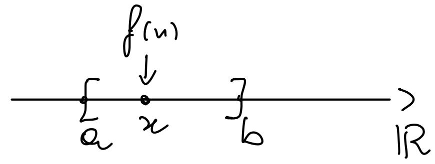
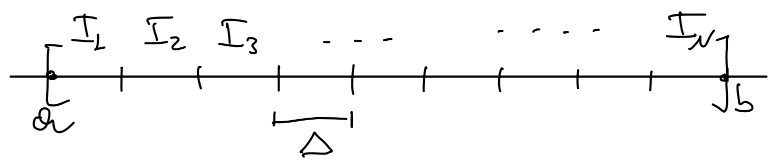
Integrali

$f: D \subseteq \mathbb{R} \to \mathbb{R}$, $[a,b] \subset D$

$$\int_a^b f(x)\,dx$$



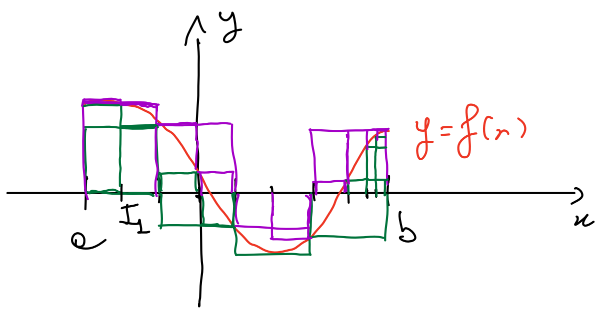
$$\int_a^b 1\,dx = b-a$$

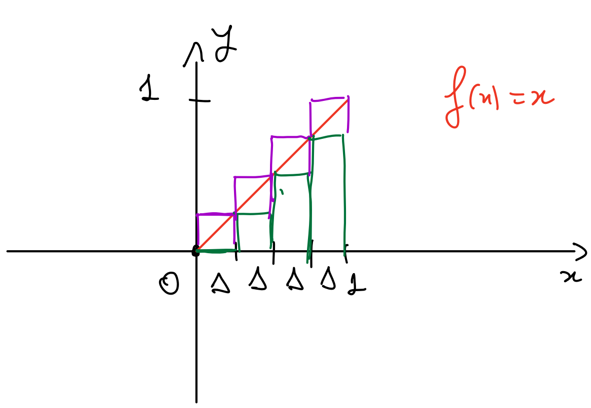


$$s(\Delta) = \sum_{i=1}^{N} |I_i| \cdot \min_{I_i} f$$

$$S(\Delta) = \sum_{i=1}^{N} |I_i| \max_{I_i} f \quad , \quad \lim_{\Delta \to 0^+} s(\Delta) = \lim_{\Delta \to 0^+} S(\Delta) =: \int_a^b f(x)\,dx$$

$$|I_i| = \Delta$$





$\Delta = \frac{1}{4}$

$$\int_0^1 x\,dx$$

$$\min_{I_1} f = f(0) = 0$$

$$\max_{I_1} f = f\left(\tfrac{1}{4}\right) = \tfrac{1}{4}$$

$$s(\Delta) = 0 + \tfrac{1}{4}\Delta + \tfrac{1}{2}\Delta + \tfrac{3}{4}\Delta$$

$$S(\Delta) = \tfrac{1}{4}\Delta + \tfrac{1}{2}\Delta + \tfrac{3}{4}\Delta + 1\cdot\Delta$$

$f(x_0) \cdot |I_i|$

Se $f|_{I_i} \equiv c$, $\quad c \cdot |I_i| = c \cdot (x_{i+1} - x_i) = F(x_{i+1}) - F(x_i)$

$I_i = (x_i, x_{i+1})$ $\qquad F(x) = c \cdot x$

$x_{i+1} - x_i = \Delta$

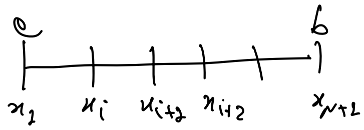
$$s(\Delta) = \sum_{i=1}^{N} |I_i| \cdot \min_{I_i} f = \sum_{i=1}^{N} \left[F^-(x_{i+1}) - F^-(x_i) \right] \xrightarrow[\Delta \to 0^+]{} \int_a^b f(x)\,dx$$

$F^-(x) = (\min_{I_i} f) \cdot x$ su I_i.

$$= \sum_{i=1}^{N} \underbrace{\frac{F^-(x_{i+1}) - F^-(x_i)}{x_{i+1} - x_i}}_{\sim (F^-)' \text{ in } I_i} \cdot (x_{i+1} - x_i) \sim \sum_{i=1}^{N} (F^-)'|_{I_i} \cdot |I_i| \xrightarrow[\Delta \to 0^+]{} \int_a^b f(x)\,dx$$

Se so che esiste $F : [a,b] \to \mathbb{R}$ tale che $F'(x) = f(x)$ allora

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0^+} \sum_{i=1}^{N} \left[F(x_{i+1}) - F(x_i) \right] = F(b) - F(a)$$



ES. $f(x) = x$ $\quad \int_0^1 x\,dx$, $\quad F(x)$ tale che $F'(x) = x$? $\quad F(x) = \frac{1}{2} x^2$

$$\int_0^1 x\,dx = F(1) - F(0) = \frac{1}{2} - 0 = \frac{1}{2}$$

· $\int_1^4 \left(\frac{1}{3} x^2 - 2x \right) dx$, $\quad F'(x) = \frac{1}{3} x^2 - 2x$, $\quad F(x) = \frac{1}{3} \cdot \frac{1}{3} x^3 - 2 \cdot \frac{1}{2} x^2 =$

$= \frac{1}{9} x^3 - x^2$

$F(4) - F(1) = \left(\frac{64}{9} - 16 \right) - \left(\frac{1}{9} - 1 \right)$.

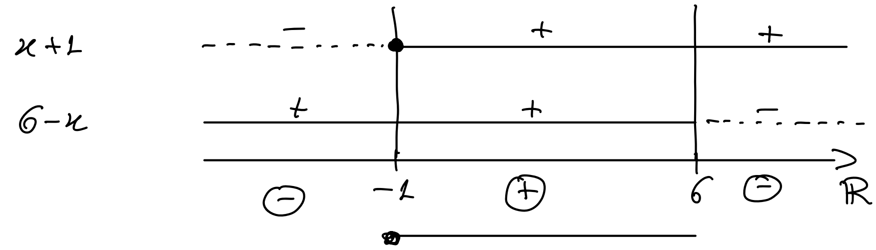
Esercizi

1. $f(x) = \sqrt[4]{\dfrac{x+1}{6-x}}$ $\qquad D = \left\{ x \in \mathbb{R} : \dfrac{x+1}{6-x} \geq 0 , \ 6 - x \neq 0 \right\} = [-1, 6)$

$$\frac{x+1}{6-x} \geq 0 \iff \begin{cases} 6 - x \neq 0 \\ x + 1 \geq 0 \\ 6 - x > 0 \end{cases} \cup \begin{cases} 6 - x \neq 0 \\ x + 1 \leq 0 \\ 6 - x < 0 \end{cases}$$

$$\iff \begin{cases} 6-x \neq 0 \\ x \geq -1 \\ x < 6 \end{cases} \cup \begin{cases} 6-x \neq 0 \\ x \leq -1 \\ x > 6 \end{cases}$$

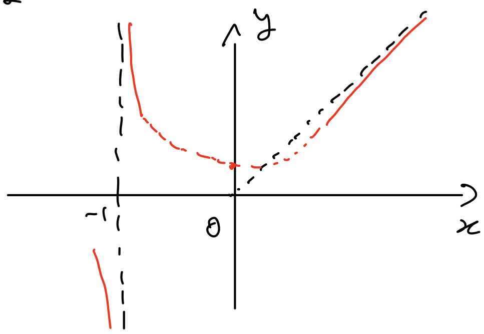
$[-1, 6)$ $\qquad\qquad$ $\emptyset$



$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \sqrt[4]{\frac{x+1}{6-x}} = 0 \quad , \quad \lim_{x \to 6^-} f(x) = \lim_{x \to 6^-} \sqrt[4]{\frac{x+1}{6-x}} = +\infty$$

$$\sqrt[4]{\frac{0^+}{7}} = 0 \qquad\qquad \sqrt[4]{\frac{7}{0^+}} = \sqrt[4]{+\infty}$$

2. $f(x) = \dfrac{x^4+1}{x^3+1}$ $\qquad$ $D = \{ x \in \mathbb{R} : x^3+1 \neq 0 \} = \mathbb{R} \setminus \{-1\}$



- $\displaystyle\lim_{x \to -1^+} \frac{x^4+1}{x^3+1} = +\infty$, $\displaystyle\lim_{x \to -1^-} \frac{x^4+1}{x^3+1} = -\infty$

$\dfrac{2}{0^+}$ $\qquad\qquad$ $\dfrac{2}{0^-}$

- $\exists\, a \neq 0,\ b \in \mathbb{R}$ t.c. $\displaystyle\lim_{x \to +\infty} \frac{f(x)}{x} = a$, $\displaystyle\lim_{x \to +\infty} [f(x) - ax] = b$ $\qquad$ asintoto obliquo $y = ax+b$

$\exists\, c \in \mathbb{R}$ t.c. $\displaystyle\lim_{x \to +\infty} f(x) = c$ $\qquad$ asintoto orizzontale $y = c$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^4+1}{x^3+1} = \lim_{x \to +\infty} \frac{x^4 (1 + 1/x^4)}{x^3 (1 + 1/x^3)} = \lim_{x \to +\infty} x \cdot \frac{1 + 1/x^4}{1 + 1/x^3} = +\infty$$

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^4+1}{x(x^3+1)} = \lim_{x\to+\infty} \frac{x^4+1}{x^4+x} = \lim_{x\to+\infty} \cancel{\frac{x^4}{x^4}} \, \frac{1+1/x^4}{1+1/x^3} = 1 = a$$

$$\lim_{x\to+\infty} \left[f(x) - x\right] = \lim_{x\to+\infty} \left[\frac{x^4+1}{x^3+1} - x\right] = \lim_{x\to+\infty} \frac{x^4+1-x(x^3+1)}{x^3+1} =$$

$$= \lim_{x\to+\infty} \frac{x^4+1-x^4-x}{x^3+1} = \lim_{x\to+\infty} \frac{-x+1}{x^3+1} = \lim_{x\to+\infty} \underbrace{\overbrace{\frac{x}{x^3}}^{\frac{1}{x^2}} \, \frac{(-1+1/x)}{(1+1/x^3)}}_{0\cdot(-1)} = 0 = b$$

$f(x)$ ha asintoto obliquo a $+\infty$ dato da $y = x$

3. $$f(x) = \begin{cases} e^{\frac{1}{x}} \cdot \sin x & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$$ in quali punti di D è continua?

$$D = \{x \in \mathbb{R} : x \neq 0\} \cup \{0\} = \mathbb{R}$$

Certamente f è continua in $\mathbb{R} \setminus \{0\}$.

Per sapere se è continua in 0, devo verificare che $\lim_{x\to 0} f(x) = f(0)$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\frac{1}{x}} \cdot \sin x$$

$$\lim_{x\to 0^+} e^{\frac{1}{x}} \sin x = \lim_{x\to 0^+} \overbrace{\frac{\sin x}{x}}^{1} \cdot \overbrace{\left(x e^{\frac{1}{x}}\right)}^{+\infty} = +\infty$$

$$e^{+\infty} \cdot \sin(0^+) = +\infty \cdot 0 \quad f.i.$$

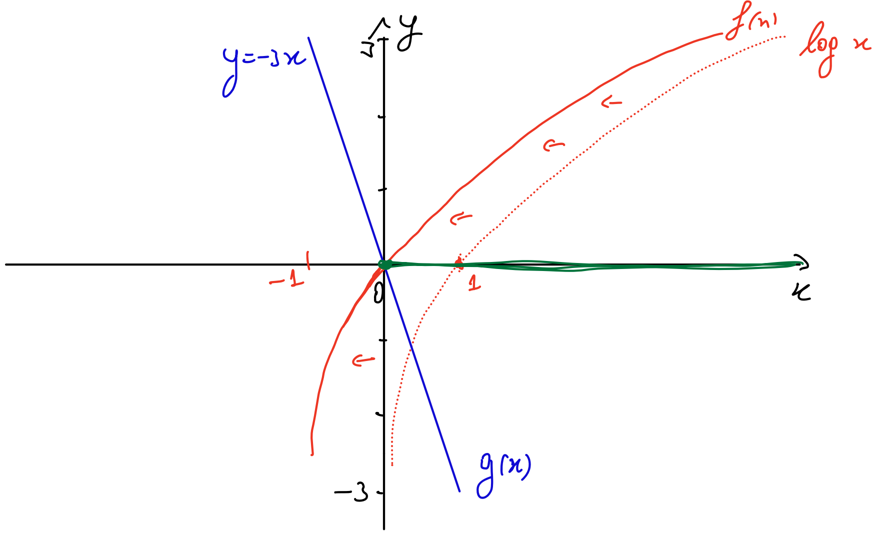
$$\lim_{\substack{x\to 0^+ \\ y = \frac{1}{x}}} x e^{\frac{1}{x}} = \lim_{y\to+\infty} \frac{1}{y} \cdot e^y = +\infty$$

f è discontinua in 0, e ha discontinuità di II specie.

$$\lim_{x\to 0^-} e^{\frac{1}{x}} \cdot \sin x = 0$$

$$e^{-\infty} \cdot \sin(0^-) = 0^+ \cdot 0 = 0$$

4. $$x + \log(x+1) \geq -2x \iff \underbrace{\log(x+1)}_{f(x)} \geq \underbrace{-3x}_{g(x)}$$



$[0, +\infty)$