Esercizio 1 $\quad X_1, \dots, X_{90}$ v.a. indipendenti di Bernoulli di parametro p

a) "0" 27 volte, "1" 63 volte. $\quad \bar{p} = \frac{63}{90} = \frac{7}{10}$

$$E[X_i] = \bar{p}$$

$\tilde{p} = \frac{7}{10}$ stima con il metodo dei momenti

b) $X_1 + \dots + X_{90}$ è una binomiale di parametri $n = 90$, $p = \frac{7}{10}$

$$P\{X_1 + \dots + X_{90} = k\} = \begin{cases} \binom{90}{k} \left(\frac{7}{10}\right)^k \left(\frac{3}{10}\right)^{90-k}, & k \in \mathbb{N},\ 0 \le k \le 90 \\ 0, & \text{alt.} \end{cases}$$

c) $P\{X_1 + \dots + X_{90} \ge 60\}$

$$\frac{X_1 + X_2 + \dots + X_{90} - 90\,E[X_i]}{\sqrt{90\, \mathrm{Var}(X_i)}} \sim Z = N(0,1)$$

$$E[X_i] = \frac{7}{10}, \quad \mathrm{Var}(X_i) = \frac{7}{10} \cdot \frac{3}{10}$$

$$P\{X_1 + \dots + X_{90} \ge 60\} = P\left\{ \frac{X_1 + \dots + X_{90} - 90 \cdot \frac{7}{10}}{\sqrt{90 \cdot \frac{21}{100}}} \ge \frac{60 - 90 \cdot \frac{7}{10}}{\sqrt{90 \cdot \frac{21}{100}}} \right\}$$

$$\sim P\left\{ Z \ge -\sqrt{\tfrac{10}{21}} \right\} = 1 - \Phi\left(-\sqrt{\tfrac{10}{21}}\right) = \Phi\left(\sqrt{\tfrac{10}{21}}\right) \sim$$

$$\sim \Phi(0.69) \sim 0.7549$$

ESERCIZIO 2

$$f(x) = \begin{cases} c\, x\, e^{-\frac{x^2}{2}}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

a) $f(x) \ge 0 \quad \forall x \in \mathbb{R}$, $\quad \int_{\mathbb{R}} f(x)\,dx = 1$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$c \ge 0 \qquad\qquad \int_0^{+\infty} c\, x\, e^{-\frac{x^2}{2}}\,dx = 1$$

$$\Updownarrow$$

$$c\left[-e^{-\frac{x^2}{2}}\right]_0^{+\infty} = c = 1$$

$$F(x) = P\{X \le x\} = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x \le 0 \\ 1-e^{-\frac{x^2}{2}}, & x \ge 0 \end{cases}$$

$$\int_0^x f(t)\,dt = \int_0^x t\, e^{-\frac{t^2}{2}}\,dt = \left[-e^{-\frac{t^2}{2}}\right]_0^x = 1 - e^{-\frac{x^2}{2}}$$

F è crescente su $\{x > 0\}$, il β-quantile r_β è l'unica soluzione di $F(r_\beta) = \beta \in (0,1)$.

$$1 - e^{-\frac{r_\beta^2}{2}} = \beta \iff r_\beta = \sqrt{2 \log \frac{1}{1-\beta}}$$

b) $\varphi_{X^2}(t) = ?$ $\qquad \varphi_{X^2}(t) = E[e^{tX^2}] = \int_{\mathbb{R}} e^{tx^2} f(x)\,dx$

$$\varphi_{X^2}(t) = \int_0^{+\infty} e^{tx^2} \cdot x\, e^{-\frac{x^2}{2}}\,dx = \frac{1}{2(t-\frac{1}{2})} \int_0^{+\infty} 2(t-\tfrac{1}{2})\, x\, e^{(t-\frac{1}{2})x^2}\,dx$$

$$= \left[\frac{1}{2(t-\frac{1}{2})}\, e^{(t-\frac{1}{2})x^2}\right]_0^{+\infty} = \begin{cases} \infty, & \text{se } t \ge \frac{1}{2} \\ \frac{1}{2(\frac{1}{2}-t)}, & \text{se } t < \frac{1}{2} \end{cases}$$

ESERCIZIO 3

Pesando 71 volte un oggetto si ottiene $\bar{m} = 140\ g$, $\bar{\sigma} = 0.4\ g$.

a) $H_0)\ \sigma^2 \geq 0.2 = \sigma_0^2$, $H_1)\ \sigma^2 < 0.2$

$$C = \left\{ \frac{(n-1)\,\bar{\sigma}^2}{\sigma_0^2} < \chi^2_{(\alpha,\,n-1)} \right\}$$

α livello del test

$n-1 = 70$

L'ipotesi è accettata al livello α se

$$\frac{(n-1)\,\bar{\sigma}^2}{\sigma_0^2} \geq \chi^2_{(\alpha,\,n-1)}$$

$\alpha = 0.2$, $n-1 = 70$, $\bar{\sigma}^2 = 0.16$, $\sigma_0^2 = 0.2$, $\chi^2_{(0.2,\,70)} \sim 59.8978$

$$\frac{70 \cdot 0.16}{0.2} = 56 \not\geq 59.8978$$

$\Rightarrow$ l'ipotesi non è accettata al livello del 20%

b) $F_{\chi^2(70)}\left(\frac{(n-1)\,\bar{\sigma}^2}{\sigma_0^2}\right) = F_{\chi^2(70)}(56) \sim 0.125$

Trovare σ_0^2 t.c. $F_{\chi^2(70)}\left(\frac{(n-1)\,\bar{\sigma}^2}{\sigma_0^2}\right) = F_{\chi^2(70)}\left(\frac{70 \cdot 0.16}{\sigma_0^2}\right) = 0.25$

$$\Leftrightarrow \frac{70 \cdot 0.16}{\sigma_0^2} = \chi^2_{(0.25,\,70)} \sim 61.6983$$

$$\Rightarrow \sigma_0^2 \sim 0.1815\ldots$$