Oss: Se f è limitata superiormente e

$$\lim_{x \to x_0} g(x) = -\infty \quad \text{allora}$$

$$\lim_{x \to x_0} (f+g)(x) = -\infty$$

Oss: Se f è limitata

e $\lim_{x \to x_0} g(x) = 0$

allora $\lim_{x \to x_0} (f \cdot g)(x) = 0$

<u>Es</u>: $\lim_{x\to\infty} \frac{\sin x}{x}$

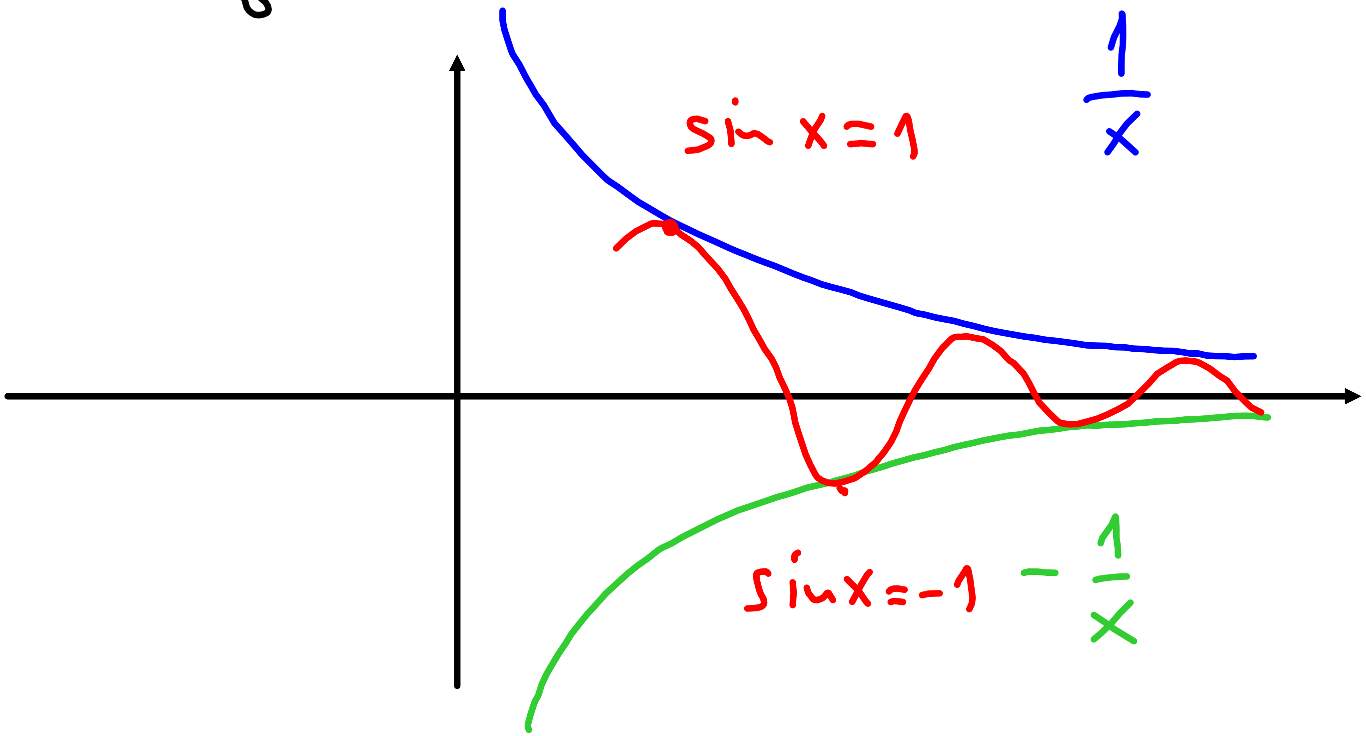
$f(x) = \sin x \qquad |\sin x| \leq 1$

$\Rightarrow$ f è limitata

$g(x) = \frac{1}{x} \qquad \lim_{x\to\infty} \frac{1}{x} = 0$

$\Rightarrow \lim_{x\to\infty} \frac{\sin x}{x} = 0$

$$\underset{\substack{\downarrow \\ 0}}{-\frac{1}{x}} \le \underset{\substack{\downarrow \\ 0}}{\frac{\sin x}{x}} \le \underset{\substack{\downarrow \\ 0}}{\frac{1}{x}}$$



Se $\lim_{x \to x_0} f(x) = 0$

f si dice infinitesima

per $x \to x_0$.

Se $\lim_{x \to x_0} f(x) = +\infty$ si

dice che f diverge positivam.

per $x \to x_0$

dir. negat. $\Rightarrow \lim_{x \to x_0} f(x) = -\infty$

Prop: Se $\lim_{x \to x_0} f(x) = 0^+$

allora $\lim_{x \to x_0} \frac{1}{f(x)} = +\infty$

Se $\lim_{x \to x_0} f(x) = 0^-$

$\Rightarrow \lim_{x \to x_0} \frac{1}{f(x)} = -\infty$

Se $\lim_{x \to x_0} f(x) = +\infty$

$\Rightarrow \lim_{x \to x_0} \frac{1}{f(x)} = 0^+$

Se $\lim_{x \to x_0} f(x) = -\infty$

$\Rightarrow \lim_{x \to x_0} \frac{1}{f(x)} = 0^-$

Se $\lim_{x \to x_0} f(x) = l$ e $l \neq 0, \pm\infty$

$\Rightarrow \lim_{x \to x_0} \frac{1}{f(x)} = \frac{1}{l}$

Limiti fondamentali.

Polinomi.

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

polinomio di grado n $(a_n \neq 0)$.

$$\lim_{x \to \infty} p(x) = ?$$

$$\lim_{x \to \infty} 3x^2 - 5x = +\infty - \infty$$

$$3x^2 - 5x = x^2\left(3 - \frac{5}{x}\right)$$

$$\downarrow \qquad \downarrow \quad \downarrow$$

$$\infty \qquad 3 - 0$$

$$\infty\left(3 - \frac{5}{+\infty}\right) = \infty\left(3 - 0^+\right) = \infty \cdot 3$$

$$= \infty$$

$$\lim_{x \to \infty} p(x) = \operatorname{sgn}(a_n)\,\infty$$

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 =$$

$$= \underset{+\infty}{\underset{\downarrow}{x^n}} \underbrace{\left(a_n + a_{n-1} \cdot \frac{1}{x} + \dots + \frac{a_1}{x^{n-1}} + \frac{a}{x^n} \right)}_{\to\ a_n + 0 + 0 + \dots + 0}$$

$$\lim_{x \to -\infty} p(x) = \begin{cases} \operatorname{sgn}(a_n)\infty & \text{se } n \text{ è pari} \\ \operatorname{sgn}(a_n)(-\infty) & \text{se } n \text{ è dispari} \end{cases}$$

Es.

$$\lim_{x \to -\infty} -5x^3 + 7x^2 = +\infty$$

$$-5(-\infty) = +\infty$$

i polinomi si comportano a $\pm\infty$ come il loro monomio di grado più alto.

Funzioni razionali

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} \begin{array}{l} \leftarrow \text{grado } n \\ \leftarrow \text{grado } m \end{array}$$

$$p(x) = a_n x^n + \dots$$

$$q(x) = b_m x^m + \dots$$

$$\frac{a_n x^n + a_{n-1} x^{n-1} + \dots}{b_m x^m + b_{m-1} x^{m-1} + \dots} =$$

$$= \frac{x^n \left(a_n + a_{n-1} \cdot \frac{1}{x} + \dots\right)}{x^m \left(b_m + b_{m-1} \cdot \frac{1}{x} + \dots\right)}$$

$\to 0$

$\to 0$

$$\lim_{x \to \infty} \frac{P(x)}{q(x)} = \lim_{x \to \infty} \frac{a_n x^n}{b_m x^m}$$

$$\lim_{x\to\infty} \boxed{\frac{a_n x^n}{b_m x^m}} = \begin{cases} \operatorname{sgn}\left(\frac{a_n}{b_m}\right)(+\infty) & \text{se } n>m \\ \frac{a_n}{b_m} & \text{se } n=m \\ 0 & \text{se } n<m \end{cases}$$

$$0^+ \quad \text{se } \frac{a_n}{b_m} > 0$$

$$0^- \quad \text{se } \frac{a_n}{b_m} < 0$$

Limite a $-\infty$

Es: $$\lim_{x \to -\infty} \frac{7x^5 + 5x^2}{-3x^3 + 2x + 1}$$

$$= \lim_{x \to -\infty} \frac{7x^5}{-3x^3} = \lim_{x \to -\infty} -\frac{7}{3} x^2$$

$$= -\frac{7}{3}(+\infty) = -\infty.$$

Limiti fondamentali

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$$

$$= \frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)}$$

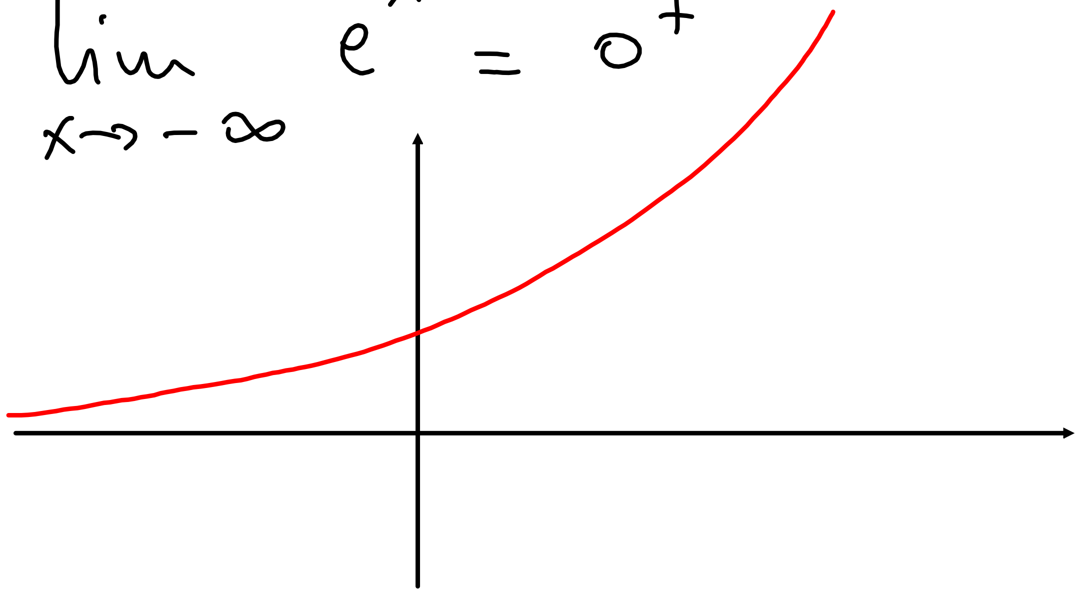
$$= \underbrace{\frac{\sin x}{x}}_{\downarrow \, 1} \cdot \underbrace{\frac{\sin x}{x}}_{\downarrow \, 1} \cdot \underbrace{\frac{1}{1+\cos x}}_{\downarrow \, \frac{1}{1+1}} \Rightarrow \frac{1}{2}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

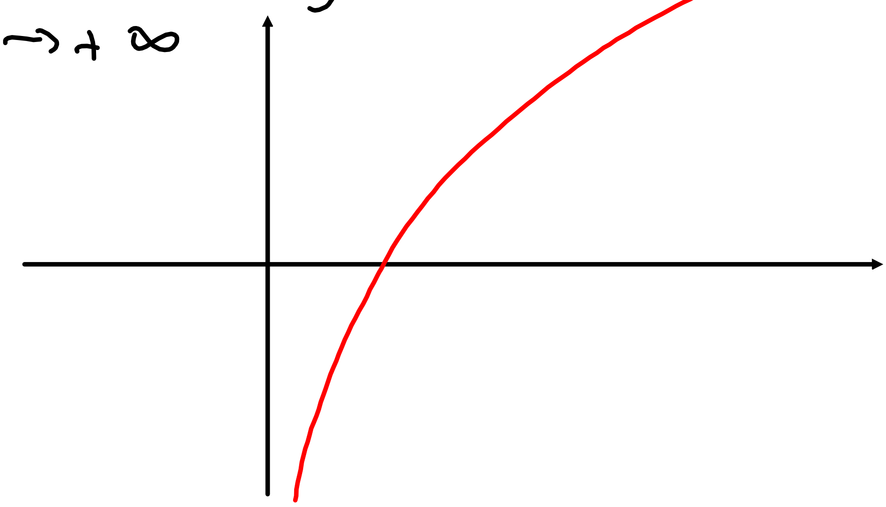
$$\lim_{x \to +\infty} e^x = +\infty$$

$$\lim_{x \to -\infty} e^x = 0^+$$



$$\lim_{x \to 0^+} \log x = -\infty$$

$$\lim_{x \to +\infty} \log x = +\infty$$



Limite della composizione

Teorema: Siano $A, B \subset \mathbb{R}$, $f : A \to B$

$g : B \to \mathbb{R}$ e sia $x_0 \in Acc(A)$. Se esiste

$\lim_{x \to x_0} f(x) = y_0$, $y_0 \in Acc(B)$, $\exists \lim_{y \to y_0} g(y)$ e

se è verificata almeno una delle seguenti

condizioni:

1) $y_0 \in B$ e g è continua in y_0
2) $\exists\, U \in \mathcal{I}(x_0)$ t.c. $f(x) \neq y_0 \quad \forall\, x \in U \cap A \setminus \{x_0\}$

allora

$$\lim_{x \to x_0} (g \circ f)(x) = \lim_{y \to y_0} g(y)$$

Es: $\lim_{x \to -\infty} \operatorname{arctg}(x^2)$

$f : \mathbb{R} \to \mathbb{R} \quad f(x) = x^2$

$g : \mathbb{R} \to \mathbb{R} \quad g(y) = \operatorname{arctg} y$

$(g \circ f)(x) = g(f(x)) = g(x^2) =$

$= \operatorname{arctg}(x^2)$

$x_0 = -\infty$.

$$y_0 = \lim_{x \to x_0} f(x) = \lim_{x \to -\infty} x^2 = +\infty$$

siamo nel caso 2) perché

$+\infty \notin B = \mathbb{R}$

$$\ell = \lim_{y \to y_0} g(y) = \lim_{y \to +\infty} \operatorname{arctg}(y)$$

$$= \pi/2$$

devo verificare che $\exists\, U \in \mathcal{I}(x_0)$ t.c. $x \in A \cap U \setminus \{x_0\} \Rightarrow f(x) \neq y_0$

ma $y_0 = +\infty$ quindi la condizione vale per un qualsiasi U

(il caso 2) è sempre verificato se $y_0 = \pm\infty$).

$\Rightarrow$ per il teorema

$$\lim_{x \to x_0} (g \circ f)(x) = \ell \quad \text{cioè}$$

$$\lim_{x \to -\infty} \operatorname{arctg}(x^2) = \pi/2 .$$

x_0 $\qquad\qquad\qquad\qquad\qquad\qquad \ell$

È un teorema di cambiamento di variabile nel limite.

$$\lim_{x \to -\infty} \operatorname{arctg}(x^2)$$

pongo $y = x^2$ (cambio variabile)

la cambio da tutte le parti

Se $x \to -\infty$ allora

$$y = x^2 \longrightarrow +\infty$$

$$\lim_{x \to -\infty} \operatorname{arctg}(x^2) = \lim_{y \to +\infty} \operatorname{arctg}(y)$$

$$= \frac{\pi}{2}.$$

Limiti fondamentali

$a > 0$

$$\lim_{x \to \infty} a^x = \begin{cases} +\infty & \text{se } a > 1 \\ 1 & \text{se } a = 1 \\ 0^+ & \text{se } 0 < a < 1 \end{cases}$$

$$\lim_{x \to -\infty} a^x$$ metto $y = -x$

$$= \lim_{y \to +\infty} a^{-y}$$

$y = -x$ se $x \to +\infty$

$\Downarrow$ $\Rightarrow y \to -\infty$

$x = -y$

$$\lim_{x \to -\infty} a^x = \lim_{y \to +\infty} a^{-y} =$$

$$= \lim_{y \to +\infty} \frac{1}{a^y} = \begin{cases} 0^+ & \text{se } a > 1 \\ 1 & \text{se } a = 1 \\ +\infty & \text{se } 0 < a < 1 \end{cases}$$

$$\lim_{\substack{x \to \infty \\ \alpha \in \mathbb{R}}} x^{\alpha} = \begin{cases} \infty & \text{se } \alpha > 0 \\ 1 & \text{se } \alpha = 0 \\ 0^{+} & \text{se } \alpha < 0 \end{cases}$$

$$\lim_{x \to \infty} \frac{a^x}{x^\alpha} = \begin{cases} +\infty & \text{se } a > 1 \\ 0^+ & \text{se } 0 < a < 1 \end{cases}$$

$a, \alpha \in \mathbb{R}$

$a > 0$

$a = 1$?

$$\frac{a^x}{x^\alpha} = \frac{1}{x^\alpha}$$

Es: $a = \frac{1}{2}$ $\quad \alpha = -3$

$$\frac{a^x}{x^\alpha} = \frac{\left(\frac{1}{2}\right)^x}{x^{-3}} = \frac{x^3}{2^x} \to 0^+$$

Es: $\lim_{x\to\infty} \frac{\log x}{x} = (*)$

cambio di variabile

$y = \log x \iff x = e^y$

se $x \to \infty \Rightarrow y \to \infty$

$(*) = \lim_{y\to\infty} \frac{y}{e^y} = 0$

Es: $\lim_{x\to+\infty} \frac{(\log x)^\beta}{x^\alpha}$ $\alpha,\beta \in \mathbb{R}$, $\alpha,\beta>0$.

cambiamento di variabile $\log x = y$. quindi $x = e^y$

se $x\to+\infty$ allora $y = \log x \to +\infty$. Il limite diventa

$$\lim_{y\to+\infty} \frac{y^\beta}{(e^y)^\alpha} = \lim_{y\to+\infty} \frac{y^\beta}{e^{y\alpha}} = \lim_{y\to+\infty} \frac{y^\beta}{(e^\alpha)^y} = 0$$

perché $e^\alpha > 1$ dato che $\alpha > 0$.

$$\lim_{x \to 0^+} (1+x)^{1/x} = \lim_{x \to 0^+} e^{\log\left[(1+x)^{1/x}\right]}$$

$$= \lim_{x \to 0^+} e^{\frac{1}{x}\log(1+x)}$$

$$= \lim_{x \to 0^+} e \qquad y = \frac{1}{x}\log(1+x)$$

$$\lim_{x \to 0^+} \frac{1}{x} \log(1+x) = 1$$

$$x \to 0^+ \Longrightarrow y \to 1$$

$$\lim_{x \to 0^+} e^{\frac{1}{x} \log(1+x)} =$$

$$= \lim_{y \to 1} e^{y} = e^{1} = e$$

Nuovi casi di indeterminazione

$f(x) > 0$

$$\lim_{x \to x_0} f(x)^{g(x)} = ?$$

quando può dare indeterm.?

$$f(x)^{g(x)} = e^{\log\left[f(x)^{g(x)}\right]}$$

$$= e^{g(x)\log(f(x))}$$

quando $g(x) \cdot \log(f(x))$

è indeterminato?

1) $g \to 0 \quad \log(f) \to +\infty$

$$\Downarrow$$

$f \to +\infty$

$f^{g} = (+\infty)^{0}$

2) $g \to 0 \quad \log(f) \to -\infty$

$$\Downarrow$$

$f^{g} = (0^{+})^{0} \qquad f \to 0^{+}$

3) $g \to \pm\infty \quad \log f \to 0$

$$\Downarrow$$

$$f \to 1$$

$$f^{g} = (1)^{\pm\infty}$$

Es: $\lim_{x \to 0^+} x \log x = 0^+ \cdot (-\infty)$?

$y = \log x \iff x = e^y$

se $x \to 0^+ \Rightarrow y \to -\infty$

$\lim_{y \to -\infty} e^y \cdot y = 0^+ (-\infty)$

$$y = -z$$

$$y \to -\infty \Rightarrow z \to \infty$$

$$\lim_{y \to -\infty} e^{y} \cdot y = \lim_{z \to \infty} e^{-z}(-z) =$$

$$= \lim_{z \to \infty} -\frac{z}{e^{z}} = -(0^{+}) = 0^{-}$$