ES. 1

$$f(x) = \begin{cases} c\,(x-1)^2, & 0<x<2 \\ 0, & \text{alt.} \end{cases}$$

(i) f è densità $\iff f(x) \geq 0 \ \forall x$ e $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$f(x) \geq 0 \iff c \geq 0$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1 \iff \int_0^2 c\,(x-1)^2\,dx = 1 \iff c = \frac{3}{2}$$

$$E[X] = \int_{-\infty}^{+\infty} x\,f(x)\,dx = \int_0^2 \frac{3}{2}\,x\,(x-1)^2\,dx = 1$$

$$\mathrm{Var}(X) = E[(X-E[X])^2] = E[X^2] - E[X]^2$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_0^2 \frac{3}{2}\,x^2\,(x-1)^2\,dx = \frac{8}{5}$$

$$\boxed{E[X] = 1} \qquad \boxed{\mathrm{Var}(X) = \frac{8}{5} - 1 = \frac{3}{5}}$$

(ii) $\bar{X}_{60} = \frac{1}{60}(X_1 + \dots + X_{60})$ $\qquad X_1, \dots, X_{60}$ indip. ed equid.

$$E[\bar{X}_{60}] = 1 \qquad \mathrm{Var}(\bar{X}_{60}) = \frac{\mathrm{Var}(X_1)}{60} = \frac{1}{100}$$

Chebishev $\qquad P\{ |\bar{X} - E[\bar{X}]| > d \} \leq \dfrac{\mathrm{Var}(\bar{X})}{d^2}$

$$P\{ 50 \leq X_1 + \dots + X_{60} \leq 70 \} = P\left\{ \frac{5}{6} \leq \bar{X}_{60} \leq \frac{7}{6} \right\} =$$

$$= P\left\{-\frac{1}{6} \le \bar{X}_{60} - 1 \le \frac{1}{6}\right\} = P\left\{ |\bar{X}_{60} - 1| \le \frac{1}{6}\right\} =$$

$$= 1 - P\left\{ |\bar{X}_{60} - 1| > \frac{1}{6}\right\} \ge 1 - \frac{1/100}{\left(1/6\right)^2} = 0.64$$

Teorema del Limite Centrale

$$\frac{X_1 + \dots + X_n - n\,E[X_k]}{\sqrt{n\,Var(X_k)}} = \frac{\sqrt{n}}{\sqrt{Var(X_k)}} \cdot \left(\bar{X} - E[X_k]\right) =$$

$$= \frac{\bar{X} - E[\bar{X}]}{\sqrt{Var(\bar{X})}} \approx Z \quad \text{con} \quad Z \sim N(0,1)$$

$$P\left\{ 50 \le X_1 + \dots + X_{60} \le 70 \right\} = P\left\{ \frac{50-60}{\sqrt{36}} \le \frac{X_1 + \dots + X_{60} - 60}{\sqrt{36}} \le \frac{70-60}{\sqrt{36}} \right\}$$

$$= P\left\{ -\frac{10}{6} \le \frac{\bar{X} - 1}{\sqrt{1/100}} \le \frac{10}{6} \right\} \approx P\left\{ -\frac{5}{3} \le Z \le \frac{5}{3} \right\} =$$

$$= \Phi\left(\frac{5}{3}\right) - \Phi\left(-\frac{5}{3}\right) = 2\,\Phi\left(5/3\right) - 1 \approx 2\,\Phi(1.66) - 1$$

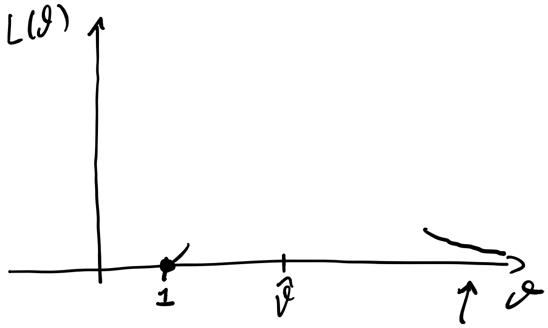
$$\approx 2 \cdot 0.951 - 1 = 0.902$$

ES. 2 Per $\vartheta > 1$ sia $f_\vartheta(x) = \begin{cases} (\vartheta - 1)\, x^{-\vartheta}, & x > 1 \\ 0, & \text{alt.} \end{cases}$

dati $x_1, \dots, x_n$ con $x_i > 1 \ \forall i$

Massima verosimiglianza.

$$L(\vartheta; x_1, \dots, x_n) = \prod_{i=1}^{n} f_\vartheta(x_i) \underset{x_i > 1 \ \forall i}{=} (\vartheta - 1)^n (x_1 \cdot \dots \cdot x_n)^{-\vartheta}$$



$x_1 \cdot \ldots \cdot x_n > 1$

$$\frac{d}{d\vartheta} L = 0 \iff \frac{d}{d\vartheta} \log L = 0$$

$$L(\vartheta) > 0 \quad \forall \vartheta > 1$$

$$\log L(\vartheta) = n \log(\vartheta - 1) - \vartheta \log(x_1 \cdot \ldots \cdot x_n)$$

$$\frac{d}{d\vartheta} \log L(\vartheta) = \frac{n}{\vartheta - 1} - \log(x_1 \cdot \ldots \cdot x_n) = 0 \iff \vartheta = 1 + \frac{n}{\log(x_1 \cdot \ldots \cdot x_n)}$$

Quindi $\hat{\vartheta} = 1 + \dfrac{n}{\log(x_1 \cdot \ldots \cdot x_n)}$ che è una buona stima

perché $\hat{\vartheta} > 1$.

<u>Metodo dei momenti</u>

$$E[X] = \int_{-\infty}^{+\infty} x f_\vartheta(x)\, dx = \int_1^{+\infty} (\vartheta - 1) x^{1-\vartheta}\, dx < +\infty \iff$$

$$\iff 1 - \vartheta < -1 \iff \vartheta > 2$$

Consideriamo quindi $\vartheta > 2$.

$$E[X] = \int_1^{+\infty} (\vartheta - 1) x^{1-\vartheta}\, dx = \lim_{M \to +\infty} \int_1^M (\vartheta - 1) x^{1-\vartheta}\, dx =$$

$$= \lim_{M \to +\infty} \left[(\vartheta - 1) \frac{x^{2-\vartheta}}{2 - \vartheta} \Big|_1^M \right] = \lim_{M \to +\infty} (\vartheta - 1) \left(\frac{M^{2-\vartheta}}{2 - \vartheta} - \frac{1}{2 - \vartheta} \right) = \frac{\vartheta - 1}{\vartheta - 2}$$

Quindi $\tilde{\vartheta}$ è la soluzione di $\dfrac{\vartheta - 1}{\vartheta - 2} = \bar{x}$

da cui $\quad \tilde{\vartheta} = \dfrac{2\bar{x}-1}{\bar{x}-1} = 2 + \dfrac{1}{\bar{x}-1}$

Si ha $\bar{x} = \frac{1}{n}(x_1+\dots+x_n) > 1$ e dunque $\tilde{\vartheta} > 2$ ha senso.

$\longleftrightarrow$

ES. 3 $\quad n = 10000 \qquad X_1,\dots,X_n \sim B(1,p)$

$x_1,\dots,x_n$ danno $\hat{p} = 0.245$

(i) $\quad H_0)\ p \geq 0.25 = p_0 \qquad H_1)\ p < 0.25$

Test unilatero destro per $p = E[X_k]$

p-value $\quad \bar{\alpha} = \Phi\left(\dfrac{\sqrt{n}}{\sqrt{p_0(1-p_0)}}(\hat{p}-p_0)\right) \approx \Phi(-1.1547) =$

$= 1 - \Phi(1.1547) \approx 1 - 0.875 = 0.125$

Dunque l'ipotesi è sufficientemente plausibile

(ii) Intervallo di fiducia al livello dell'80% bilatero

$$I = \left[\hat{p} - \frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}\, q_{1-\alpha/2}\ ,\ \hat{p} + \frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}\, q_{1-\alpha/2}\right]$$

$1-\alpha = 0.8 \iff \alpha = 0.2 \iff 1-\alpha/2 = 0.9$

Precisione della stima $= \dfrac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}\, q_{0.9} \approx 0.005$

con $q_{0.9} \approx 1.285$, $\hat{p} = 0.245$, $n = 10000$