Algebra delle matrici

$A \in \mathcal{M}(m \times n)$, $\underline{x} \in \mathbb{R}^n$, $A\underline{x} \in \mathbb{R}^m$

ES $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} \in \mathcal{M}(2 \times 3)$, $B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}(3 \times 3)$

$$\mathcal{M}(2\times 3) \ni AB = \begin{pmatrix} 2\cdot 1 + 1\cdot 2 + 0\cdot 0 & 2\cdot 0 + 1\cdot 2 + 0\cdot 1 & 2\cdot(-1) + 1\cdot 3 + 0\cdot 0 \\ 1\cdot 1 - 1\cdot 2 + 2\cdot 0 & 1\cdot 0 + -1\cdot 2 + 2\cdot 1 & 1\cdot(-1) - 1\cdot 3 + 2\cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 & 1 \\ -1 & 0 & -4 \end{pmatrix}$$

$A \in \mathcal{M}(m \times n)$, $B \in \mathcal{M}(n \times k) \Rightarrow AB \in \mathcal{M}(m \times k)$

$A \in \mathcal{M}(n \times n)$, $B \in \mathcal{M}(n \times n) \Rightarrow AB \in \mathcal{M}(n \times n)$

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$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad BA \in \mathcal{M}(n \times n)$

Domanda Esiste $I \in \mathcal{M}(n\times n)$ tale che $AI = IA = A$ $\forall A \in \mathcal{M}(n \times n)$?

matrice identità

ES $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ $\quad I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$AI = \begin{pmatrix} 2a + c & 2b + d \\ -a + 3c & -b + 3d \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

$a = 1$, $c = 0$, $b = 0$, $d = 1$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad\quad I\underline{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\quad\quad\quad\quad \underline{e_1} \quad \underline{e_2}$

$$I \in \mathcal{M}(3\times 3), \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \det I = 1.$$
$$\quad\quad\quad\quad\quad\quad\quad \underline{e}_1 \quad \underline{e}_2 \quad \underline{e}_3$$

Domanda: Quali matrici quadrate hanno un'inversa?
Per quali $A \in \mathcal{M}(n\times n)$, esiste $A^{-1} \in \mathcal{M}(n\times n)$ tale che
$$A A^{-1} = A^{-1} A = I$$

- Data $A \in \mathcal{M}(2\times 2)$, cerchiamo $A^{-1} \in \mathcal{M}(2\times 2)$ tale che
$$A A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \text{ tale che } A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ e } A\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

rango(A), rango $\left(A \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right)$ devono essere uguali

$$\text{rango } A = \text{rango}\left(A \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right) = \begin{cases} 0 & \text{NO} \\ 1 & \Leftrightarrow \operatorname{Imm} A = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 2 & \Leftrightarrow \det A \neq 0 \end{cases}$$

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rango A, rango $\left(A \,\middle|\, \begin{matrix} 0 \\ 1 \end{matrix}\right)$ devono essere uguali

$$\text{rango } A = \text{rango}\left(A \,\middle|\, \begin{matrix} 0 \\ 1 \end{matrix}\right) = \begin{cases} 0 & \text{NO} \\ 1 & \Leftrightarrow \operatorname{Imm} A = \operatorname{Span}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 2 & \Leftrightarrow \det A \neq 0 \end{cases}$$

Quindi dobbiamo avere $\det A \neq 0 \Leftrightarrow$ rango $A = 2$. Allora le soluzioni hanno dim $= 2 - 2 = 0$. Quindi
$$A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{e} \quad A\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
hanno entrambi un'unica soluzione.

Troviamola. Applico l'algoritmo a $\left(A \middle| \begin{matrix}1\\0\end{matrix}\right)$ e $\left(A \middle| \begin{matrix}0\\1\end{matrix}\right)$ e avrò $\left(\begin{matrix}1 & 0\\0 & 1\end{matrix} \middle| \begin{matrix}a\\b\end{matrix}\right)$ e $\left(\begin{matrix}1 & 0\\0 & 1\end{matrix} \middle| \begin{matrix}c\\d\end{matrix}\right)$, e le soluzioni sono $\begin{pmatrix}a\\b\end{pmatrix}$ e $\begin{pmatrix}c\\d\end{pmatrix}$. Quindi $A^{-1} = \begin{pmatrix}a & c\\b & d\end{pmatrix}$

Algoritmo Data A, se $\det A \neq 0$, troviamo A^{-1} applicando l'algoritmo di Gauss-Jordan a $(A \mid I)$, che risulta in $(I \mid A^{-1})$.

ES $A = \begin{pmatrix}2 & 1\\-1 & 0\end{pmatrix}$, $\det A = (2 \cdot 0) - (1 \cdot (-1)) = 1 \neq 0$

$$B = \left(\begin{matrix}2 & 1\\-1 & 0\end{matrix} \middle| \begin{matrix}1 & 0\\0 & 1\end{matrix}\right)$$
$$\qquad A \qquad\quad I$$

$B_2 \to 2B_2 + B_1$, $B = \left(\begin{matrix}2 & 1\\0 & 1\end{matrix} \middle| \begin{matrix}1 & 0\\1 & 2\end{matrix}\right)$

$B_1 \to B_1 - B_2$, $B = \left(\begin{matrix}2 & 0\\0 & 1\end{matrix} \middle| \begin{matrix}0 & -2\\1 & 2\end{matrix}\right)$

$B_1 \to \frac{1}{2} B_1$, $B = \left(\begin{matrix}1 & 0\\0 & 1\end{matrix} \middle| \begin{matrix}0 & -1\\1 & 2\end{matrix}\right)$
$$\qquad I \qquad\quad A^{-1}$$

$A = \begin{pmatrix}2 & 1\\-1 & 0\end{pmatrix}$, $A^{-1} = \begin{pmatrix}0 & -1\\1 & 2\end{pmatrix}$

$$AA^{-1} = \begin{pmatrix}2 \cdot 0 + 1 \cdot 1 & 2 \cdot (-1) + 1 \cdot 2\\-1 \cdot 0 + 0 \cdot 1 & (-1)(-1) + 0 \cdot 2\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}_{I}$$

$$A^{-1}A = \begin{pmatrix}0 \cdot 2 + (-1)(-1) & 0 \cdot 1 + (-1) \cdot 0\\1 \cdot 2 + 2 \cdot (-1) & 1 \cdot 1 + 2 \cdot 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

Prop Data $A \in M(n \times n)$, la matrice inversa $A^{-1} \in M(n \times n)$ esiste se e solo se $\det A \neq 0$. E se $\det A \neq 0$, per trovare A^{-1}, riduciamo $(A \mid I)$ a $(I \mid A^{-1})$ tramite l'algoritmo di Gauss-Jordan.

ES $A = \begin{pmatrix} 2 & 1 & k+2 \\ 0 & 1 & k \\ 1 & -1 & 1 \end{pmatrix}$. Dire per quali $k \in \mathbb{R}$ esiste A^{-1} e trovarla.

$$\det A = (2 + k + 0) - (k+2 - 2k + 0) = 2 + k - 2 + k =$$
$$= 2k$$

$$\begin{matrix} 2 & 1 & k+2 & 2 & 1 \\ 0 & 1 & k & 0 & 1 \\ 1 & -1 & 1 & 1 & -1 \end{matrix}$$

A^{-1} esiste per $k \neq 0$.

$$B = \left(\begin{array}{ccc|ccc} 2 & 1 & k+2 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \qquad \underline{k \neq 0}$$

$B_3 \to 2B_3 - B_1$,
$$B = \left(\begin{array}{ccc|ccc} 2 & 1 & k+2 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 1 & 0 \\ 0 & -3 & -k & -1 & 0 & 2 \end{array}\right)$$

$B_3 \to B_3 + 3B_2$,
$$B = \left(\begin{array}{ccc|ccc} 2 & 1 & k+2 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 1 & 0 \\ 0 & 0 & 2k & -1 & 3 & 2 \end{array}\right)$$

$B_1 \to B_1 - B_2$,
$$B = \left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & k & 0 & 1 & 0 \\ 0 & 0 & 2k & -1 & 3 & 2 \end{array}\right)$$

$B_2 \to k B_1 - B_3$, $\quad B = \left(\begin{array}{ccc|ccc} 2k & 0 & 0 & k+1 & -k-3 & -2 \\ 0 & 1 & k & 0 & 1 & 0 \\ 0 & 0 & 2k & -1 & 3 & 2 \end{array}\right)$

$B_2 \to 2 B_2 - B_3$, $\quad B = \left(\begin{array}{ccc|ccc} 2k & 0 & 0 & k+1 & -k-3 & -2 \\ 0 & 2 & 0 & 1 & -1 & -2 \\ 0 & 0 & 2k & -1 & 3 & 2 \end{array}\right)$

$B_1 \to \frac{1}{2k} B_1$, $\quad B_2 \to \frac{1}{2} B_2$, $\quad B_3 \to \frac{1}{2k} B_3$

$$B = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{k+1}{2k} & -\frac{k+3}{2k} & -\frac{1}{k} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & -\frac{1}{2k} & \frac{3}{2k} & \frac{1}{k} \end{array}\right)$$

$$\qquad I \qquad\qquad\qquad A^{-1}$$

$A = \begin{pmatrix} 2 & 1 & k+2 \\ 0 & 1 & k \\ 1 & -1 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} \frac{k+1}{2k} & -\frac{k+3}{2k} & -\frac{1}{k} \\ \frac{1}{2} & -\frac{1}{2} & -1 \\ -\frac{1}{2k} & \frac{3}{2k} & \frac{1}{k} \end{pmatrix}$

$$AA^{-1} = \begin{pmatrix} 2\cdot\frac{k+1}{2k} + \frac{1}{2} - \frac{k+2}{2k} & -\frac{k+3}{k} - \frac{1}{2} + \frac{3(k+2)}{2k} & -\frac{2}{k} - 1 + \frac{k+2}{k} \\ 0 + \frac{1}{2} - k\cdot\frac{1}{2k} & 0 - \frac{1}{2} + \frac{3}{2} & 0 - 1 + 1 \\ \frac{k+1}{2k} - \frac{1}{2} - \frac{1}{2k} & -\frac{k+3}{2k} + \frac{1}{2} + \frac{3}{2k} & -\frac{1}{k} + 1 + \frac{1}{k} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{1}{k} + \frac{1}{2} - \frac{1}{2} - \frac{1}{k} & -1 - \frac{3}{k} - \frac{1}{2} + \frac{3}{2} + \frac{3}{k} & -\frac{2}{k} - 1 + 1 + \frac{2}{k} \\ 0 & 1 & 0 \\ \frac{1}{2} + \frac{1}{2k} - \frac{1}{2} - \frac{1}{2k} & -\frac{1}{2} - \frac{3}{2k} + \frac{1}{2} + \frac{3}{2k} & 1 \end{pmatrix} = I$$

Def Data $A \in M(n \times n)$, un vettore $\underline{v} \in \mathbb{R}^n \setminus \{\underline{0}\}$ si chiama <u>autovettore</u> con <u>autovalore</u> $\ell \in \mathbb{R}$ se $A\underline{v} = \ell \underline{v}$.

$$\ell \underline{v} = \ell \cdot I \underline{v} \quad , \quad A\underline{v} = \ell \underline{v} \iff A\underline{v} = \ell(I\underline{v}) \iff$$

$$\iff A\underline{v} - \ell(I\underline{v}) = \underline{0} \iff (A - \ell I)\underline{v} = \underline{0}$$

Domanda Per quali $\ell \in \mathbb{R}$, $(A - \ell I)\underline{x} = \underline{0}$ ha soluzioni $\neq \underline{0}$?

R. Per quelli per cui $\det(A - \ell I) = 0$

ES $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ autovalori e autovettori di A?

$$A - \ell I = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} - \ell \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} =$$

$$= \begin{pmatrix} 2-\ell & 1 \\ -1 & -\ell \end{pmatrix}$$

$$\det(A - \ell I) = (2-\ell)(-\ell) - 1 \cdot (-1) = \ell^2 - 2\ell + 1$$

Gli autovalori di A sono i valori ℓ per cui $\det(A - \ell I) = 0$, ossia $\ell^2 - 2\ell + 1 = 0$.

Quindi $\ell^2 - 2\ell + 1 = 0 \iff (\ell - 1)^2 = 0 \iff \ell = 1$.

Gli autovettori con autovalore 1 sono le soluzioni di

$$(A - 1 \cdot I)\underline{x} = \underline{0}$$

$$\left[\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \underline{x} = \underline{0}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \underline{x} = 0$$

$$C = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right)$$

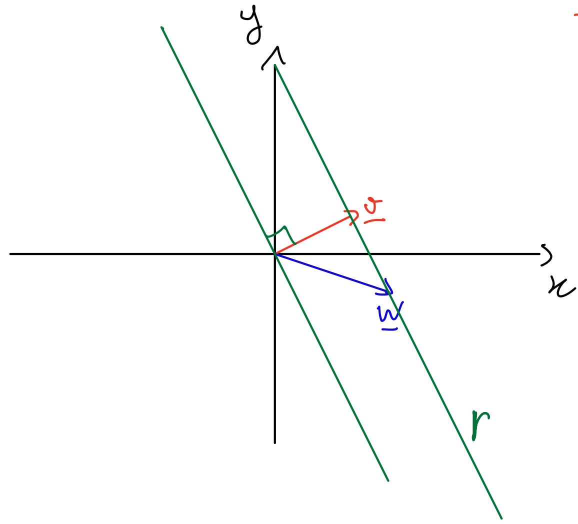
$C_2 \to C_2 + C_1$, $\quad C = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \qquad x + y = 0$

$$\begin{cases} x + \lambda = 0 \\ y = \lambda \end{cases} \qquad \begin{cases} x = -\lambda \\ y = \lambda \end{cases} \qquad W = \operatorname{Span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cdot(-1) + 1\cdot 1 \\ (-1)\cdot(-1) + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A\,\underline{v} = \qquad\qquad = \underset{\substack{\shortparallel \\ 1}}{\ell}\,\underline{v}$$

Trovare la retta in $\mathbb{R}^2$ ortogonale a $\underset{\underline{v}}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$ e passante per il punto $\underset{\underline{w}}{\begin{pmatrix} 3 \\ -1 \end{pmatrix}}$.



$$r = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \Big/ ax + by = c \right\} = \underline{u} + \operatorname{Span}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$\Updownarrow$ Trovare $a, b, c \in \mathbb{R}$ e $\Updownarrow$ Trovare $\underline{u}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \mathbb{R}^2$