Integrazione per parti

$f, g : I \to \mathbb{R}$, I intervallo

f continua , $g \in C^1$.

Sia F una primitiva di f

allora

$$\int f g \, dx = F g - \int F g' \, dx \, .$$

<u>dim</u>: $(Fg)' = F'g + Fg' =$

$= fg + Fg'$

integro

$$\int (Fg)'\,dx = \int fg\,dx + \int Fg'\,dx$$

$$Fg = \int fg\,dx + \int Fg'\,dx$$

$$\int fg\,dx = Fg - \int Fg'\,dx$$

$\square$

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Es:

$$\int \underset{g}{x} \underset{f}{\sin x}\, dx =$$

$$g' = 1 \,, \quad F = -\cos x$$

$$= -\cos x \cdot x - \int -\cos x \cdot 1 \, dx =$$

$$= -x \cos x + \sin x + c \,.$$

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Es:

$$\int \log x \, dx =$$

$$= \int \underset{f}{1} \cdot \underset{g}{\log x} \, dx = \qquad F = x \qquad g' = \frac{1}{x}$$

$$= x \cdot \log x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx =$$

$$= x \log x - x + c \; .$$

$$\underline{E}s: \quad \int \underset{f}{x^n} \underset{g}{\log x}\, dx = \quad (n \neq -1)$$

$$F = \frac{x^{n+1}}{n+1} \qquad g' = \frac{1}{x}$$

$$= \frac{x^{n+1}}{n+1} \log x - \int \frac{x^{n+\cancel{1}}}{n+1} \cdot \frac{1}{\cancel{x}}\, dx =$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \cdot \frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2} + c \; .$$

Es: $\boxed{\int \cos^2 x \, dx} =$

$$= \int \underset{f}{\cos x} \; \underset{g}{\cos x} \, dx$$

$F = \sin x \qquad g' = - \sin x$

$$= \sin x \cdot \cos x - \int \sin x\,(-\sin x)\,dx$$

$$= \sin x \cos x + \int \sin^2 x\,dx =$$

$$= \sin x \cos x + \int 1 - \cos^2 x\,dx =$$

$$= \boxed{\sin x \cos x + x - \int \cos^2 x\,dx}$$

$$\int \cos^2 x\, dx = \sin x \cos x + x - \int \cos^2 x\, dx$$

$$2\int \cos^2 x\, dx = \sin x \cos x + x + c$$

$$\int \cos^2 x\, dx = \frac{\sin x \cos x + x}{2} + c$$

Oss: $D\left[\log f(x)\right]$ $\quad \underline{f>0}$

$$= \frac{1}{f(x)} \cdot f'(x)$$

$$D(\log f)^2 = 2\log f \cdot \frac{f'}{f}$$

Es: $\int \operatorname{arctg} x \, dx =$

$$= \int \underset{f}{1} \cdot \underset{g}{\operatorname{arctg} x} \, dx =$$

$$= x \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} \, dx$$

$$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$$= x \operatorname{arctg} x - \frac{1}{2}\log(1+x^2) + c$$

Integrazione per sostituzione

I, J intervalli di $\mathbb{R}$

$f : I \to \mathbb{R}$ continua

$\varphi : J \to I$ di classe C^1

allora, se F è una primitiva di f

$$\int (f \circ \varphi)\, \varphi' \, dx = (F \circ \varphi) + c$$

$$= \int f(t)\,dt \Big|_{t=\varphi(x)}$$

dim: $(F\circ\varphi)' = (F'\circ\varphi)\cdot\varphi'$

$= (f\circ\varphi)\,\varphi'$ integrate

$\Rightarrow\ F\circ\varphi = \int (f\circ\varphi)\,\varphi'\,dx\,.$ $\square$

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<u>Es.</u> $\int x e^{x^2} dx =$

$\varphi(x) = x^2 \qquad \varphi'(x) = 2x, \; x = \frac{\varphi'}{2}$

$= \int \frac{\varphi'(x)}{2} e^{\varphi(x)} dx =$

poniamo $f(t) = \frac{e^t}{2}$

$= \int f(\varphi(x))\,\varphi'(x)\,dx$

F primitiva di f

$$F(t) = \int \frac{e^t}{2}\, dt = \frac{1}{2} e^t$$

$$\Rightarrow \int (f \circ \varphi)\, \varphi'(x)\, dx = (F \circ \varphi) + c$$

$$= \frac{1}{2} e^{\varphi(x)} + c = \frac{1}{2} e^{x^2} + c.$$

verifica $D\left(\frac{1}{2} e^{x^2}\right) = \frac{1}{\cancel{2}} \cdot \cancel{2}x\, e^{x^2}.$

Metodo pratico per la sostituzione.

$$\int x e^{x^2} dx$$

pongo $x^2 = t = t(x)$

$$\Rightarrow \frac{dt}{dx} = 2x \Rightarrow dt = 2x\, dx$$

$$x\, dx = \frac{dt}{2}$$

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sostituisco nell'integrale compreso nel dx

$$\int x e^{x^2} dx = \int e^t \frac{dt}{2} =$$

$$= \frac{e^t}{2} + c \underset{\uparrow}{=} \frac{e^{x^2}}{2} + c$$

$t = x^2$

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Es: $\displaystyle\int \frac{tg\,x}{\cos^2 x}\,dx =$

$\displaystyle = \int \frac{\sin x}{\cos^3 x}\,dx$ $\qquad \cos x = t$

$$\frac{dt}{dx} = -\sin x$$

$\Rightarrow \quad \sin x\,dx = -dt$

$\displaystyle = \int \frac{-dt}{t^3} = -\int t^{-3}\,dt =$

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$$= -\left(-\frac{1}{2}\right) t^{-2} + c = \frac{1}{2t^2} + c =$$

$$= \frac{1}{2\cos^2 x} + c \qquad t = \cos x$$

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Es: $\int \sqrt{1-x^2}\, dx =$

$x = \sin t \qquad \frac{dx}{dt} = \cos t$

$dx = \cos t \, dt$

$= \int \sqrt{1-\sin^2 t} \cdot \cos t \, dt$

$= \int \sqrt{\cos^2 t} \cos t \, dt =$

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$$= \int |\cos t| \cos t \, dt$$

supponiamo che sia $\cos t > 0$

$$= \int \cos^2 t \, dt =$$

$$\frac{t + \sin t \cos t}{2} + c =$$

$x = \sin t \Rightarrow t = \arcsin x$

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$$= \frac{\arcsin x + \sin(\arcsin x)\cos(\arcsin x)}{2} + c$$

$$= \frac{\arcsin x + x\cos(\arcsin x)}{2} + c$$

$= x$

$$\cos t = \sqrt{1 - \sin^2 t}$$

$$\cos(\arcsin x) = \sqrt{1 - \sin^2(\arcsin x)}$$

$$= \sqrt{1 - x^2}$$

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$$= \frac{\arcsin x + x\sqrt{1-x^2}}{2} + c .$$

Es: $D(\arcsin)$

$f(x) = \sin x \,, \quad f^{-1}(y) = \arcsin y$

$$D(f^{-1}(y)) = \frac{1}{f'(f^{-1}(y))} =$$

$f'(x) = \cos x$

$$= \frac{1}{\cos(f^{-1}(y))} = \frac{1}{\cos(\arcsin y)} =$$

$$= \frac{1}{\sqrt{1-\sin^2(\arcsin y)}} = \frac{1}{\sqrt{1-y^2}}.$$

quindi

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c.$$

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Es: $D(\arccos x) =$

$$= \frac{-1}{\sqrt{1-x^2}} \cdot$$

Oss: $D(\arcsin x + \arccos x)$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

$\Rightarrow \arcsin x + \arccos x =$ costante

quanto vale la costante?

la calcolo per $x=0$.

costante = arcsin 0 + arccos 0 =

$$= 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$$

$$= \frac{\pi}{2} - \arccos x + c =$$

$$= - \arccos x + d$$

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Integrazione di funzioni razionali con denominatore di 2° grado.

$$\int \frac{ax+b}{x^2+cx+d}\,dx$$

(caso 1) $\displaystyle\int \frac{dx}{(x-a)^2}$

$x-a=t \quad dx=dt$

$$= \int \frac{dt}{t^2} = \int t^{-2}\,dt = -t^{-1} + c$$

$$= \frac{-1}{x-a} + c$$

2) $$\int \frac{dx}{(x-a)(x-b)} \quad \text{con } a \neq b.$$

cerco di scrivere

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

determino A e B.

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$$\frac{A}{x-a}+\frac{B}{x-b}=\frac{A(x-b)+B(x-a)}{(x-a)(x-b)}=$$

$$=\frac{Ax-Ab+Bx-Ba}{(x-a)(x-b)}=$$

$$=\frac{\overset{0}{\boxed{(A+B)}}x\overset{1}{-Ab-Ba}}{(x-a)(x-b)}=\frac{0\cdot x+1}{(x-a)(x-b)}$$

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$$\begin{cases} A+B=0 \\ -Ab-Ba=1 \end{cases} \qquad B=-A$$

$$-Ab+Aa=1$$

$$\Downarrow$$

$$A(a-b)=1$$

$$B=\frac{1}{b-a} \qquad A=\frac{1}{a-b} \qquad a\neq b.$$

$$\frac{1}{(x-a)(x-b)}=\frac{\frac{1}{a-b}}{x-a}+\frac{\frac{1}{b-a}}{x-b}=$$

$$= \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)$$

$$\Rightarrow \int \frac{dx}{(x-a)(x-b)} =$$

$$= \frac{1}{a-b} \int \frac{1}{x-a} - \frac{1}{x-b}\, dx =$$

$$= \frac{1}{a-b}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x-b}\right) =$$

$$= \frac{1}{a-b}\left(\log|x-a| - \log|x-b|\right) + c$$

$$= \frac{1}{a-b} \log\left|\frac{x-a}{x-b}\right| + c$$

$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + c.$$

$$\int \frac{dx}{k^2+x^2} = \qquad k \neq 0.$$

$$= \int \frac{dx}{k^2\left(1+\frac{x^2}{k^2}\right)} = \frac{1}{k^2}\int \frac{dx}{1+\frac{x^2}{k^2}}$$

$$\frac{x}{k} = t \qquad dx = k\,dt$$

$$= \frac{1}{k^{\cancel{2}}} \int \frac{\cancel{k}\,dt}{1+t^2} =$$

$$= \frac{1}{k} \operatorname{arctg} t + c =$$

$$= \frac{1}{k} \operatorname{arctg}\left(\frac{x}{k}\right) + c$$

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3) nessuna radice reale.

$$x^2 + px + q = x^2 + px + \frac{p^2}{4} - \frac{p^2}{4} + q =$$

$$= \left(x + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)$$

se non ci sono radici reali

$$q - \frac{p^2}{4} > 0.$$

quindi

$$\int \frac{dx}{x^2+px+q} = \int \frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)} =$$

$$t = x+\frac{p}{2} \qquad dt = dx$$

$$= \int \frac{dt}{t^2+\left(q-\frac{p^2}{4}\right)} \qquad \text{che è}$$

$$\left(q-\frac{p^2}{4}\right) > 0$$

del tipo $\displaystyle\int \frac{dt}{t^2+k^2}$