Teorema: $A \subset \mathbb{R}$, $x_0 \in Acc(A)$

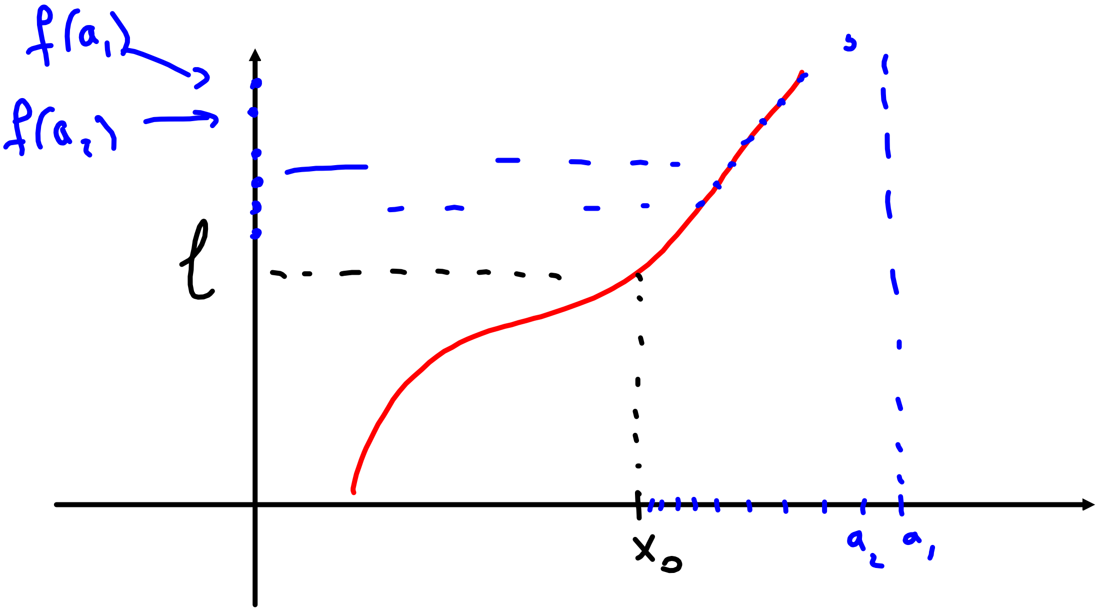
$f: A \to \mathbb{R}$. Allora

$\lim_{x \to x_0} f(x) = l$ se e solo se

$\lim_{n \to \infty} f(a_n) = l \quad \forall \{a_n\}$

t.c. $\lim_{n \to \infty} a_n = x_0$ e

$a_n \neq x_0$ definitivamente.


f(a1)
f(a2)
ℓ
x0
a2
a1

È utile per dimostrare che un limite non esiste.

Es: $\lim_{x \to \infty} \sin x$

Scelgo una successione

$a_n = n\pi$

$\lim_{n \to \infty} a_n = +\infty$

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \sin(n\pi) = 0$$

ora scelgo

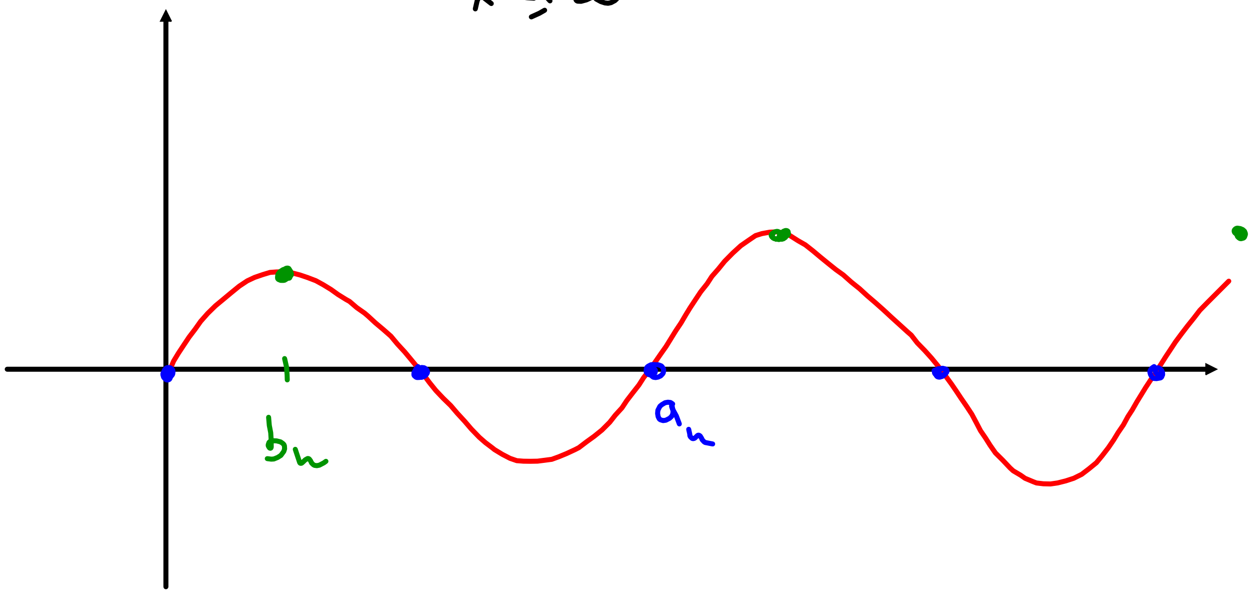
$$b_n = \frac{\pi}{2} + 2n\pi$$

anche $\lim_{n\to\infty} b_n = +\infty$

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \sin\left(\frac{\pi}{2} + 2n\pi\right) =$$

$$= \lim_{n\to\infty} \sin\frac{\pi}{2} = 1$$

Allora $\nexists \lim\limits_{x \to \infty} \sin x$



<u>Teorema</u>: $n_0 \in \mathbb{N}$ $f : [n_0, +\infty) \longrightarrow \mathbb{R}$.
$\forall n \in \mathbb{N}$, $n \geq n_0$ poniamo
$a_n = f(n)$. Se esiste
$\lim_{x \to +\infty} f(x) = \ell$ allora
$\exists \lim_{n \to +\infty} a_n = \ell$.

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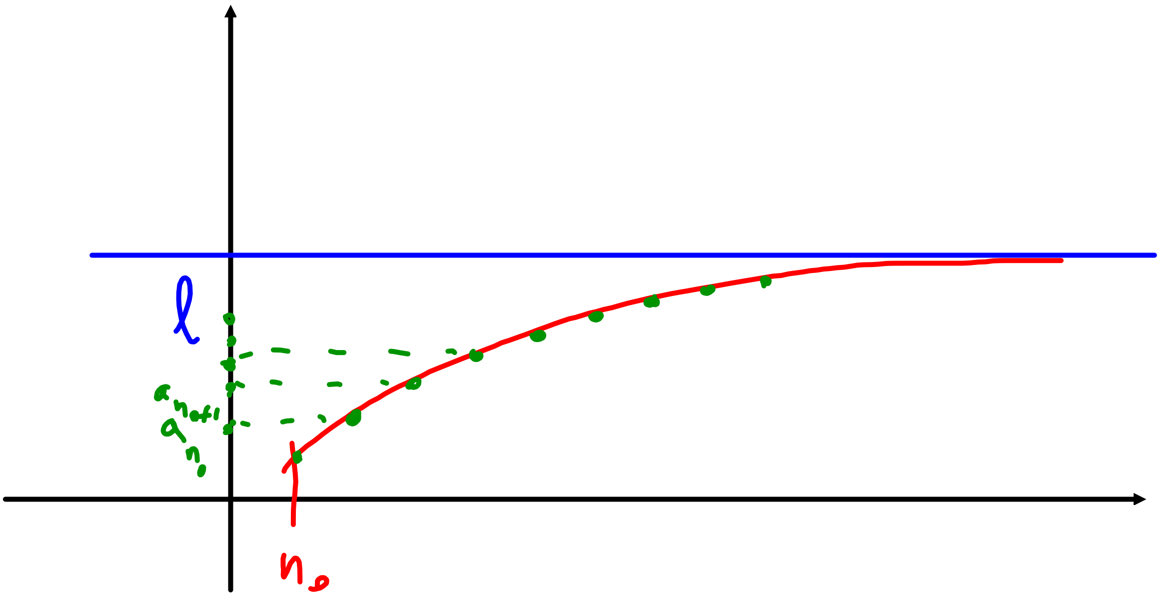

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Es: $\lim_{n\to\infty} e^{n} = +\infty$

perché $\lim_{x\to\infty} e^{x} = +\infty$.

Es: $\lim_{n\to\infty} n \sin \frac{1}{n}$

$f(x) = x \sin \frac{1}{x}$

$\lim_{x\to\infty} x \sin \frac{1}{x} = x\left[\frac{1}{x} + o\left(\frac{1}{x^2}\right)\right] = 1.$

oppure, direttamente

$$n \sin \frac{1}{n} = n \left[\frac{1}{n} + o\left(\frac{1}{n^2}\right)\right] \to 1$$

ho usato la sostituzione

$$\sin t = t + o(t^2) \text{ se } t \to 0$$

$t = \frac{1}{n}$ lo posso fare

perché se $n \to +\infty \Rightarrow \frac{1}{n} \to 0$.

Il <u>viceversa in generale è</u> falso

Es: $f(x) = \sin(\pi x)$

$$\lim_{x\to\infty} f(x) = \nexists$$

ma $f(n) = \sin(n\pi) = 0 \ \forall n$

$$\Rightarrow \exists \lim_{n\to\infty} f(n) = 0 .$$

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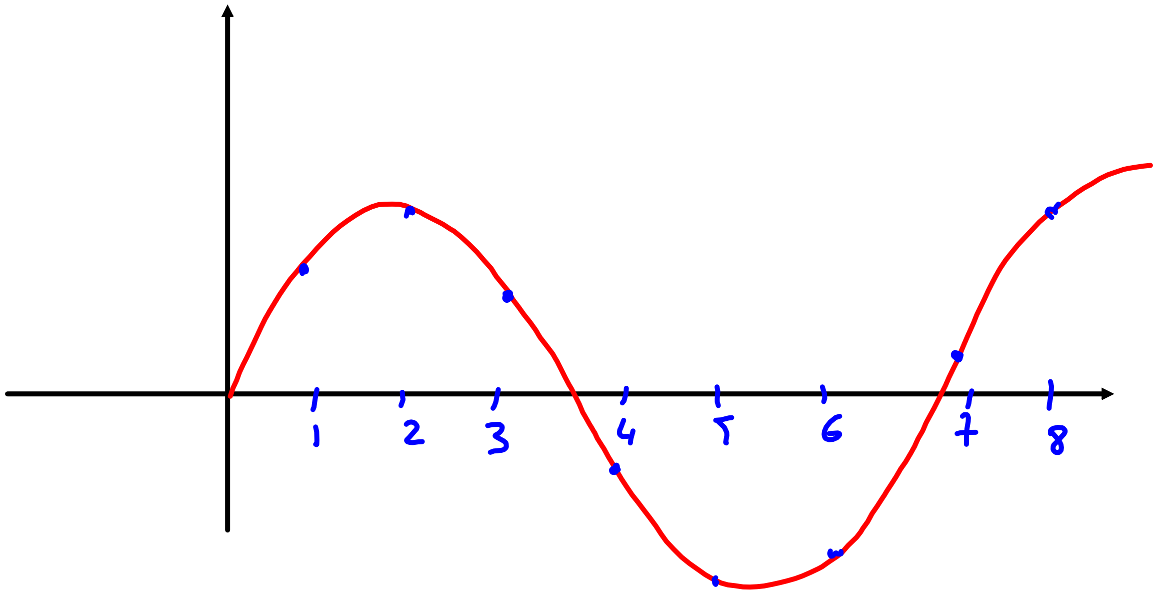
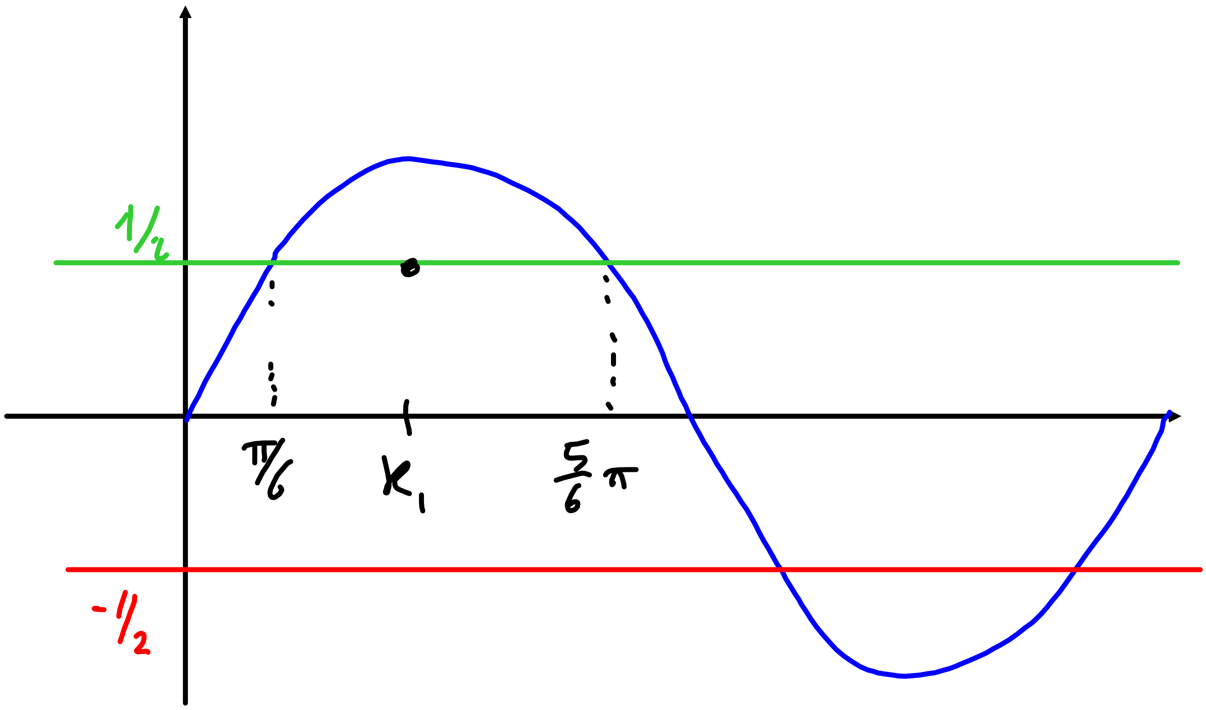
Es: $\lim_{n\to\infty} \sin n$




1/2
-1/2
π/6
x1
5/6 π

risolvo la disequazione

$$\sin x > \frac{1}{2}$$

$\sin x = \frac{1}{2}$ $\quad x = \frac{\pi}{6}$, $x = \frac{5}{6}\pi$

$\Rightarrow$ $\sin x > \frac{1}{2}$ in $\left(\frac{\pi}{6}, \frac{5}{6}\pi\right)$

l'intervallo è di lunghezza

$$\frac{5}{6}\pi - \frac{\pi}{6} = \frac{2}{3}\pi > 2$$

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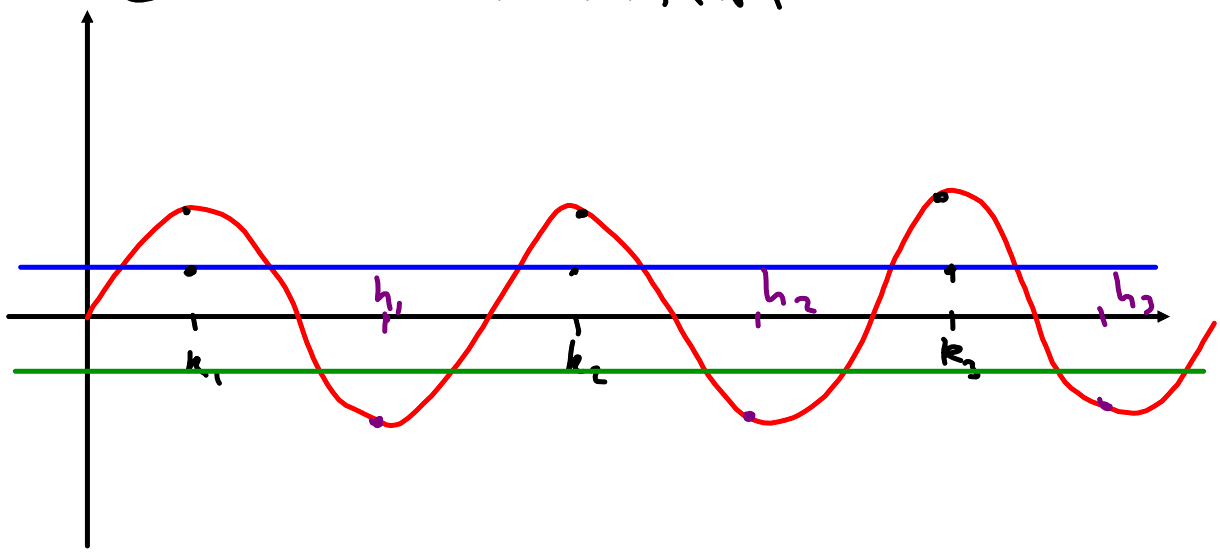
nell'intervallo ci sono almeno
2 numeri interi



posso costruire una sottosuccessione

t.c. $\sin(k_n) > \frac{1}{2} \quad \forall n.$

Per lo stesso motivo posso

costruire una sottosuccessione

h_n t.c.

$\sin(h_n) < -\frac{1}{2}$

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Se esistesse $\lim_{n\to\infty} \sin n = \ell$

allora dovrebbe essere

$$\lim_{n\to\infty} \sin(k_n) = \ell \geq \frac{1}{2}$$

$$\lim_{n\to\infty} \sin(h_n) = \ell \leq -\frac{1}{2}$$

assurdo.

Es.: $a_n = \dfrac{\sin(n^2) + \sin(n) + 3}{n^2 + n + 1}$

$$\lim_{n\to\infty} a_n = \frac{\text{limitato}}{\infty} = 0$$

$n^2 + n + 1 > 0$

$\sin(n^2) + \sin(n) + 3 \geq -1 - 1 + 3 = 1 > 0$

$a_n > 0 \quad \forall n \Rightarrow \{a_n\}$ ha max

$\inf a_n = 0$.

se avesse minimo sarebbe $\min a_n = 0$ ma è impossibile perché $a_n > 0$.

Es: $a_n = n^2 e^{-1/n} \sin n$

$$\lim_{n\to\infty} e^{-1/n} = e^{-\frac{1}{\infty}} = e^0 = 1$$

$$\lim_{n\to\infty} n^2 e^{-1/n} = \infty \cdot 1 = \infty$$

$\exists\ k_n \to \infty$ t.c. $\sin(k_n) > \frac{1}{2}$

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$$\Rightarrow a_{k_n} = (k_n)^2 e^{-1/k_n} \sin(k_n)$$

$$> (k_n)^2 e^{-1/k_n} \cdot \frac{1}{2} \longrightarrow +\infty$$

$$\underbrace{(k_n)^2 e^{-1/k_n}}_{\to +\infty}$$

Allo stesso modo posso trovare

$h_n \to \infty$ t.c. $\sin(h_n) < -\frac{1}{2}$

$$a_{h_n} = (h_n)^2 e^{-1/h_n} \sin(h_n) <$$

$$< -(h_n)^2 e^{-1/h_n} \cdot \frac{1}{2} \longrightarrow -\infty$$

$$a_{k_n} \longrightarrow +\infty, \quad a_{h_n} \longrightarrow -\infty$$

$$\Rightarrow \nexists \lim_{n\to\infty} a_n .$$

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<u>Teorema</u>: Sia $\{a_n\}$ una successione e $\{a_{k_n}\}$, $\{a_{h_n}\}$ due sottosucc. estratte t.c.

$$\{k_n : n \in \mathbb{N}\} \cup \{h_n : n \in \mathbb{N}\} = \mathbb{N}$$

(saturano tutti gli indici).

Se $\exists \lim_{n \to \infty} a_{k_n} = l$ e se

$$\exists \lim_{n\to\infty} a_{h_n} = l \quad \text{allora}$$

$$\exists \lim_{n\to\infty} a_n = l .$$

Es:
$$a_n = \frac{\log(n+1)^{(-1)^n}}{n^3}, \quad n \geq 1$$

indici pari
$$a_{2n} = \frac{\log(2n+1)^{(-1)^{2n}}}{(2n)^3} = \frac{\log(2n+1)}{8n^3}$$

$$\lim_{n\to\infty} a_{2n} = 0$$

n dispari

$$a_{2n+1} = \frac{\log(2n+2)^{(-1)^{2n+1}}}{(2n+1)^3} =$$

$$= \frac{[\log(2n+2)]^{-1}}{(2n+1)^3} = \frac{1}{[\log(2n+2)](2n+1)^3}$$

$$\lim_{n\to\infty} a_{2n+1} = 0$$

$\Rightarrow$ dato che $\{2n\}$ e $\{2n+1\}$

saturano tutto $\mathbb{N}$

$$\Rightarrow \lim_{n \to +\infty} a_n = 0.$$

max? min?

$a_n > 0 \quad \forall n \geq 1 \Rightarrow \{a_n\}$ ha max

$\log(n+1) > 0$

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$\inf a_n = 0$

$\Rightarrow \min\{a_n\} \nexists$ perché $\underline{\underline{a_n > 0}}$.

Criterio del rapporto

Se $a_n > 0$ definitivamente

e $\exists \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

allora

1) se $0 \le l < 1 \Rightarrow \lim_{n\to\infty} a_n = 0$

2) se $l > 1 \Rightarrow \lim_{n\to\infty} a_n = +\infty$

Oss: Se $l = 1$ non si applica il criterio.

Es: $a_n = \left(\frac{1}{2}\right)^n$

criterio del rapporto

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^n} = \frac{2^n}{2^{n+1}} = \frac{1}{2} < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0 .$$

Es: $a_n = 2^n$

criterio del rapporto

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} = 2 > 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = +\infty .$$

Es: $\lim_{n \to \infty} n! = +\infty$

$$n! = n(n-1)(n-2)\cdots$$
$$\geq n$$

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Es: $k \in \mathbb{N}$ fissato, $k \geq 1$.

$$\lim_{n \to \infty} \frac{n!}{n^k} = \frac{\infty}{\infty}$$

criterio del rapporto.

$$a_n = \frac{n!}{n^k} \qquad a_{n+1} = \frac{(n+1)!}{(n+1)^k}$$

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$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^k} \cdot \frac{n^k}{n!} =$$

$$= \frac{(n+1)\cancel{!}}{\cancel{n!}} \cdot \underbrace{\boxed{\frac{n^k}{(n+1)^k}}}_{\to 1} \to \infty \cdot 1 = \infty$$

$l = +\infty$

$l > 1$

$$\Rightarrow \lim_{n\to\infty} a_n = +\infty$$

Es: $\lim_{n\to\infty} \frac{n!}{b^n}$ $b>1$

a_n

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\cancel{!}}{b^{\cancel{n+1}}} \cdot \frac{\cancel{b^n}}{\cancel{n!}} = \frac{n+1}{b} \to +\infty$$

$\ell = +\infty > 1$

$\Rightarrow \lim_{n\to\infty} a_n = +\infty$

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Es: $\lim_{n \to \infty} \frac{n^n}{n!}$