)), \tag{2.6}$$

where $W^{s,u}_{loc}(\underline{x}_0)$ are the local manifolds in $B_\varepsilon(\underline{x}_0)$ for some $\varepsilon > 0$.

It is interesting to analyse the possible intersection of the global stable and unstable manifolds. By the local uniqueness of the solutions to an ODE, the two global manifolds cannot intersect transversally. In $\mathbb{R}^2$ they can coincide or end up at another saddle fixed point, giving rise to a homoclinic or two heteroclinic orbits respectively. In $\mathbb{R}^n$ with $n \geq 3$ more interesting phenomena occurs, and some imply the existence of "chaotic" phenomena (see Section 3.4).