Esercizio

$$\begin{cases} x + ky = 4 - k \\ kx + 4y = 4 \end{cases} \quad , \quad k \in \mathbb{R}$$

k	rango(A)	rango(C)	W	
$\neq \pm 2$	2	2	$\dim = 2-2=0$	unica soluz.
$+2$	1	1	$\dim = 2-1=1$	rette
-2	1	2	$\emptyset$	

$$A = \begin{pmatrix} 1 & k \\ k & 4 \end{pmatrix} \in \mathcal{M}(2\times 2) \qquad C = \left(\begin{array}{cc|c} 1 & k & 4-k \\ k & 4 & 4 \end{array}\right) \in \mathcal{M}(2\times 3)$$

$k=0$

$$C = \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 4 & 4 \end{array}\right), \quad C_2 \to \tfrac{1}{4} C_2 \quad C = \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right)$$

$$\begin{cases} x = 4 \\ y = 1 \end{cases}$$

$k \neq 0$

$$C = \left(\begin{array}{cc|c} 1 & k & 4-k \\ k & 4 & 4 \end{array}\right)$$

$$C_2 \to C_2 - kC_1 \qquad C = \left(\begin{array}{cc|c} 1 & k & 4-k \\ 0 & 4-k^2 & 4-k(4-k) \end{array}\right)$$

$k=+2$

$$C = \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right) \qquad x + 2y = 2$$

$$\begin{cases} y = \lambda \\ x + 2\lambda = 2 \end{cases} \qquad \begin{cases} x = 2 - 2\lambda \\ y = \lambda \end{cases}$$

$$W = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R} \right\}$$

$\boxed{k=-2}$ $\quad C = \left(\begin{array}{cc|c} 1 & -2 & 6 \\ 0 & 0 & 16 \end{array}\right)$ $\quad \nexists$ soluzioni

$\boxed{k \neq 0, +2, -2}$ $\quad C = \left(\begin{array}{cc|c} 1 & k & 4-k \\ 0 & 4-k^2 & 4-k(4-k) \end{array}\right)$

$C_1 \longrightarrow (4-k^2)C_1 - k\,C_2$

$$C = \left(\begin{array}{cc|c} 4-k^2 & 0 & (4-k)(4-k^2) - k(4-4k+k^2) \\ 0 & 4-k^2 & 4-4k+k^2 \end{array}\right)$$

$C_1 \longrightarrow \frac{1}{4-k^2} C_1, \qquad C_2 \longrightarrow \frac{1}{4-k^2} C_2$

$$C = \left(\begin{array}{cc|c} 1 & 0 & 4-k-\dfrac{k(4-4k+k^2)}{4-k^2} \\ 0 & 1 & \dfrac{4-4k+k^2}{4-k^2} \end{array}\right)$$

$$\begin{cases} x = 4-k-\dfrac{k(4-4k+k^2)}{4-k^2} \\ y = \dfrac{4-4k+k^2}{4-k^2} \end{cases}$$

ES

$$\begin{cases} y - z = 1 \\ 2x-4y+4z = -2 \\ x - y + z = 0 \end{cases} \qquad A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -4 & 4 \\ 1 & -1 & 1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$A \in M(3\times 3)$ $\qquad$ rango $A = $
- 0 NO
- 1 NO
- 2 $\leftarrow$
- 3 NO

rango $A \geq 2 \iff \exists$ sottomatrice 2×2 con $\det \neq 0$.

$$\det\begin{pmatrix} 0 & 1 \\ 2 & -4 \end{pmatrix} = 0\cdot(-4) - 1\cdot 2 = -2 \neq 0$$

rango $A \geq 3 \iff \det A \neq 0$

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -4 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{matrix} 0 & 1 \\ 2 & -4 \\ 1 & -1 \end{matrix}$$

$$\det A = (0 + 4 + 2) - (4 + 0 + 2) = 0$$

$\Rightarrow$ rango $A \leq 2$ $\Rightarrow$ rango $A = 2$

$$C = \left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 2 & -4 & 4 & -2 \\ 1 & -1 & 1 & 0 \end{array}\right) \in \mathcal{M}(3 \times 4), \quad \overset{\text{rango } A}{\overset{\shortparallel}{2}} \leq \text{rango } C \leq 3$$

$$C^2 = -C^3$$

$$\det \begin{pmatrix} 1 & -1 & 1 \\ -4 & 4 & -2 \\ -1 & 1 & 0 \end{pmatrix} = 0$$

$$\det \begin{pmatrix} 0 & -1 & 1 \\ 2 & 4 & -2 \\ 1 & 1 & 0 \end{pmatrix} = (0 + 2 + 2) - (4 + 0 + 0) = 0$$

$$\begin{pmatrix} 0 & -1 & 1 \\ 2 & 4 & -2 \\ 1 & 1 & 0 \end{pmatrix} \begin{matrix} 0 & -1 \\ 2 & 4 \\ 1 & 1 \end{matrix}$$

$$\det \begin{pmatrix} 0 & 1 & 1 \\ 2 & -4 & -2 \\ 1 & -1 & 0 \end{pmatrix} = 0$$

Tutte le sottomatrici 3×3 di C hanno $\det = 0$. Quindi rango $C \leq 2$.

Quindi rango $C = 2$ perché rango $C \geq$ rango $A = 2$.

rango C = rango A quindi il sistema ammette soluzione W

che è un sottospazio affine di dim $= 3 - \text{rango } A = 1$.

$$\begin{cases} y - z = 1 \\ 2x - 4y + 4z = -2 \\ x - y + z = 0 \end{cases} \quad , \qquad C = \left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 2 & -4 & 4 & -2 \\ 1 & -1 & 1 & 0 \end{array}\right)$$

<u>Esercizio</u>. Trovare W. $\left(\underline{\text{Sol}} \quad W = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\text{sol partic.}} + \underbrace{\text{Span}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{\text{sol. del sistema omog.}} \right)$

<u>ES</u>

$$\begin{cases} 3x_1 + x_3 - x_4 = 0 \\ x_2 - 2x_3 - x_4 = -4 \\ 4x_1 + x_2 + 2x_3 - 3x_4 = 0 \\ 3x_1 - x_2 + x_4 = 1 \end{cases}$$

$$A = \begin{array}{c} (+) \\ (-) \\ (+) \\ (-) \end{array} \begin{pmatrix} 3 & 0 & 1 & -1 \\ 0 & 1 & -2 & -1 \\ 4 & 1 & 2 & -3 \\ 3 & -1 & 0 & 1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 0 \\ -4 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{rango } A = 4 \iff \det A \neq 0$$

$$\det A = (+1)(3) \cdot \det\begin{pmatrix} 1 & -2 & -1 \\ 1 & 2 & -3 \\ -1 & 0 & 1 \end{pmatrix} + (-1) \cdot (0) \cdot \det(\quad) +$$

$$+ (+1) \cdot (4) \cdot \det\begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & 0 & 1 \end{pmatrix} + (-1) \cdot (3) \cdot \det\begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & -1 \\ 1 & 2 & -3 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & -2 & -1 \\ 1 & 2 & -3 \\ -1 & 0 & 1 \end{pmatrix}\begin{matrix} 1 & -2 \\ 1 & 2 \\ -1 & 0 \end{matrix} = (2-6+0)-(2+0-2) = -4$$

$$\det\begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & 0 & 1 \end{pmatrix}\begin{matrix} 0 & 1 \\ 1 & 2 \\ -1 & 0 \end{matrix} = (0+1+0)-(-2+0+1) = 2$$

$$\det\begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & -1 \\ 1 & 2 & -3 \end{pmatrix}\begin{matrix} 0 & 1 \\ 1 & 2 \\ 1 & 2 \end{matrix} = (0-1-2)-(+2+0-3) = -2$$

$$\det A = 3\cdot(-4) + 4\cdot(2) + (-3)\cdot(-2) =$$
$$-12+8+6 = 2 \neq 0$$

$\Rightarrow$ rango $A = 4$

$\Rightarrow$ rango $C = 4$, $C \in \mathcal{M}(4\times 5)$

Il sistema ammette almeno una soluzione perché rango A = rango C

e W ha $\dim = \underset{n}{\underset{\uparrow}{4}} - \underset{\text{rango } A}{\underset{\uparrow}{4}} = 0$. La soluzione è unica.

<u>Oss</u> $A\underline{x} = \underline{b}$, $A : \mathbb{R}^n \to \mathbb{R}^m$

Se $n=m$ e $\det A \neq 0$, allora $\forall\, \underline{b}$ $\exists!\, \underline{x}$ soluzione di $A\underline{x} = \underline{b}$.

<u>Esercizio</u> Trovare l'unica soluzione $\left(W = \left\{ \begin{pmatrix} 1 \\ 8 \\ 3 \\ 6 \end{pmatrix} \right\} \right)$

$$C = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x_1 \\ 0 & 1 & 0 & 0 & x_2 \\ 0 & 0 & 1 & 0 & x_3 \\ 0 & 0 & 0 & 1 & x_4 \end{array}\right)$$

ES Trovare i vettori di $\mathbb{R}^3$ che siano ortogonali alle rette

$$r_1 = \text{Span}\begin{pmatrix}1\\1\\0\end{pmatrix} \quad e \quad r_2 = \text{Span}\begin{pmatrix}1\\2\\1\end{pmatrix}.$$

Un vettore $\begin{pmatrix}x\\y\\z\end{pmatrix}$ è ortogonale a $r_1 = \text{Span}\begin{pmatrix}1\\1\\0\end{pmatrix} \iff \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} = 0$

$\iff x\cdot 1 + y\cdot 1 + z\cdot 0 = 0 \iff x + y = 0$

Un vettore $\begin{pmatrix}x\\y\\z\end{pmatrix}$ è ortogonale a $r_2 = \text{Span}\begin{pmatrix}1\\2\\1\end{pmatrix} \iff \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix} = 0$

$\iff x\cdot 1 + y\cdot 2 + z\cdot 1 = 0 \iff x + 2y + z = 0$

Soluzioni di $\begin{cases} x + y = 0 \\ x + 2y + z = 0 \end{cases}$ $\quad A = \begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 1\end{pmatrix}$, rango $A = \cancel{0}\ \cancel{1}\ 2$

$\det\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix} = 1 \neq 0$

$\Downarrow$

$\dim W = 3 - 2 = 1$

$$C = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0\\ 1 & 2 & 1 & 0\end{array}\right),$$

$C_2 \to C_2 - C_1$, $\quad C = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\end{array}\right)$

$C_1 \to C_1 - C_2 \quad C = \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & 1 & 1 & 0\end{array}\right)$

$\begin{cases} x - z = 0 \\ y + z = 0 \end{cases}$, $\quad z = \lambda$, $\quad \begin{cases} x - \lambda = 0 \\ y + \lambda = 0 \\ z = \lambda \end{cases}$, $\quad \begin{cases} x = \lambda \\ y = -\lambda \\ z = \lambda \end{cases}$

$$W = \text{Span}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Trovare le soluzioni di in $\mathbb{R}^2$ $\begin{cases} x+2y=1 \\ 2x-y=0 \\ x+y=k \end{cases}$

$$\mathcal{M}(3\times 2) \ni A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix}$$

$$\text{rango } A = \begin{matrix} \cancel{0} \\ \cancel{1} \\ 2 \end{matrix} \quad , \quad \det \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = -1-4 = -5 \neq 0$$

$$C = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & k \end{array}\right) \in \mathcal{M}(3\times 3)$$

$$\text{rango } C = \begin{matrix} 2 & \text{se } k = 3/5 \\ 3 & \text{se } k \neq 3/5 \end{matrix}$$

$$\det C = \det \left(\begin{array}{ccc} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & k \end{array}\right) \begin{array}{cc} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{array} = (-k+0+2)-(-1+0+4k)$$
$$= 3-5k$$

Il sistema ammette soluzione solo se $k = 3/5$, e W ha dim $= 2-2 = 0$. $W = \{\text{punto}\}$

$$C = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & 3/5 \end{array}\right) \quad , \quad C_3 \to 5C_3 \qquad C = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 5 & 5 & 3 \end{array}\right)$$

$$C_2 \to C_2 - 2C_1 \qquad C = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & -2 \\ 5 & 5 & 3 \end{array}\right)$$

$$C_3 \to C_3 - 5C_1 \qquad C = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & -2 \\ 0 & -5 & -2 \end{array}\right)$$

$C_3 \to C_3 + C_2 \qquad C = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{array}\right)$

$C_1 \to 5C_1 + 2C_2 \qquad C = \left(\begin{array}{cc|c} 5 & 0 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{array}\right)$

$C_1 \to \frac{1}{5} C_1, \quad C_2 \to -\frac{1}{5} C_2, \qquad C = \left(\begin{array}{cc|c} 1 & 0 & 1/5 \\ 0 & 1 & 2/5 \\ 0 & 0 & 0 \end{array}\right)$

$$\begin{cases} x = 1/5 \\ y = 2/5 \end{cases}$$