Chapter 1

Introduction and basic concepts

1.1 Dynamical systems

Definition 1.1. A *dynamical system* is a triple $(X, G, \mathcal{S})$ defined as the action $\mathcal{S}$ of a semigroup G with identity e on a set X, that is a function

$$\mathcal{S} : G \times X \to X$$

such that $\mathcal{S}(e, x) = x$ for all $x \in X$, and $\mathcal{S}(g_1, \mathcal{S}(g_2, x)) = \mathcal{S}(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and all $x \in X$.

In the following, the set X is assumed to be a locally compact connected metric space.

Two main examples of dynamical system are given in the following definitions.

Definition 1.2. A *discrete-time dynamical system* is defined by the action of $\mathbb{N}_0$ on a set X defined through the iterations of a map $T : X \to X$ by

$$\mathcal{S}(n, x) = T^n(x),$$

where $T^n = T \circ \cdots \circ T$ is the composition of T with itself n times. A discrete-time dynamical system is denoted by the triple $(X, \mathbb{N}_0, T)$.

If the map T is invertible, the system can be extended to the action of the group $\mathbb{Z}$ on X. Examples of a discrete-time dynamical system are sequences defined by a recurrence relation. Let $\{x_n\}$ be a sequence of real numbers defined by

$$x_0 = a \in \mathbb{R}, \qquad x_n = f(x_{n-1}) \quad \forall\, n \geq 1,$$

for a real-valued function f. This corresponds to the dynamical system defined on $X = \mathbb{R}$ through the iterations of the map $f : \mathbb{R} \to \mathbb{R}$, that is $x_n = f^n(a)$.

Definition 1.3. A *continuous-time dynamical system* is defined by the action of $\mathbb{R}$ on a set $X \subset \mathbb{R}^n$ defined through the flow $\phi_t(\underline{x})$ of an autonomous ordinary differential equation $\dot{\underline{x}}(t) = F(\underline{x})$, that is

$$\mathcal{S}(t, \underline{x}) = \phi_t(\underline{x}),$$

where $\phi_t(\underline{x})$ is the solution of an ordinary differential equation[1] with initial condition $\underline{x}$, and $\phi_t : X \to X$ is a continuous function. A continuous-time dynamical system is denoted by the triple $(X, \mathbb{R}, \phi)$.

Definiton 1.3 includes the case of non-autonomous differential equations by using the standard procedure of "enlarging" the space of variables. Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ define a time-dependent vector field $F(t, \underline{x})$ on $\mathbb{R}^n$ and consider the Cauchy problem

$$\begin{cases} \dot{\underline{x}}(t) = F(t, \underline{x}(t)) \\ \underline{x}(0) = \underline{x}_0 \end{cases}$$

If we let $\underline{y} = (\underline{x}, t) \in \mathbb{R}^{n+1}$ and $\tilde{F}(\underline{y}) = (F(t, \underline{x}), 1)$ be a vector field on $\mathbb{R}^{n+1}$, the previous non-autonomous Cauchy problem is equivalent to the autonomous problem

$$\begin{cases} \dot{\underline{y}}(t) = \tilde{F}(\underline{y}(t)) \\ \underline{y}(0) = (\underline{x}_0, 0) \end{cases}$$

A similar procedure can be applied to the case of sequences defined by a recurrence relation depending on n.

Analogously, it is known that ordinary differential equations of order greater than one can be reduced to systems of ordinary differential equations of order one, hence again included in Definition 1.3. The same is true for the discrete-time case. The following example shows how the procedure works.

[1] All ordinary differential equations we consider are assumed to have the property of local uniqueness of solutions and time-interval of existence of solutions given by $\mathbb{R}$ up to reparametrization.

Example 1.1. Let us consider the sequence $\{x_n\}$ defined as follows

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 1, \quad x_n = x_{n-1} + 2^{n-3}\, x_{n-2} + x_{n-3} \quad \forall\, n \geq 4.$$

We define the vector $\underline{y}_n = (x_n, x_{n-1}, x_{n-2}, n) \in \mathbb{R}^4$. Then using the previous recurrence we have

$$\underline{y}_{n+1} = \Big(x_n + 2^{n-2}\, x_{n-1} + x_{n-2}\, ,\, x_n\, ,\, x_{n-1}\, ,\, n+1\Big) = T(\underline{y}_n) \quad \forall\, n \geq 3$$

with initial condition set to be $y_3 = (1,1,0,3)$ and $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T(a,b,c,d) = \Big(a + 2^{d-2}\, b + c\, ,\, a\, ,\, b\, ,\, d+1\Big).$$

The idea of an action of a semigroup on a set X can be used in more abstract contexts. Here we show only one example of algebraic nature that will be studied in more details in part IV of this book.

Example 1.2. Let X be a group, G be $\mathbb{R}$, and consider the action S on X given by multiplication for a one-parameter subgroup of X. For example, if $X = SL(2,\mathbb{R})$ the action of $\mathbb{R}$ defined by

$$S(t,x) = x \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in SL(2,\mathbb{R})$$

represents the geodesic flow on the hyperbolic Poincaré half-plane (see Chapter 9).

1.2 Basic notions

Definition 1.4. Given a dynamical system $(X, G, \mathcal{S})$, the *orbit* of a point $x \in X$ is the set $\mathcal{O}(x) := \{\mathcal{S}(g,x) \,:\, g \in G\}$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, the orbit of a point $x \in X$ is the set

$$\mathcal{O}(x) = \{T^n(x) \,:\, n \in \mathbb{N}_0\}\,. \tag{1.1}$$

If the map T is invertible, then we can consider the action of the group $\mathbb{Z}$ on X and define the *forward orbit* and *backward orbit* of a point $x \in X$ by

$$\mathcal{O}^+(x) := \{T^n(x) \,:\, n \geq 0\}\,, \quad \mathcal{O}^-(x) := \{T^n(x) \,:\, n \leq 0\}\,.$$

The orbit $\mathcal{O}(x)$ is then given by $\mathcal{O}^+(x) \cup \mathcal{O}^-(x)$.

For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the *forward orbit* and *backward orbit* of a point $\underline{x} \in X$ are defined by

$$\mathcal{O}^+(\underline{x}) := \bigcup_{t\geq 0} \phi_t(\underline{x}), \qquad \mathcal{O}^-(\underline{x}) := \bigcup_{t\leq 0} \phi_t(\underline{x}), \tag{1.2}$$

and the orbit is $\mathcal{O}(\underline{x}) = \mathcal{O}^+(\underline{x}) \cup \mathcal{O}^-(\underline{x})$.

Definition 1.5. Given a dynamical system $(X, G, \mathcal{S})$, the *centralizer* of a point $x \in X$ is the sub-semigroup

$$\mathcal{C}(x) := \{g \in G \,:\, S(g, x) = x\}\,.$$

A point x is called *fixed* if $\mathcal{C}(x) = G$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, a point $x \in X$ is fixed if and only if $T(x) = x$. If x is not a fixed point but its centralizer is not G, x is called *periodic* and the minimum positive element in $\mathcal{C}(x)$ is the *minimal period* of x. For a fixed point $\mathcal{O}(x) = \{x\}$, and for a periodic point of minimal period p

$$\mathcal{O}(x) = \left\{x,\, T(x),\, T^2(x),\, \ldots,\, T^{p-1}(x)\right\}.$$

For a non-invertible map there might be points which are not periodic but are pre-images of a periodic point. For such points x, the centralizer contains only the identity of G, but there exists $k \geq 1$ such that $\mathcal{C}(T^k(x))$ has a minimal positive element p. These points are called *pre-periodic with minimal period p*.

For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$ given by the solutions to $\dot{\underline{x}}(t) = F(\underline{x})$, a point $\underline{x} \in X$ is fixed if and only if $F(\underline{x}) = \underline{0}$. If $\underline{x}$ is not a fixed point but its centralizer is not trivial, $\underline{x}$ is called *periodic* and the minimum positive element in $\mathcal{C}(\underline{x})$ is the *minimal period* of T. A periodic point $\underline{x}$ of minimal period $T > 0$ satisfies

$$\phi_{t+T}(\underline{x}) = \phi_t(\underline{x}), \quad \forall\, t \in \mathbb{R},$$

and

$$\phi_{t+s}(\underline{x}) \neq \phi_t(\underline{x}), \quad \forall\, s \in (0, T),\, t \in \mathbb{R}.$$

For a fixed point $\mathcal{O}(\underline{x}) = \{\underline{x}\}$. For a periodic point of minimal period T

$$\mathcal{O}(\underline{x}) = \bigcup_{0\leq t\leq T} \phi_t(\underline{x}),$$

and its orbits is called *a periodic orbit of period T*.

Definition 1.6. Given a dynamical system $(X, G, \mathcal{S})$, a set $A \subset X$ is called *invariant* if for each $x \in A$ it holds $\mathcal{S}(g, x) \in A$ for all $g \in G$.

For a continuous-time dynamical system one can introduce a weaker notion. We say that a subset A of X is *forward invariant* if for each $\underline{x} \in A$ it holds $\phi_t(\underline{x}) \in A$ for all $t \geq 0$. Analogously A is called *backward invariant* if the same relation holds for all $t \leq 0$. By definition, A is *invariant* if the previous relation holds for all $t \in \mathbb{R}$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, we consider more situations. We say that a subset A of X is *forward invariant* if $T(A) \subseteq A$, A is called *fully invariant* if $T(A) = A$, A is called *completely invariant* if $T^{-1}(A) = A$. The different notions are useful in different approaches.

Finally, if the action of the group G on X can be interpreted in terms of time evolution, we can introduce notions about the forward and backward evolution of an orbit. In more general situations, one studies the set of all the possible limit points of an orbit as the sequence of the elements of the group acting varies.

Definition 1.7. For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, the *ω-limit set* of a point $x \in X$ is the set

$$\omega(x) := \{y \in X \,:\, \exists\, n_k \to +\infty \text{ such that } T^{n_k}(x) \to y \text{ as } k \to \infty\}.$$

Definition 1.8. For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the *α-limit set* of a point $\underline{x} \in X$ is the set

$$\alpha(\underline{x}) := \left\{\underline{y} \in X \,:\, \exists\, t_k \to -\infty \text{ such that } \phi_{t_k}(\underline{x}) \to \underline{y} \text{ as } k \to \infty\right\}.$$

Analogously the *ω-limit set* of a point $\underline{x} \in X$ is the set

$$\omega(\underline{x}) := \left\{\underline{y} \in X \,:\, \exists\, t_k \to +\infty \text{ such that } \phi_{t_k}(\underline{x}) \to \underline{y} \text{ as } k \to \infty\right\}.$$

Proposition 1.1. *Given a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, let $\underline{x} \in X$ such that $\mathcal{O}^+(\underline{x})$ is bounded. Then the set $\omega(\underline{x})$ is non-empty, compact and invariant. If $\mathcal{O}^-(\underline{x})$ is bounded, the same holds for the set $\alpha(\underline{x})$.*

Proof (see [Gl94]). Given a point $\underline{x}$ with bounded forward orbit, let us consider a strictly increasing sequence $\{\tau_j\}_{j=0}^{\infty}$ of times in $\mathbb{R}^+$ with $\tau_0 = 0$ and $\tau_j \to +\infty$, and let $\underline{x}_j := \phi_{\tau_j}(\underline{x})$. We first show that

$$\omega(\underline{x}) = \bigcap_{j=0}^{\infty} \overline{\mathcal{O}^+(\underline{x}_j)}. \tag{1.3}$$

By the definition of the ω-limit set, it is immediate that $\omega(\underline{x}) \subset \overline{\mathcal{O}^+(\underline{x}_j)}$ for all $j \geq 0$. Hence it remains to show that if $\underline{y} \in \cap_{j=0}^\infty \overline{\mathcal{O}^+(\underline{x}_j)}$ then $\underline{y} \in \omega(\underline{x})$. By definition of closure of a set, for all $j \geq 0$ there exists a sequence $\{\underline{\xi}_n^j\}_n$ of points in $\mathcal{O}^+(\underline{x}_j)$ such that $\underline{\xi}_n^j \to \underline{y}$, hence there exists a sequence $\{t_n^j\}_n$ such that $\phi_{t_n^j}(\underline{x}_j) \to \underline{y}$. In particular we have proved that there exists a strictly increasing diverging sequence $\{\tau_j\}_{j=0}^\infty$ and sequences $\{t_n^j\}_n$ such that

$$\phi_{\tau_j + t_n^j}(\underline{x}) \xrightarrow[n\to\infty]{} \underline{y}, \quad \forall\, j \geq 0.$$

From $\{\tau_j + t_n^j\}_{j,n}$ we can then extract a diverging sequence $\{\tilde{t}_k\}_k$ such that $\phi_{\tilde{t}_k}(\underline{x}) \to \underline{y}$ as $k \to \infty$. Hence $\underline{y} \in \omega(\underline{x})$, and (1.3) is proved.

The first properties of $\omega(\underline{x})$ follow from (1.3). The sets $\{\overline{\mathcal{O}^+(\underline{x}_j)}\}_j$ define a decreasing sequence of non-empty closed sets, which are bounded because $\mathcal{O}^+(\underline{x})$ is bounded. Hence $\omega(\underline{x})$ is a non-empty compact set. It remains to prove that it is invariant.

Let $\underline{y} \in \omega(\underline{x})$, and let $\{t_k\}_k$ be a positively diverging sequence such that $\phi_{t_k}(\underline{x}) \to \underline{y}$ as $k \to \infty$. By the properties of a continuous-time dynamical system

$$\phi_{t+t_k}(\underline{x}) = \phi_t(\phi_{t_k}(\underline{x})) \xrightarrow[k\to\infty]{} \phi_t(\underline{y}), \quad \forall\, t \in \mathbb{R}.$$

Hence we have shown that $\phi_t(\underline{y}) \in \omega(\underline{x})$ for all $t \in \mathbb{R}$. This concludes the proof for the ω-limit set.

The proof for the α-limit set follows along the same lines. □

Proposition 1.2. *Given a discrete-time dynamical system* $(X, \mathbb{N}_0, T)$*, let* $x \in X$ *such that* $\mathcal{O}(x)$ *is bounded. Then the set* $\omega(x)$ *is non-empty and compact. If* T *is continuous then* $\omega(x)$ *is fully invariant.*

Proof. We can repeat the proof of Proposition 1.1 to show that the ω-limit set is non-empty and compact. In particular the proof follows from the analogue of (1.3).

Let $T : X \to X$ be a continuous map with respect to a topological structure on X. Then given $y \in \omega(x)$, and being $\{n_k\}_k$ the diverging sequence of naturals for which $T^{n_k} \to y$ as $k \to \infty$, we have

$$T^{n_k+1}(x) = T(T^{n_k}(x)) \xrightarrow[k\to\infty]{} T(y).$$

Hence $T(y) \in \omega(x)$, and $\omega(x)$ is a positively invariant set. On the other hand, since $\mathcal{O}(x)$ is bounded, the sequence $\{T^{n_k-1}(x)\}_k$ admits a convergent sub-sequence $\{T^{n_{k_j}-1}(x)\}_j$ with limit point z. Hence $z \in \omega(x)$. Again by continuity of T we find

$$T(z) = T\Big(\lim_{j\to\infty} T^{n_{k_j}-1}(x)\Big) = \lim_{j\to\infty} T^{n_{k_j}}(x) = y$$

since n_{k_j} is a subsequence of n_k. Hence $y \in T(\omega(x))$, and $\omega(x)$ is then fully invariant. □

We remark that the ω-limit set is not completely invariant in general. It is sufficient to think of the case in which the ω-limit set is a fixed point with more than one pre-image.

Definition 1.9. For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the orbit of a point $\underline{y}$ is called *homoclinic* if there exists a fixed point $\underline{x}$ such that

$$\alpha(\underline{y}) = \omega(\underline{y}) = \{\underline{x}\}.$$

If there exist two distinct fixed points $\underline{x}_1$, $\underline{x}_2$ such that

$$\alpha(\underline{y}) = \{\underline{x}_1\} \quad \text{and} \quad \omega(\underline{y}) = \{\underline{x}_2\},$$

then the orbit of the point $\underline{y}$ is called *heteroclinic*.

Definition 1.9 can be adapted verbatim to the case of a discrete-time dynamical system $(X, \mathbb{N}_0, T)$ with invertible T.

1.3 Examples

Here we collect the main examples of discrete dynamical systems that will be used in the following.

Example 1.3 (The roots). Sequences defined by a recurrence are the first very basic example of a discrete-time dynamical system. Let $c > 0$, $k \in [1, 3]$, and consider the sequence $\{a_n\}$ defined by

$$\begin{cases} a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n^k}\right), & \forall\, n \geq 0 \\ a_0 \in (0, +\infty) \end{cases}$$

It is an exercise to prove that for all $a_0 \in \mathbb{R}^+$ it holds $\lim_n a_n = c^{\frac{1}{k+1}}$. This can be read as a result about the asymptotic behaviour of the orbits of points in $\mathbb{R}^+$ for the dynamical system defined by the map

$$T_{c,k} : \mathbb{R}^+ \to \mathbb{R}^+, \quad T_{c,k}(x) = \frac{1}{2}\left(x + \frac{c}{x^k}\right).$$

In fact one can prove that $\omega(x) = c^{\frac{1}{k+1}}$ for all $x \in \mathbb{R}^+$.

Example 1.4 (Rotations of the circle). Let us consider the action of $\mathbb{Z}$ on S^1 given by the rotation of an angle $2\pi\alpha$, for $\alpha \in \mathbb{R}$, that is

$$\mathcal{S}(n, z) = z\, e^{2\pi i n\alpha} \in S^1, \quad \forall\, z \in S^1,\, n \in \mathbb{Z}.$$

By writing $S^1 = \{z \in \mathbb{C} \,:\, z = e^{2\pi i x}, x \in \mathbb{R}\}$, we make the identification of S^1 with $[0,1]/(0 \sim 1)$, the unit interval with end points identified. The rotation of angle $2\pi\alpha$ can then be written as a map on S^1 as

$$R_\alpha : S^1 \to S^1, \quad R_\alpha(x) = x + \alpha \ (\mathrm{mod}\ 1) = \{x + \alpha\}. \tag{1.4}$$

Proposition 1.3. *If α is rational all orbits of R_α are periodic of the same minimal period. If α is irrational all orbits of R_α are dense.*

Proof. If $\alpha = p/q \in \mathbb{Q}$ with $(p, q) = 1$, then $R^q_\alpha(x) = \{x + q\alpha\} = x$ for all $x \in [0, 1)$. In addition, if $n \in \mathbb{N}$ and $n < q$, we can write $n\alpha = np/q = m+r/q$ with $m \in \mathbb{Z}$ and $r/q \in \mathbb{Q} \cap (0, 1)$. Hence $R^n_\alpha(x) = \{x + r/q\} \neq x$. It follows that all orbits are periodic of the minimal period q.

Let's now assume that α is irrational. Since R_α is an isometry, it is enough to show that one orbit is dense. In fact, we prove that forward orbits are dense by considering $\{R^n_\alpha(0)\}_{n\geq 0}$.

Let $x \in S^1$, then we show that for any $\varepsilon > 0$ there exists $\bar{n}$ such that $R^{\bar{n}}_\alpha(0) \in (x - \varepsilon\,,\, x + \varepsilon)$. First, by Proposition B.3, we find $p, q \in \mathbb{N}$ such that $0 < q\alpha - p < \varepsilon$. This means that $R^q_\alpha(0) \in (-\varepsilon, \varepsilon)$. If we now consider the points $\{k(q\alpha - p)\}_{k\geq 0}$, it follows that there exists $K > 0$ such that the points $\{k(q\alpha - p)\}_{0\leq k\leq K}$ create a partition of $[0, 1]$ into intervals of length less than ε. Therefore for all $x \in S^1$

$$\min_{0\leq k\leq K} d(x, k(q\alpha - p)) < \varepsilon$$

and the minimum is achieved for some value $\bar{k}$. Hence choosing $\bar{n} = \bar{k}q$ the proof is finished. □

A consequence of the proposition is that if α is rational, then all points have their own periodic orbit as α-limit and ω-limit sets. Instead, if α is irrational, then $\alpha(x) = \omega(x) = S^1$ for all x.

Example 1.5 (The tent maps). It is a family of maps

$$T_s : [0, 1] \to [0, 1] \quad \text{with } s \in (0, 2)$$

defined as

$$T_s(x) = \begin{cases} s\,x, & \text{if } x \in [0, \frac{1}{2}]; \\ s\,(1 - x), & \text{if } x \in [\frac{1}{2}, 1]. \end{cases} \tag{1.5}$$

Example 1.6 (The logistic maps). It is a family of maps

$$T_\lambda : [0,1] \to [0,1] \quad \text{with } \lambda \in (0,4]$$

defined as

$$T_\lambda(x) = \lambda\, x\, (1-x). \tag{1.6}$$

Example 1.7 (Linear endomorphisms of the circle). It is a family of maps

$$T_m : S^1 \to S^1 \quad \text{with } m \in \mathbb{N},\, m \geq 2$$

where again we think of S^1 as $[0,1]/(0 \sim 1)$, defined as

$$T_m(x) = \{mx\}. \tag{1.7}$$

Special cases are $m = 2$ which is also called the *Bernoulli map* and is related with the binary expansion of real numbers, and $m = 10$ which is related with the decimal expansion of real numbers.

Example 1.8 (Symbolic dynamics). We now introduce an abstract system. Let $\mathcal{A}$ be a finite or countable alphabet and denote by $N \in \mathbb{N} \cup \{\infty\}$ the number of symbols. Let $\Omega_\mathcal{A}$ be the set of all infinite strings with symbols from $\mathcal{A}$, that is

$$\Omega_\mathcal{A} = \mathcal{A}^{\mathbb{N}_0} = \{\omega = (\omega_i)_{i\in\mathbb{N}_0} \,:\, \omega_i \in \mathcal{A}\ \forall\, i \in \mathbb{N}_0\}\,.$$

If $N < \infty$, the space X is compact when endowed with the product topology or with the metric

$$d_\theta(\omega, \tilde\omega) := \theta^{\min\{i\in\mathbb{N}_0\,:\,\omega_i \neq \tilde\omega_i\}}\,, \quad \text{for a fixed } \theta \in (0,1). \tag{1.8}$$

The space $\Omega_\mathcal{A}$ is totally disconnected and a basis of the product topology is given by the *cylinders*: for $k \in \mathbb{N}$, $i_1, i_2, \dots, i_k \in \mathbb{N}_0$, and $a_1, a_2, \dots, a_k \in \mathcal{A}$, we define

$$C_{i_1,i_2,\dots,i_k}(a_1, a_2, \dots, a_k) := \big\{\omega \in \Omega_\mathcal{A} \,:\, \omega_{i_j} = a_j\ \forall\, j = 1, \dots, k\big\}\,.$$

In particular, we use the notations $C(a) = C_1(a)$ and

$$C_{i_1,i_2,\dots,i_k}(\omega) = C_{i_1,i_2,\dots,i_k}(\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_k})$$

for a fixed $\omega \in \Omega_\mathcal{A}$.

On $\Omega_{\mathcal{A}}$ we consider the discrete dynamical system given by the action of the continuous map

$$\sigma : \Omega_{\mathcal{A}} \to \Omega_{\mathcal{A}}, \quad (\sigma(\omega))_i = \omega_{i+1} \quad \forall\, i \in \mathbb{N}_0.$$

The system $(\Omega_{\mathcal{A}}, \mathbb{N}_0, \sigma)$ is called *full shift on* $\mathcal{A}$.

In some situations it is useful to consider a sub-system of the full shift. A first easy example is given by considering infinite strings which cannot contain a given set of words of finite length. For example, let $M = (m_{ij}) \in M(N \times N, \{0,1\})$, a $N \times N$ matrix with coefficients in the set $\{0,1\}$ and rows and columns indexed by $\mathcal{A}$. We set

$$\Omega_{\mathcal{A},M} := \left\{ \omega \in \mathcal{A}^{\mathbb{N}_0} \,:\, m_{\omega_i \, \omega_{i+1}} = 1 \;\forall\, i \in \mathbb{N}_0 \right\},$$

that is, saying that the transition from $a \in \mathcal{A}$ to $b \in \mathcal{A}$ is allowed iff $m_{ab} = 1$, the set $\Omega_{\mathcal{A},M}$ contains the infinite strings in $\mathcal{A}^{\mathbb{N}_0}$ which contain only allowed transitions. It is immediate to verify that $\Omega_{\mathcal{A},M}$ is forward invariant for the action of σ and it is fully invariant if for each $b \in \mathcal{A}$ there exists $a \in \mathcal{A}$ with $m_{ab} = 1$. Hence we can restrict the action of σ to $\Omega_{\mathcal{A},M}$, and the dynamical system $(\Omega_{\mathcal{A},M}, \mathbb{N}_0, \sigma)$ is called *subshift of finite type on* $\mathcal{A}$.

Finally, by considering bi-infinite strings $\mathcal{A}^{\mathbb{Z}}$, one can consider the action of σ on $\mathcal{A}^{\mathbb{Z}}$ and on $\mathcal{A}^{\mathbb{Z}}_M$. In this case the map σ is invertible and the dynamical systems $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z}, \sigma)$ and $(\mathcal{A}^{\mathbb{Z}}_M, \mathbb{Z}, \sigma)$ are called double full shift and double subshift of finite type, respectively.

Example 1.9 (Toral automorphisms). Let $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus. Given a matrix $A \in M(2 \times 2, \mathbb{Z})$ with $\det(A) = \pm 1$, the linear map $\mathbb{R}^2 \ni \underline{x} \mapsto A\,\underline{x}$ may be projected onto a continuous automorphisms of $\mathbb{T}^2$ given by

$$\mathbb{T}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto T_A(x,y) := A \begin{pmatrix} x \\ y \end{pmatrix} \mod \mathbb{Z}^2.$$

The most famous example is the so-called *Arnold's Cat map*, which is the toral automorphism given by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Example 1.10 (The standard map). Let us consider an electron with charge e moving horizontally in a cyclotron thanks to the action of a vertical magnetic field of constant modulus B, and subject to a time-dependent voltage drop $V\sin(\omega t)$ across a narrow azimuthal gap. Let E denote the energy of the electron, then the period of rotation is given by $T = 2\pi\,\frac{E}{eBc}$. We measure energy and time (E,t) just before every voltage drop, hence after one circuit we obtain

$$E' = E - e\,V\sin(\omega t)\,, \qquad t' = t + \frac{2\pi}{eBc}\,E'\,.$$

Using the variables $x := \frac{\omega}{2\pi}t$ and $y := \frac{\omega}{eBc}E$, and setting $k := 2\pi\frac{\omega V}{Bc}$, we have defined the map

$$\tilde{T} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}\,, \quad \tilde{T}(x,y) = \Big(x + y - \frac{k}{2\pi}\,\sin(2\pi x)\,,\, y - \frac{k}{2\pi}\,\sin(2\pi x)\Big)\,.$$

Note that $\tilde{T}(x+1,y) = \tilde{T}(x,y) + (1,0)$, hence given the projection $\pi : \mathbb{R}\times\mathbb{R} \to S^1\times\mathbb{R}$ defined as $\pi(x,y) = (x - \lfloor x \rfloor, y)$, it follows that the map

$$T : S^1\times\mathbb{R} \to S^1\times\mathbb{R}\,, \quad T(x,y) = \Big(x + y - \frac{k}{2\pi}\,\sin(2\pi x)\,,\, y - \frac{k}{2\pi}\,\sin(2\pi x)\Big) \tag{1.9}$$

satisfies $\pi\circ\tilde{T} = T\circ\pi$. Hence $\tilde{T}$ is a lift of T. The map T is known as the *(Chirikov) standard map.*

Note also that $T(x,y+1) = T(x,y) + (0,1)$, hence the standard map can be considered as acting on $\mathbb{T}^2$.

Example 1.11 (Birkhoff billiards). Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with C^3 boundary[2]. Let us normalize the set to $|\partial\Omega| = 1$ and fix the positive orientation of the boundary.

The *mathematical billiard* is the continuous dynamical system given by the frictionless motion of a pointwise ball inside Ω, with elastic specular reflections at $\partial\Omega$. The phase space is then given by $\Omega\times S^1$, since the velocity of the ball is preserved in modulus.

A convenient simpler description of the system is given by the Poincaré map of the flow on the set $\partial\Omega\times[0,\pi]$, described by the evolution of the couples (position,angle) of the subsequent collisions of the ball with the boundary of the set. For each collision, its position can be described by the arc-length coordinate $s \in S^1$ and its angle by the angle $\vartheta \in [0,\pi]$ between the trajectory of the ball after the collision and the oriented tangent vector to $\partial\Omega$ at the collision point. We have thus described a map

$$T : S^1\times(0,\pi) \to S^1\times(0,\pi)$$

[2]Thanks to [Ha77] this assumption avoid accumulation of collision times

which can be continuously extended to $S^1 \times [0, \pi]$ by $T(s, 0) = (s, 0)$ and $T(s, \pi) = (s, \pi)$.

This map may be defined with some cautions for more general domains $\Omega \subset \mathbb{R}^2$ (see [CM06]).

Example 1.12 (Mechanics and Billiards). Let m_1 and m_2 be two distinct point masses moving frictionless on the interval $[0, 1]$, subject to perfectly elastic collisions among them and with two infinite ideal walls at the extremes of the interval. Let $x_1, x_2 \in [0, 1]$ with $x_1 \leq x_2$, and $v_1, v_2 \in \mathbb{R}$, denote the positions and velocities of the masses, and introduce the variables $q_1 := \sqrt{m_1}\, x_1$ and $q_2 := \sqrt{m_2}\, x_2$, and $u_1 = \sqrt{m_1}\, v_1$ and $u_2 := \sqrt{m_2}\, v_2$. The invariances of the kinetic energy K and of the linear momentum P of the system read in the new variables as

$$u_1^2 + u_2^2 = 2K\,, \quad \sqrt{m_1}\, u_1 + \sqrt{m_2}\, u_2 = P\,.$$

In the new variables, the configuration space is given by the triangle

$$A = \left\{(q_1, q_2) \in \mathbb{R}^2 \,:\, q_1 \geq 0\,,\, q_2 \leq \sqrt{m_2}\,,\, \sqrt{m_2}\, q_1 \leq \sqrt{m_1}\, q_2\right\}\,.$$

A trajectory $(q_1(t), q_2(t))$ satisfies the following constraints:

$$\dot{q}_1^2(t) + \dot{q}_2^2(t) = 2K \qquad \forall\, t$$

(hence the motion occurs with constant speed);

$$\sqrt{m_1}\, \dot{q}_1 + \sqrt{m_2}\, \dot{q}_2 = P \qquad \forall\, t$$

(hence the velocity vector of the motion has fixed scalar product with the vector $(\sqrt{m_1}, \sqrt{m_2})$).

These properties imply that the motion $(q_1(t), q_2(t))$ in A can be described by the orbit of a mathematical billiard ball inside A.

1.4 Exercises

1.1. Let $T : [0, 1] \to [0, 1]$ be defined by

$$T(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \in (0, 1]; \\ 1, & \text{if } x = 0. \end{cases}$$

Show that for all $x \in [0, 1]$ the ω-limit set $\omega(x)$ is non-empty but not forward invariant.

1.2. In Example 1.12 let the masses move in $[0, +\infty)$, and consider the motion with initial positions $q_1(0) < q_2(0)$ and velocities $u_1(0) = 0$ and $u_2(0) = -1$. If $m_2 \geq m_1$, how many collisions among the two balls and among mass m_1 and the wall at $x = 0$ will occur? What happens if $m_2 = 100^n m_1$?