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Istituzioni di Analisi – LECTURE 47

Note Title 22/11/2021

→ Compact operators

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→ RAYLEIGH quotient

Def. Let X and Y be metric spaces. A function $f : X \to Y$ is called $\boxed{\text{COMPACT}}$ if the image of every bounded set in X is a relatively compact subset of Y.

In terms of sequences: if $\{x_n\} \subseteq X$ is bounded, then $\{f(x_n)\}$ admits a converging subsequence.

Rmk Compact $\not\Rightarrow$ continuous [consider any function whose image is 7 points].

Prop. (Linear + compact $\Rightarrow$ strong – strong continuous)

Let X and Y be normed vector spaces. Let $f : X \to Y$ be a function.

Let us assume that

(i) f is linear

(ii) f is compact.

Then f is SS continuous, and actually also Lip. cont., and in particular

$$x_n \to x_\infty \text{ in } X \quad \Rightarrow \quad f(x_n) \to f(x_\infty) \text{ in } Y.$$

(strong convergence)

$\boxed{\text{Proof}}$ Since f is linear, we can assume WLOG that $x_\infty = 0$.

Let $x_n \to 0$. We know that it is bounded.

Assume that $x_n \neq 0$ for every $n \in \mathbb{N}$.

Then consider $\frac{x_n}{\|x_n\|_X} = v_n$. Then $\|v_n\| = 1$ and therefore v_n is bounded

$\Rightarrow$ $f(v_n)$ admits a converging subsequence

$\Rightarrow$ $\|f(v_n)\|_Y \leq M$ for every $n \in \mathbb{N}$

$$\|f(v_n)\|_Y = \frac{\|f(x_n)\|_Y}{\| \quad {}_m\|_X} \leq M$$

$\Rightarrow$ $\|f(x_n)\|_Y \leq M \, \underset{\substack{\downarrow \\ 0}}{\|x_n\|_X}$ (true for free even if $x_n = 0$)

$\Rightarrow$ $\|f(x_n)\|_Y \to 0$

This proves that every subsequence $f(x_{n_k})$ has a further subsequence s.t. $f(x_{n_{k_i}}) \to 0$.
By the sub-sub lemma this implies that that $f(x_n) \to 0$.

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<u>Def</u>. (Symmetric operator)
Let H be a Hilbert space. A function $f : H \to H$ is called $\boxed{\text{symmetric}}$ if

$$\boxed{\langle f(v), w \rangle = \langle v, f(w) \rangle \qquad \forall (v, w) \in H^2}$$

<u>Exercise</u> If f is symmetric, then f is LINEAR

<u>Hint</u>
$$\begin{aligned} \langle f(v_1+v_2), w \rangle &= \langle v_1+v_2, f(w) \rangle \\ &= \langle v_1, f(w) \rangle + \langle v_2, f(w) \rangle \\ &= \langle f(v_1), w \rangle + \langle f(v_2), w \rangle \\ &\ldots \end{aligned}$$

Prop. (Lin. + symm. + cpt. $\Rightarrow$ weak-strong continuous)
Let H be a Hilbert space and let $f : H \to H$ be a function.
Let us assume that
(i) f is linear
(ii) f is compact
(iii) f is symmetric
Then f is ws continuous, namely (if x_n is bounded and)

$$x_n \underset{\text{weak}}{\longrightarrow} x_\infty \text{ in } H \quad \Rightarrow \quad f(x_n) \underset{\text{strong}}{\longrightarrow} f(x_\infty) \text{ in } H$$

$\boxed{\text{Proof}}$

$$\lim_{n\to+\infty} \langle f(x_n), w\rangle \underset{\substack{\nearrow \\ f \text{ symm}}}{=} \lim_{n\to+\infty} \langle x_n, f(w)\rangle$$

$$\underset{\substack{\nearrow \\ x_n \to x_\infty}}{=} \langle x_\infty, f(w)\rangle$$

$$\underset{\substack{\nearrow \\ f \text{ symm}}}{=} \langle f(x_\infty), w\rangle$$

This proves that $f(x_n) \longrightarrow f(x_\infty)$ weakly

Take any subsequence $\{f(x_{n_k})\}$. Due to compactness, this admits a further subsequence

$$f(x_{n_{k_i}}) \longrightarrow y_\infty \quad \text{strongly}$$

But we know that

$$f(x_{n_{k_i}}) \rightharpoonup f(x_\infty)$$

and therefore $y_\infty = f(x_\infty)$.
The conclusion follows from the sub-sub lemma

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Rmk One day the same statement is true without $\{x_n\}$ bounded and even without assuming f is symmetric.

RAYLEIGH QUOTIENT Let $f : H \to H$

$$\boxed{q(v) := \frac{\langle f(v), v \rangle}{\|v\|^2} \qquad \forall v \in H \setminus \{0\}}$$

Theorem (Variational characterization of eigenvalues/eigenvectors)

Let $f : H \to H$ as above. Let us assume that

(i) f is linear

(ii) f is symmetric

(iii) f is compact

Let $V \subseteq H$ be a f-invariant $\underset{\uparrow}{\text{closed}}^{\text{strongly}}$ subspace ($f(v) \in V$ for every $v \in V$).

Then the following are true

(1) There exists

$$\max \{ |q(v)| : v \in V \setminus \{0\} \}$$

and it coincides with

$$\sup \{ |\langle f(v), v \rangle| : \|v\| = 1, v \in V \}$$

and with

$$\sup \{ |\langle f(v), v \rangle| : \|v\| \leq 1, v \in V \}$$

(2) Consider any maximum point $v_0 \in V \setminus \{0\}$ and let $|q(v_0)|$ be the maximum value.

Then

$$f(v_0) = \underset{\uparrow \text{eigenvalue}}{q(v_0)} \cdot \overset{\swarrow \text{eigenvector}}{v_0}$$

without absolute value

Rmk We do not say that $q(v)$ admits max and min.

This is true only in FINITE dimension

Exercise Find example

$\boxed{\text{Proof}}$ (i) We claim that

$$\sup\{|q(v)| : v \in V \setminus \{0\}\} \underset{\substack{\uparrow \\ q(v) \text{ is } 0\text{-homogeneous}}}{=} \sup\{|\langle f(v), v\rangle| : v \in V, \|v\| = 1\}$$

$$\leq \sup\{|\langle f(v), v\rangle| : v \in V, \|v\| \leq 1\}$$

The key point is that the last one is an equality, because for every $w \in V$ with $\|w\| < 1$ we find that $\frac{w}{\|w\|}$ is a better competitor because

$$\left|\left\langle f\left(\frac{w}{\|w\|}\right), \frac{w}{\|w\|}\right\rangle\right| \underset{\substack{\uparrow \\ \text{lin.}}}{=} \frac{1}{\|w\|^2} |\langle f(w), w\rangle| \underset{\substack{\uparrow \\ \|w\|^2 < 1}}{\geq} |\langle f(w), w\rangle|$$

[Some discussion needed if $w = 0$]

The second important point is that the last one is a maximum, namely

$$\max\{|\langle f(v), v\rangle| : v \in V, \|v\| \leq 1\} \quad \text{exists}$$

Take $\{v_n\}$ a sup-izing sequence. Then there exists

$$v_{n_k} \longrightarrow v_\infty \quad \text{weakly}$$

By the ws continuity we know that

$$f(v_{n_k}) \longrightarrow f(v_\infty) \quad \text{strongly}$$

$$\langle f(v_{n_k}), v_{n_k}\rangle \underset{\substack{\uparrow \\ 1 \text{ weak } + \\ 1 \text{ strong}}}{\longrightarrow} \langle f(v_\infty), v_\infty\rangle$$

(2) Let v_0 be a maximum point for $|q(v)|$.

Then v_0 is either max o min for $q(v)$

Write

$$q(v_0+tv) = \frac{\langle f(v_0+tv), v_0+tv\rangle}{\|v_0+tv\|^2}$$

$$= \frac{\langle f(v_0), v_0\rangle + 2t\langle f(v_0), v\rangle + t^2\langle f(v), v\rangle}{\|v_0\|^2 + 2t\langle v_0, v\rangle + t^2\|v\|^2}$$

$$0 = q'(0) = \frac{2\langle f(v_0), v\rangle\|v_0\|^2 - 2\langle v_0, v\rangle\langle f(v_0), v_0\rangle}{\|v_0\|^4}$$

$\forall v \in V$

$$\langle \underbrace{f(v_0)\|v_0\|^2 - \langle f(v_0), v_0\rangle v_0}_{=0}, v\rangle = 0 \qquad \forall v \in V$$

$$f(v_0) = \frac{\langle f(v_0), v_0\rangle}{\|v_0\|^2} v_0 = q(v_0)\, v_0$$

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Questions → Where did we use that we have $|q(v)|$?

→ where did we use that V is f-invariant ?

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Note Title 22/11/2021

SPECTRAL THEOREM (for compact operators)

Theorem Let H be a separ. Hilbert space. Let $f : H \to H$ and let us assume that
(infinite dimension)

(i) f is linear

(ii) f is compact

(iii) f is symmetric

Then the following statements are true.

(1) H admits a Hilber basis $\{v_n\}$ made by eigenvectors of f

$$f(v_n) = \lambda_n v_n \qquad \forall n \geq 1$$

(2) The sequence $\{\lambda_n\}$ is such that

$$|\lambda_n| \longrightarrow 0$$

(3) For every $n \geq 1$, the eigenspace of λ_n has FINITE dimension unless $\lambda_n = 0$

Proof Step 1 Eigenvectors with different eigenvalues are orthogonal

$f(v) = \lambda v \qquad f(w) = \mu w$

$$\lambda \langle v, w \rangle = \langle \lambda v, w \rangle = \langle f(v), w \rangle = \langle v, f(w) \rangle$$
$$= \langle v, \mu w \rangle = \mu \langle v, w \rangle$$

$$\leadsto \quad \underbrace{(\lambda - \mu)}_{\neq 0} \langle v, w \rangle = 0$$

Step 2 If V is f-invariant, then $V^\perp$ is again f-invariant

$w \in V^\perp$ and $v \in V$. Then

$$\langle f(w), v \rangle = \langle \underset{\in V^\perp}{w}, \underset{\in V}{f(v)} \rangle = 0 \quad \Rightarrow \quad f(w) \in V^\perp$$

Step 3 The idea is to consider a sequence of f-invariant subspaces

$$H = V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$$

and in each subspace we apply the variational characterization.

$V_1 = H \rightsquigarrow$ there exists v_1 and λ_1 s.t. $f(v_1) = \lambda_1 v_1$

Set $V_2 = \{ v \in H : \langle v, v_1 \rangle = 0 \}$. This is f-invariant

($\perp$ of Span(v_1))

$\rightsquigarrow$ there exists v_2 and λ_2 s.t. $f(v_2) = \lambda_2 v_2$

Note that $\underset{\uparrow \atop \sup\{|q(v)| : v \in V_2 \setminus \{0\}\}}{|\lambda_2|} \leq \overset{}{\underset{\uparrow \atop \sup\{|q(v)| : v \in V_1 \setminus \{0\}\}}{|\lambda_1|}}$

... Suppose we have defined $v_1, \dots, v_k$. Then set

$$V_{k+1} = \{ v \in V : \langle v, v_i \rangle = 0 \text{ for every } i = 1, \dots, k \}$$

$$= \operatorname{Span}(v_1, \dots, v_k)^{\perp} \quad \text{and therefore } f\text{-invariant}$$

Actually there are two cases.

→ There exists $k \geq 1$ such that $q(v) \equiv 0$ in V_k

→ $q(v) \not\equiv 0$ for every $k \geq 1$.

$\boxed{\text{Case 1}}$ $q(v) \equiv 0$ in V_k. Then $V_k = \operatorname{Ker}(f)$

Hyp. $\langle f(v), v \rangle = 0 \quad \forall v \in V$

Th $f(v) = 0 \quad \forall v \in V$

Here we used symmetry (think of rotation by 90° degrees in $\mathbb{R}^2$)

$$0 = \langle f(v+w), v+w \rangle$$

$$\underset{\overset{\uparrow}{\text{symmetry}}}{=} \langle \underset{\overset{\shortparallel}{0}}{f(v), v \rangle} + 2\langle f(v), w \rangle + \underset{\overset{\shortparallel}{0}}{\langle f(w), w \rangle} = 2\langle f(v), w \rangle$$

for every v and w in V $\rightsquigarrow$ $f(v) = 0$
because $f(v) \in V \cap V^{\perp}$

If we are in this case than we take $v_1, \dots, v_{k-1}$ and we add a orthonormal basis of $\ker(f)$

↑ this is in turn a separable Hilbert space

(general fact: any subset of a separ. metric space is separ.)

$\boxed{\text{Case 2}}$ We obtain an infinite sequence $\{v_n\}$ s.t.

$$f(v_n) = \lambda_n v_n$$

This is NOT YET an orthonormal basis.

We observe that $|\lambda_{n+1}| \leq |\lambda_n|$ for every $n \geq 1$

The claim is that $\boxed{\lambda_n \to 0.}$

If this is not the case, then $|\lambda_n| \geq \nu_0 > 0$

Consider $\{v_n\}$. It is a bounded sequence. Therefore

$\{f(v_n)\} = \{\lambda_n v_n\}$ has a converging subseq.

On the other hand

$$\| \lambda_n v_n - \lambda_m v_m \|^2 = \lambda_n^2 \underset{\overset{\shortparallel}{1}}{\| v_n \|^2} + \lambda_m^2 \underset{\overset{\shortparallel}{1}}{\| v_m \|^2} + 2 \lambda_n \lambda_m \underset{\overset{\shortparallel}{0}}{\langle v_n, v_m \rangle}$$

$$\geq 2\nu_0^2$$

$\Rightarrow$ $\{\lambda_n v_n\}$ does NOT admit Cauchy subsequences

$\boxed{\text{Step 4}}$ Assume that $\lambda_m \to 0$ and consider

$$V_\infty := V_1 \cap V_2 \cap V_3 \cap \dots$$

$$= \text{Clos}(\text{Span}(v_1, \dots, v_k, \dots))^\perp$$

The claim is that V_∞ is Ker (f) and therefore it is enough to add a Hilbert basis of Ker (f).

Clearly $V_\infty \subseteq V_k$ for every $k \geq 1$, and therefore

$$\sup\{|q(v)| : v \in V_\infty \setminus \{0\}\} \leq \sup\{|q(v)| : v \in V_k \setminus \{0\}\}$$
$$= |\lambda_k| \to 0$$

$\rightsquigarrow$ $q(v) \equiv 0$ in V_∞ $\rightsquigarrow$ (as before) $f(v) \equiv 0$ in V_∞.

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Example Let $\{e_n\}$ be a Hilbert basis in H

Define

$$f(e_n) = \begin{cases} \frac{1}{n} e_n & \text{if } n \text{ IS ODD} \\ 0 & \text{if } n \text{ IS EVEN} \end{cases}$$

It can be verified that, if we extend f by linearity, the resulting operator is compact with an infinite dimensional kernel.

Example (Compactness is needed) $H = L^2((0,1))$

Consider $A : H \to H$ defined by

$$[Au](x) = x\,u(x)$$

$\boxed{\text{Fact 1}}$ It is well-defined, linear and Lip. cont.

$$\|x u(x) - x v(x)\|^2_{L^2} = \int_0^1 \underset{\leq 1}{x^2} (u(x)-v(x))^2 dx$$
$$\leq \int_0^1 (u(x)-v(x))^2 dx$$
$$= \|u(x)-v(x)\|^2_{L^2}$$

$\boxed{\text{Fact 2}}$ It is symmetric

$$< Au, v > = \int_0^1 x u(x) v(x) dx = \int_0^1 u(x) \cdot x v(x) dx = < u, Av >$$

$\boxed{\text{Fact 3}}$ A admits no eigenvalue

Assume $Au = \lambda u \rightsquigarrow x u(x) = \lambda u(x)$ for a.e. $x \in (0,1)$

$u(x)(x-\lambda) = 0$

$\rightsquigarrow u(x) = 0$ almost everywhere

$\boxed{\text{Fact 4}}$ A is not compact (if it were, then eigenvalues would exist)

<u>Exercise</u> Find $\{u_n\} \subseteq L^2((0,1))$ bounded s.t. $\|u_n\| = 1$ but $\{x u_n(x)\}$ has NO subseq. that converges STRONGLY. (No Cauchy subseq.)

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Istituzioni di Analisi – LECTURE 49

Note Title 24/11/2021

APPROXIMATION OF COMPACT OPERATORS

Prop. Setting: X and Y are metric spaces

$f_n : X \to Y$

$f_\infty : X \to Y$

Assumptions:

(i) Y is COMPLETE ← [Exercise: show that this is needed]

(ii) f_n is a compact operator for every $n \geq 1$

(iii) $f_n \to f_\infty$ unif. on bounded subsets of X.

Then f_∞ is compact.

Proof Let $B \subseteq X$ be any bounded set. We need to prove that $f_\infty(B)$ is relatively compact in Y.

Since Y is complete, it is enough to prove that $f_\infty(B)$ is totally bounded.

Idea: $f_\infty(B)$ is close to $f_n(B)$, which is totally bounded

Details.

- We are given $\varepsilon > 0$.
- There exists n large enough so that
$$d_Y(f_\infty(x), f_n(x)) \leq \underset{\uparrow}{\tfrac{1}{2}\varepsilon} \qquad \forall x \in B$$
- There exist finitely many points $y_1, \dots, y_k$ such that
$$f_n(x) \in \bigcup_{i=1}^{k} B(y_i, \underset{\uparrow}{\tfrac{1}{2}\varepsilon}) \qquad \forall x \in B$$
- By the triangular inequality $f_\infty(x) \in \bigcup_{i=1}^{k} B(y_i, \underset{\uparrow}{\varepsilon}) \quad \forall x \in B.$

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Let H be a $\boxed{\text{SEPARABLE}}$ Hilbert space, and let $\{e_n\}$ be a Hilbert basis.

Let

$$H_n := \text{Span}(e_1, \dots, e_n) \quad \leftarrow \text{vector space of dim. } n$$

Let

$$P_n : H \longrightarrow H_n$$

denote the orthogonal projection.

If X is any metric space, and $f : X \to H$ is any function, we can set

$$f_n(x) = P_n(f(x)) \qquad \forall x \in X$$

Then

$$f_n : X \longrightarrow H_n \quad (\text{finitely dim. approx. of } f)$$

<u>Question</u>: in which sense $f_n \to f$?

Trivial answer: pointwise convergence, namely $f_n(x) \to f(x)$ for every $x \in X$.

$\boxed{\text{Prop}}$ (Approximation of compact operators in Hilbert space)

Setting as above.

(1) If f is compact, then $f_n \to f$ uniformly on bounded subsets of X

(2) If f is continuous, then f_n is continuous (trivial)

(2) If X is a normed space, and f is linear, then f_n is linear (trivial)

$\boxed{\text{Proof}}$ Let $B \subseteq X$ be a bounded subset, and let $\varepsilon > 0$.
By assumption $f(B) \subseteq H$ is rel. compact and therefore totally bounded, and therefore there exists $v_1, \dots, v_k$ in H such that

$$f(B) \subseteq \bigcup_{i=1}^{k} B(v_i, \tfrac{1}{2}\varepsilon)$$

For every $i = 1, \dots, k$ there exists $z_i \in \underset{\underset{\text{FINITE LIN COMB.}}{\uparrow}}{\mathrm{Span}}(e_1, \dots, e_n, \dots)$ such that

$$\|v_i - z_i\| \le \tfrac{1}{2}\varepsilon$$

Finally, there exists $n_0 \ge 1$ such that $z_i \in H_{n_0}$ for every $i = 1, \dots, k$.

Claim: $\|f(x) - f_n(x)\| \le \varepsilon \quad \forall n \ge n_0 \quad \forall x \in B$

Given $x \in B$, find i such that $f(x) \in B(v_i, \frac{1}{2}\varepsilon)$ and observe that

$$\|f(x) - f_n(x)\| = \|f(x) - \overset{\overset{H_n}{\cup}}{P_n}(f(x))\|$$

$$\le \|f(x) - \underset{\underset{H_n}{\cap}}{z_i}\|$$

$P_n(f(x))$ minimizes the distance from H_n

$$\le \|f(x) - v_i\| + \|v_i - z_i\|$$

$$\le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \quad ☺$$

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$\boxed{\text{Rmk}}$ This result motivates some results "IF and ONLY IF" such as

" $f : X \to H$ is < assumption > if and only if f can be approximated by a sequence $f_n : X \to H_n$ satisfying < assumption > "

For example "f linear, continuous compact"
"f_n linear, continuous, compact"

compactness follows from boundedness if the target space has finite dim

NONLINEAR PROJECTION

Theorem Let Y be a normed space

Let $K \subseteq Y$ be a compact subset

Let $\varepsilon > 0$.

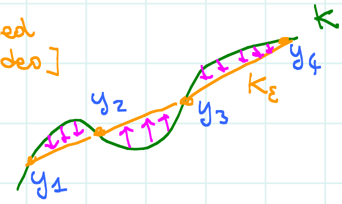
Then there exists $K_\varepsilon \subseteq Y$ and $P_\varepsilon : K \to K_\varepsilon$ such that

(1) K_ε is the convex envelope of a finite number of points of K

(2) P_ε is continuous

(3) $\| P_\varepsilon(y) - y \| \leq \varepsilon$ for every $y \in K$

[ε removed after video]



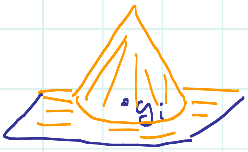
Proof Since K is compact, there exist $y_1, \dots, y_k$ such that

$$K \subseteq \bigcup_{i=1}^{k} B(y_i, \varepsilon) = A_\varepsilon$$

Let K_ε be the convex envelope of these points

$$K_\varepsilon = \left\{ \sum_{i=1}^{k} \lambda_i y_i \ : \ \lambda_i \geq 0 \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

For every $i = 1, \dots, k$ we introduce the function the "small cone function"



$$c_i(y) := \max \{ 0, \varepsilon - \| y - y_i \| \}$$

Let us consider $S(y) := \sum_{i=1}^{k} c_i(y) \neq 0$ for every $y \in A_\varepsilon$.

For every $y \in A_\varepsilon$ let us define

$$P_\varepsilon(y) := \underbrace{\sum_{i=1}^{k} \frac{c_i(y)}{S(y)} y_i}_{\in K_\varepsilon \text{ because sum of coeff.} = 1} \qquad \forall y \in A_\varepsilon$$

We only need to check that $\|P_\varepsilon(y)-y\| \le \varepsilon$ for every $y \in K$

$$\|P_\varepsilon(y)-y\| = \left\| \sum_{i=1}^{k} \frac{c_i(y)}{S(y)} y_i - y \right\|$$

$$= \left\| \sum_{i=1}^{k} \frac{c_i(y)}{S(y)} (y_i - y) \right\|$$

$$\le \sum_{i=1}^{k} \frac{c_i(y)}{S(y)} \underbrace{\|y_i - y\|}$$

↑ this is less than ε when $c_i(y) \neq 0$

$$\le \sum_{i=1}^{k} \frac{c_i(y)}{S(y)} \varepsilon$$

$$= \varepsilon .$$

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Rmk In many statements P_ε or better $P_{1/n}$ plays the role of the projection $P_n : H \to H_n$ in Hilbert.

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Istituzioni di Analisi – LECTURE 50

Note Title 24/11/2021

Fixed point theorems from BROUWER to SCHAUDER

Theorem (BROUWER) Let $R>0$ and let $\overline{B}(0,R)$ be a closed ball in $\mathbb{R}^d$. Let

$$f : \overline{B}(0,R) \to \overline{B}(0,R)$$

be a continuous function.

Then there exists at least one $x \in \overline{B}(0,R)$ such that

$$f(x) = x.$$

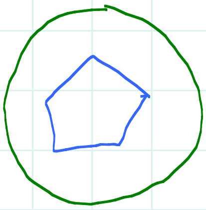
$\boxed{\text{Proof}}$ Geometry courses (some algebraic topology is needed)

Corollary (Modified Brouwer theorem)

Let $D \subseteq \mathbb{R}^d$ be a CONVEX COMPACT set

Let $f : D \to D$ be a continuous functions.

Then f admits at least one fixed point.



$\boxed{\text{Proof}}$ Take $R>0$ s.t. $D \subseteq \overline{B}(0,R)$

Consider $P_D : \mathbb{R}^d \to D$ be the projection onto the convex set D.

Define

$$\hat{f} : \overline{B}(0,R) \to \overline{B}(0,R)$$

as

$$\hat{f}(x) := f(P_D(x))$$

Apply Brouwer to $\hat{f}$ and obtain $x \in \overline{B}(0,R)$ such that

$$x = \hat{f}(x) \in D$$

and therefore also $x \in D$.

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Bad news: there is no Brouwer theorem in infinite dimension

Example Let H be the usual separable Hilbert space with Hilbert basis $\{e_n\}$.
Let $(x_1, x_2, \dots)$ denote the components of $x \in H$.
Let us define

$$f(x) = \left((1-\|x\|)^{1/2}, x_1, x_2, x_3, \dots \right) \qquad \forall x \in \bar{B}(0,1)$$

It is possible to check that f is continuous, and that $f: \bar{B}(0,1) \to \bar{B}(0,1)$ and actually $\|f(x)\| = 1$ for every $x \in \bar{B}(0,1)$.
On the other hand, f has no fixed point

Any fixed point has to satisfy $x_1 = x_2 = x_3 = \dots$ and therefore $\underset{\substack{\uparrow \\ \text{unique vector with all equal components}}}{x=0}$, but $f(0) \neq 0$

SCHAUDER FIXED POINT THEOREM

Setting: $\underset{\substack{\uparrow \\ \text{normed space}}}{Y} \supseteq \underset{\substack{\uparrow \\ \text{convex set}}}{C} \supseteq \underset{\substack{\uparrow \\ \text{compact set}}}{K}$

$f: C \to K$ continuous function

Then f admits at least one fixed point (of course in K).

Idea: approximate f with function f_n defined in space with finite dimension, when we can apply Brouwer

$\boxed{\text{Proof}}$ Apply the nonlinear projection lemma with $\varepsilon := \frac{1}{n}$.
We obtain

$$P_n : K \longrightarrow \underset{\substack{\uparrow \\ \text{convex envelope of a finite} \\ \text{number of points of } K}}{K_n}$$

Observe that $K_n \subseteq C$ because C is convex, and therefore we can define

$$f_n : K_n \longrightarrow K_n$$

as

$$f_n(\underset{\substack{\uparrow \\ \in C}}{x}) = P_n(\underbrace{f(x)}_{\in K})$$

We observe that f_n is continuous and K_n is contained in a vector space with finite dimension Y_n = Span of the points whose convex envelope is K_n.
We observe that K_n is convex and compact, and from Brouwer we deduce that there exists $x_n \in K_n \subseteq C$ s.t.

$$f_n(x_n) = x_n.$$

Consider $\{\underset{\substack{\shortparallel \\ y_n}}{f(x_n)}\} \subseteq K$ and therefore $y_{n_k} \longrightarrow y_\infty \in K$

<u>Claim</u>: y_∞ is a fixed point of f.

Before we prove that $x_{n_k} \longrightarrow y_\infty$:

$$\| x_{n_k} - y_\infty \| \underset{\substack{\uparrow \\ \text{fixed point}}}{=} \| f_{n_k}(x_{n_k}) - y_\infty \|$$

$$\le \underset{\substack{\shortparallel \\ \underbrace{\| P_{n_k}(f(x_{n_k})) - f(x_{n_k}) \|}_{\le \frac{1}{n_k} \text{ because of the property of the projection}}}}{\| f_{n_k}(x_{n_k}) - f(x_{n_k}) \|} + \underbrace{\| \overset{\substack{y_{n_k} \\ \shortparallel}}{f(x_{n_k})} - y_\infty \|}_{\to 0}$$

At this point

$$f(y_\infty) = f\left(\lim_{k\to+\infty} x_{n_k}\right) = \lim_{k\to+\infty} f(x_{n_k}) = \lim_{k\to+\infty} y_{n_k} = y_\infty.$$

— o — o —

$\boxed{\text{Classical application}}$ $\boxed{\text{PEANO's theorem for diff. equations}}$

Statement Let $d \geq 1$ be an integer

Let $t_0 \in \mathbb{R}$, $u_0 \in \mathbb{R}^d$

Let $\delta_0 > 0$ and $r_0 > 0$

Let $f : \underbrace{[t_0-\delta_0, t_0+\delta_0] \times \overline{B}(u_0, r_0)}_{R} \to \mathbb{R}^d$ continuous

Consider the Cauchy pbm

$$u'(t) = f(t, u(t))$$

$$u(t_0) = u_0$$

Let

$$M := \max\{|f(t,u)| : (t,u) \in R\}$$

and let

$$\delta_1 \leq \delta_0 \qquad \delta_1 M \leq r_0$$

Then there exists at least one solution

$$u : [t_0-\delta_1, t_0+\delta_1] \to \overline{B}(u_0, r_0)$$

Several proofs

- approximate f by Lipschitz functions
- delay problems "à la Tonelli"
- Schauder

} see lectures "Analisi 2"

As usual, a solution is a fixed point of the VOLTERRA operator

$$[\Phi(u)](t) := u_0 + \int_{t_0}^{t} f(s, u(s))\, ds \qquad \forall t \in [t_0-\delta_1, t_0+\delta_1]$$

Functional setting

$Y := \{$ continuous functions $u : [t_0-\delta_1, t_0+\delta_1] \to \mathbb{R}^d \}$

↑ normed vector space (sup norm)

$C := \{$ cont. functions $u : [t_0-\delta_1, t_0+\delta_1] \to \overline{B}(u_0, r_0) \}$

↑ convex set

$K := \{$ functions in C that are Lip. cont. with constant $\leq M \}$

↑ compact set because of Ascoli – Arzelà

We just need to prove that $\Phi : C \longrightarrow K$

(here we need the definition of M and condition $\delta_1 M \leq r_0$)

We need also that Φ is continuous wrt the norm of Y

(true because integrals pass to the limit wrt uniform convergence).

Schauder $\Rightarrow$ at least one fixed point. ☺

— o — o —

Istituzioni di Analisi — LECTURE 51

Note Title 25/11/2021

Setting: X and Y are NORMED SPACES (almost everything works up to locally convex topological vector spaces)

Questions:
① $\exists\, f : X \to Y$ linear and $f \not\equiv 0$
② $\exists\, f : X \to Y$ linear and CONTINUOUS and $f \not\equiv 0$
③ $\exists\, f : X \to Y$ linear and NON-CONTINUOUS

Answer ① YES, but we need CHOICE AXIOM (CA)

Proof CA $\Rightarrow$ existence of HAMEL BASIS in X (algebraic basis)

$\uparrow$ every $x \in X$ is FINITE linear combination

Let $\{x_i\}_{i \in I}$ be such a basis. Define $f(x_i) = y_i$ $\uparrow$ elements of Y

and extend by linearity

—o—o—

Prop (7 shades of continuity for LINEAR MAPS)

Let $F : X \to Y$ be a linear map.
Then the following are equivalent.
(i) f is ε/δ continuous
(ii) f is sequentially continuous
(iii) f (every ball in X) is bounded in Y
(i') f is ε/δ continuous in $x=0$
(ii') f is seq. cont. in $x=0$
(iii') $f(B_X(0,1))$ is bounded in Y
(vii) f is LIPSCHITZ CONTINUOUS

$\boxed{\text{Proof}}$ Trivial : (i) $\Rightarrow$ (i'), (ii) $\Rightarrow$ (ii'), (iii) $\Rightarrow$ (iii')

(vii) $\Rightarrow$ all the rest

Standard : (i) $\Leftrightarrow$ (ii) (general fact in metric spaces)

(i') $\Rightarrow$ (iii') Apply ε/δ cont. in $x=0$ with $\varepsilon = 1$.

We find $\delta > 0$ such that $f\,(B_X(0,\delta)) \subseteq B_Y(0,1)$

By linearity

$$f\,(B_X(0,1)) = f\left(\tfrac{1}{\delta} B_X(0,\delta)\right) \underset{\substack{\uparrow \\ \text{linearity}}}{\subseteq} \tfrac{1}{\delta} B_Y(0,1)$$

(i) $\Rightarrow$ (vii) As above there exists $\delta > 0$ such that

$$f\,(\bar{B}_X(0,1)) \subseteq \tfrac{1}{\delta} \bar{B}_Y(0,1)$$

For every $x \in X$ with $x \neq 0$ consider $\frac{x}{\|x\|_X} \in \bar{B}_X(0,1)$

then

$$f(x) \underset{\substack{\uparrow \\ \text{linearity}}}{=} \|x\|_X f\left(\tfrac{x}{\|x\|_X}\right)$$

and therefore

$$\|f(x)\|_Y = \|x\|_X \, \underbrace{\left\|f\left(\tfrac{x}{\|x\|}\right)\right\|_Y}_{\in \bar{B}_Y(0,\frac{1}{\delta})} \leq \tfrac{1}{\delta} \|x\|_X .$$

By linearity

$$\|f(x_2) - f(x_1)\|_Y = \|f(x_2 - x_1)\|_Y \leq \underset{\substack{\uparrow \\ \text{Lip. constant}}}{\tfrac{1}{\delta}} \|x_2 - x_1\|_X .$$

—o—o—

$\boxed{\text{Answer ③}}$ YES if dim $(X) = +\infty$ and we accept C.A.

$\boxed{\text{Proof}}$ Consider $Y = \mathbb{R}$. We construct $f : X \to \mathbb{R}$ linear and NOT continuous.

Let $\{x_n\} \subseteq X$ be linearly indep. vectors (can assume $\|x_n\|_X = 1$)

CA $\Rightarrow$ we can complete them to Hamel basis

Define $f(x_n) = n$ for every $n \geq 1$

f (rest of the basis) = what we want (for ex. 0)

and extend by linearity.

The image of $B_X(0,1)$ is unbounded.

— o — o —

Def. (PSEUDO-NORM)

Let X be a VECTOR SPACE. A pseudo-norm is a function

$$p : X \to \mathbb{R}$$

with two properties

(i) (subadditivity)

$$\boxed{p(x_1 + x_2) \leq p(x_1) + p(x_2) \qquad \forall (x_1, x_2) \in X^2}$$

(ii) (positively homogeneous)

$$\boxed{p(\lambda x) = \lambda p(x) \qquad \forall x \in X \quad \forall \lambda > 0}$$

Rmk There are no sign conditions

Theorem (HAHN-BANACH, analytic form)
Let X be a vector space. Let p be a pseudo-norm on X. Let $E \subseteq X$ be a vector subspace. Let $f : E \to \mathbb{R}$ be a linear map such that

$$f(x) \leq p(x) \qquad \forall x \in E$$

Then there exists $\hat{f} : X \to \mathbb{R}$ linear such that

$$\hat{f}(x) = f(x) \qquad \forall x \in E$$

$$\hat{f}(x) \leq p(x) \qquad \forall x \in X$$

$\boxed{\text{Proof}}$ Let $\mathcal{Y}$ denote the set of extensions of f. More precisely an element of $\mathcal{Y}$ is a pair $(\underset{\substack{\uparrow\\ \text{subspace}}}{F}, \underset{\substack{\nwarrow\\ \text{lin. map on } F}}{g})$ such that

$$E \subseteq F$$

$$g(x) = f(x) \quad \forall x \in E \qquad + \qquad g(x) \leq p(x) \quad \forall x \in F$$

- $\mathcal{Y}$ is nonempty (it contains (E, f))

- $\mathcal{Y}$ is partially ordered by inclusion $(F_1, g_1) \leq (F_2, g_2)$ if
  $F_1 \subseteq F_2$ and $g_2(x) = g_1(x) \quad \forall x \in F_1$

- If $(F_i, g_i)_{\underset{\substack{\uparrow\\ \text{not necessarily countable}}}{i \in I}}$ is a totally ordered subset of $\mathcal{Y}$
  then it admits an upper bound in $\mathcal{Y}$ defined as

$$\hat{F} := \bigcup_{i \in I} F_i \qquad \hat{g}(x) = g_i(x) \quad \text{if } x \in F_i$$

[check that this is a well defined element of $\mathcal{Y}$]

ZORN LEMMA $\Rightarrow$ $\mathcal{Y}$ admits a maximal element.

Let (F, g) denote this element. Claim: $F = X$.
Assume it is not the case, namely $x_0 \in X \setminus F$.
In this case we extend g to $F \oplus \mathbb{R} x_0$.
Is it possible? We have to define

$$g(v + \alpha x_0) \qquad \forall v \in F \quad \forall \alpha \in \mathbb{R}$$
$$= g(v) + \alpha\, g(x_0)$$

We need to find $g(x_0)$ in such a way that

$$g(v) + \alpha\, g(x_0) \leq p(v + \alpha x_0) \qquad \forall v \in F \quad \forall \alpha \in \mathbb{R}$$

There are three cases

- $\alpha = 0 \rightsquigarrow$ easy
- $\alpha > 0 \rightsquigarrow \quad g(v + \alpha x_0) \leq p(v + \alpha x_0)$

$$\cancel{\alpha}\, g\Big(\underset{\substack{\uparrow \\ \text{generic element of } F}}{\tfrac{v}{\alpha}} + x_0\Big) \leq \cancel{\alpha}\, p\Big(\frac{v}{\alpha} + x_0\Big)$$

Therefore we need $\boxed{g(y + x_0) \leq p(y + x_0)} \qquad \forall y \in F$

- $\alpha < 0 \rightsquigarrow \alpha = -\beta$ with $\beta > 0 \rightsquigarrow \quad g(v - \beta x_0) \leq p(v - \beta x_0)$

$$\cancel{\beta}\, g\Big(\frac{v}{\beta} - x_0\Big) \leq \cancel{\beta}\, p\Big(\frac{v}{\beta} - x_0\Big)$$

$$\boxed{g(z - x_0) \leq p(z - x_0)} \qquad \forall z \in F$$

The two conditions are

$$g(y)+g(x_0)\le p(y+x_0)\qquad \forall y\in F$$
$$g(z)-g(x_0)\le p(z-x_0)\qquad \forall z\in F$$

$$g(z)-p(z-x_0)\le g(x_0)\le p(y+x_0)-g(y)\qquad \forall (y,z)\in F^2$$

We can find $g(x_0)$ if and only if

$$g(z)-p(z-x_0)\overset{?}{\le} p(y+x_0)-g(y)\qquad \forall (y,z)\in F^2$$

namely

$$g(z)+g(y)\overset{?}{\le} p(y+x_0)+p(z-x_0)\qquad \forall (y,z)\in F^2$$

$$g(z)+g(y)\underset{\substack{\uparrow\\ g \text{ linear}\\ \text{on } F}}{=} g(z+y)\underset{\substack{\uparrow\\ \text{estimate true in } F}}{\le} p(z+y)$$
$$= p(z-x_0+y+x_0)$$
$$\underset{\substack{\uparrow\\ p \text{ is subadditive}}}{\le} p(z-x_0)+p(y+x_0)$$

This gives a contradiction, so that $F=X$.

— o — o —

Istituzioni di Analisi — LECTURE 52

Note Title 25/11/2021

Prop Let V be a normed space. Let $v_0 \in V$ be any element. Then there exists $f : V \to \mathbb{R}$ linear such that

$$f(v_0) = \|v_0\|_V$$

and

$$|f(v)| \leq \|v\|_V \qquad \forall v \in V.$$

Proof Consider the (pseudo)-norm $p(v) = \|v\|$.

Consider $E = \operatorname{Span}(v_0)$ (if $v_0 = 0$, then $f \equiv 0$)

and consider $f : E \to \mathbb{R}$ defined as

$$f(\alpha v_0) := \alpha \|v_0\|$$

Let us check that

$$f(v) \leq p(v) \quad \text{for every } v \in E$$

namely if $v = \alpha v_0$

$$\alpha \|v_0\| \leq \|\alpha v_0\| \qquad \forall \alpha \in \mathbb{R}$$

which is trivial.

HB $\Rightarrow$ we can extend f to V in such a way that

$$f(v) \leq p(v) = \|v\|_V \qquad \forall v \in V$$

We need that

$$|f(v)| \leq p(v) = \|v\|_V \qquad \forall v \in V$$

If $f(v) \geq 0$ this is trivial

If $f(v) < 0$ then

$$|f(v)| = -f(v) = f(-v) \leq p(-v) = \|-v\|_V = \|v\|_V$$

— o — o —

Def. In the sequel we call any such f $\boxed{\text{an}}$ ALIGNED FUNCTIONAL of v_0.

$\boxed{\text{Answer ②}}$ If V and W are normed vector spaces, then there exists a nontrivial $f : V \to W$ that is linear and continuous

$\boxed{\text{Proof}}$

$$V \longrightarrow \mathbb{R} \longrightarrow W$$

aligned functional of any $v_0 \neq 0$

just map 1 to any $w_0 \neq 0$ in W

SPACE OF CONTINUOUS LINEAR OPERATORS

Defn Let V and W be normed spaces.
We call $\mathcal{L}(V,W)$ the space of linear and continuous functions $f : V \to W$.
For every $f \in \mathcal{L}(V,W)$ we set

$$\boxed{\|f\|_{\mathcal{L}(V,W)} := \sup \{ \|f(v)\|_W : v \in V \text{ and } \|v\|_V \leq 1 \}}$$

Prop. Notations as above. Then $\mathcal{L}(V,W)$ is a vector space and $\|f\|_{\mathcal{L}(V,W)}$ is a norm in $\mathcal{L}(V,W)$, such that

$$\|f(v)\|_W \leq \|f\|_{\mathcal{L}(V,W)} \cdot \|v\|_V \qquad \forall v \in V$$

Moreover, if W is complete (Banach), then also $\mathcal{L}(V,W)$ is complete (Banach).

$\boxed{\text{Proof}}$ Standard: vector space + norm + inequality
Let us check the completeness when W is complete.
Let $\{f_n\} \subseteq \mathcal{L}(V,W)$ be any Cauchy sequence, wrt the operator norm

Step 1 For every fixed $v \in V$ we have that $\{f_n(v)\}$ is a Cauchy sequence in W

$$\|f_n(v)-f_m(v)\|_W = \|(f_n-f_m)(v)\|_W$$

$$\le \underbrace{\|f_n-f_m\|_{\mathcal{L}(V,W)}}_{\le \varepsilon \text{ if } n \text{ and } m \text{ are large}} \cdot \|v\|_V$$

Step 2 From step 1 we deduce that there exists $f_\infty : V \to W$ such that

$$f_n(v) \longrightarrow f_\infty(v) \qquad \forall v \in V$$

It is easy to see that

- f is linear (pointwise limit of lin. functions)
- f is continuous (pointwise limit of equi-Lip. functions is Lip. cont. The seq. $\{f_n\}$ is equi-Lip because Cauchy seq. are always bounded and the norm in $\mathcal{L}(V,W)$ is the Lip. constant).

Step 3 We have to prove that $f_n \to f_\infty$ wrt the norm of $\mathcal{L}(V,W)$.

$$\|(f_n-f_\infty)(v)\|_W \le \varepsilon \qquad \forall v \in V \text{ with } \|v\|_V \le 1$$

if n is large enough.

Since it is a Cauchy sequence, we know that $\|f_n-f_m\|_{\mathcal{L}(V,W)} \le \varepsilon$ for every $n \geq n_0$, $m \geq n_0$, namely

$$\|f_n(v)-f_m(v)\|_W \le \varepsilon \qquad \forall v \in V \text{ with } \|v\|_V \le 1$$

Let $m \to +\infty$.

— o — o —

Defn (TOPOLOGICAL DUAL)
Let V be a normed space. The top. dual V' of V is the set of all continuous functions $f : V \to \mathbb{R}$.
It is always a Banach space (even if V is NOT complete) wrt the norm

$$\boxed{\|f\|_{V'} := \sup\{|f(v)| : \|v\|_V \leq 1\}}$$

$\boxed{\text{Prop}}$ (Dual characterization of the norm)

Let V be a normed space. Then for every $v \in V$ it is true that

$$\|v\|_V = \sup\{|f(v)| : f \in V' \text{ and } \|f\|_{V'} \leq 1\}$$

$$= \max\{|f(v)| : \quad " \quad \}$$

$\boxed{\text{Proof}}$ $\boxed{\geq}$ $$|f(v)| \leq \underbrace{\|f\|_{V'}}_{\leq 1} \cdot \|v\|_V \leq \|v\|_V$$

$\boxed{\leq}$ Aligned functional realizes the maximum.

—o—o—

Simple application of proof by duality

Consider $(a,b) \subseteq \mathbb{R}$ and $f : (a,b) \to \mathbb{R}^d$. Then

$$\left\| \int_a^b f(t)\,dt \right\| \leq \int_a^b \|f(t)\|\,dt$$

(Standard result with the standard norm)

How to prove the result with a different norm?
$\|(x,y)\| = (x^6+y^6)^{1/6}$

$\boxed{\text{Proof by duality}}$ Let $L : \mathbb{R}^d \longrightarrow \mathbb{R}$ be any linear map

Then

$$L\left(\int_a^b f(t)\,dt\right) \underset{\uparrow}{=} \int_a^b L(f(t))\,dt$$

$\int$ commutes with L

and therefore

$$\left| L\left(\int_a^b f(t)\,dt\right)\right| = \left|\int_a^b L(f(t))\,dt\right|$$

$$\underset{\nearrow}{\leq} \int_a^b |L(f(t))|\,dt$$

scalar function

$$\leq \int_a^b \underset{\substack{\uparrow\\ \text{dual}}}{\|L\|_{V'}} \cdot \|f(t)\|_V\,dt \quad \uparrow \mathbb{R}^d \text{ with the given norm}$$

$$\leq \int_a^b \|f(t)\|_V\,dt \qquad \text{if } \|L\|_{V'} \leq 1$$

We take the supremum as L varies in V' with $\|L\|_{V'} \leq 1$ and in the LHS we obtain the norm.

— o — o —

<u>Example</u> $V = L^1((0,+\infty))$

Define $L : V \to \mathbb{R}$ as

$$L(u) = \int_0^{+\infty} u(x) \arctan x \, dx$$

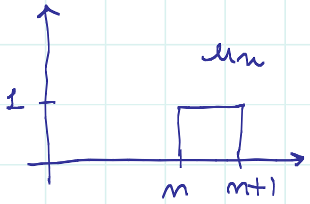
Can prove: L is linear and continuous because

$$|L(u)| \leq \frac{\pi}{2} \|u\|_V$$

Therefore $L \in V'$ and $\|L\|_{V'} = \frac{\pi}{2}$

but sup is not realized.

In order to prove that $\|L\|_{V'} = \frac{\pi}{2}$ consider



Example 2 $\qquad V = L^1((0,2))$

$$L : V \to V$$
$$u(x) \to x\,u(x)$$

Then $\|L\|_{\mathcal{L}(V,V)} = 2$ and again it is not realized

— o — o —

Istituzioni di Analisi — LECTURE 53

Note Title 29/11/2021

Defn Let V be normed space, and let V' be the topological dual.

(1) If $\{v_n\} \subseteq V$ and $v_\infty \in V$, we say that

$$v_n \rightharpoonup v_\infty \quad \text{weakly in } V$$

if

$$f(v_n) \underset{\uparrow}{\longrightarrow} f(v_\infty) \qquad \forall f \in V'$$

convergence of numbers

(2) If $\{f_n\} \subseteq V'$ and $f_\infty \in V'$, we say that

$$f_n \overset{*}{\rightharpoonup} f_\infty \quad \text{weakly * in } V'$$

if

$$f_n(v) \underset{\uparrow}{\longrightarrow} f_\infty(v) \qquad \forall v \in V$$

convergence of numbers

Rmk Weak * convergence is actually pointwise conv. of functions.

Rmk One day weak conv. will coincide with weak convergence as previously defined in Hilbert spaces.

Basic properties
① Uniqueness of the limit
② Linearity
③ Strong ⇒ weak
④ $v_n \rightharpoonup v_\infty \Rightarrow v_{n_k} \rightharpoonup v_\infty$
and the same for weak *
⑤ Sort of sub-sub-lemma.

$\boxed{\text{Proof of uniqueness}}$ Assume that $u_n \rightharpoonup u_\infty$ and
$u_n \rightharpoonup \hat{u}_\infty$

Then for every $f \in V'$

$$f(u_n) \to f(u_\infty)$$
$$f(u_n) \to f(\hat{u}_\infty)$$

and hence $f(u_\infty - \hat{u}_\infty) = 0 \quad \forall f \in V'$

Now use f = aligned functional of $u_\infty - \hat{u}_\infty$ (we need HB and therefore CA)

— o — o —

Theorem (LSC of the norm)

Assume $u_n \rightharpoonup u_\infty$ weakly in V
$f_n \rightharpoonup f_\infty$ weakly $*$ in V'

Then

$$\liminf_{n \to +\infty} \|u_n\|_V \geq \|u_\infty\|_V$$

$$\liminf_{n \to +\infty} \|f_n\|_{V'} \geq \|f_\infty\|_{V'}$$

$\boxed{\text{Dim}}$ $\boxed{V}$ By definition $|f(u_n)| \leq \|f\|_{V'} \cdot \|u_n\|_V$
and therefore

$$|f(u_\infty)| = \lim_{n \to +\infty} |f(u_n)| \leq \overbrace{\|f\|_{V'}}^{\leq 1} \cdot \liminf_{n \to +\infty} \|u_n\|_V$$

$$\leq \liminf_{n \to +\infty} \|u_n\|_V$$

provided that $\|f\|_{V'} \leq 1$

Now consider f = aligned functional of u_∞ and obtain $\|u_\infty\|_V$ in the LHS.

$\boxed{V'}$ Observe that $|f_n(v)| \leq \|f_n\|_{V'} \cdot \|v\|_V \leq \|f_n\|_{V'}$
if $\|v\|_V \leq 1$

Therefore for every $v \in V$ with $\|v\|_V \leq 1$ it turns out that

$$|f_\infty(v)| = \lim_{n\to+\infty} |f_n(v)| \leq \liminf_{n\to+\infty} \|f_n\|_{V'}$$

Now consider the sup of LHS among all such v.

— o — o —

Theorem (Weak $*$ compactness of balls)
Let V be a normed space, and let V' be the top. dual.
Assume V is $\boxed{\text{SEPARABLE}}$
Let $\{f_n\} \subseteq V'$ be a $\boxed{\text{BOUNDED}}$ sequence.
Then there exists

$$f_{n_k} \xrightarrow{*} f_\infty \quad \text{weakly} * \text{ in } V'.$$

Proof We follow the same road map as in Hilbert spaces.
Let us choose a countable dense set $D \subseteq V$.

Step 1 There exists $n_k \to +\infty$ such that

$$\boxed{f_{n_k}(v) \quad \text{has a limit} \quad \forall v \in D}$$

Let us $D = \{d_1, d_2, d_3, \dots\}$. Observe that

$$|f_n(d_1)| \leq \underbrace{\|f_n\|_{V'}}_{\leq M} \cdot \|d_1\|_V$$

$\leadsto$ existence of conv. subsequence

Then we consider $d_2 \in D$ and we obtain a further subsequence
... we conclude by the usual diagonal argument

Step 2 We show that $\underset{\substack{\uparrow \\ \text{the same subsequence that} \\ \text{is good for } D}}{f_{n_k}}(v) \longrightarrow \text{something} \quad \forall v \in V$

It is enough to prove that $\{f_{n_k}(v)\}$ is a Cauchy seq. of numbers.

$$|f_{n_k}(v) - f_{n_\ell}(v)| \leq |f_{n_k}(v) - f_{n_k}(d)| \qquad ①$$
$$+ |f_{n_k}(d) - f_{n_\ell}(d)| \qquad ②$$
$$+ |f_{n_\ell}(d) - f_{n_\ell}(v)| \qquad ③$$

① and ③ are small if d is close enough to v, ② is small if ℓ and k are large enough.

Formally:

→ we are given $\varepsilon > 0$

→ fix $d \in D$ such that $M \|d - v\|_V \leq \frac{1}{4}\varepsilon$ so that

$$|f_n(v) - f_n(d)| \leq \underbrace{\|f_n\|_{V'}}_{\leq M} \cdot \|v - d\|_V \leq \frac{1}{4}\varepsilon$$

→ once that $d \in D$ has been fixed, we choose $\ell_0 \in \mathbb{N}$ such that ② is $\leq \frac{1}{4}\varepsilon$ for every $k \geq \ell_0$, $\ell \geq \ell_0$ (possible because $\{f_{n_k}(d)\}$ is a Cauchy seq.)

Step 3 At the end of step 2 we know that

$$\lim_{k \to +\infty} f_{n_k}(v) = \text{something} =: f_\infty(v) \qquad \forall v \in V$$

We need that $f_\infty \in V'$. This is easy

→ f_∞ is linear (pointwise limit of linear functions)

→ f_∞ is continuous (pointwise limit of M-Lip. cont. funct.)

—o—o—

Exercise Try to repeat the argument in order to show that ball is V are weakly compact and see what happens.

Rmk (About LSC of the norm)
Both in V and in V' the norm is the supremum of linear functions. As a general fact, the sup of cont. functions is LSC.

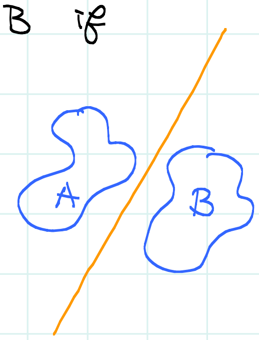
— o — o —

SEPARATION RESULTS

Def Let V be a normed space, and let A and B be two non-empty subsets.

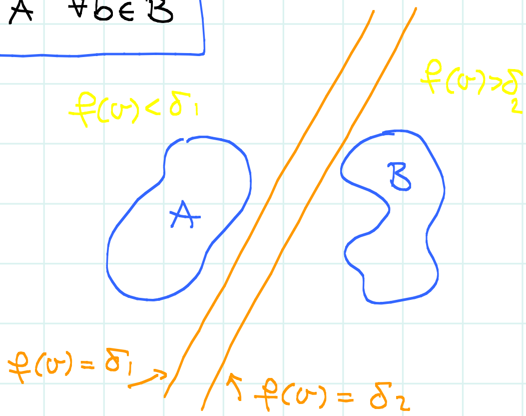
(1) We say that $f \in V'$ separates weakly A and B if

$$\boxed{f(a) < f(b) \qquad \forall a \in A \quad \forall b \in B}$$



(2) We say that $f \in V'$ separates strongly A and B if there exists $\delta_1 < \delta_2$ such that

$$\boxed{f(a) \leq \delta_1 < \delta_2 \leq f(b) \qquad \forall a \in A \quad \forall b \in B}$$



Theorem 1 (Weak separation of convex sets)

Let us assume that

(i) A and B are nonempty convex sets

(ii) $A \cap B = \emptyset$

(iii) A is $\boxed{\text{OPEN}}$

Then a weak separation exists

Theorem 2 (Strong separation of convex sets)

Let us assume that

(i) as before

(ii) as before

(iii) A is $\boxed{\text{CLOSED}}$ and B is $\boxed{\text{COMPACT}}$

Then a strong separation exists.

Corollary (Strong + convex $\Rightarrow$ weak)

Let us assume that

(i) A is strongly closed

(ii) A is convex

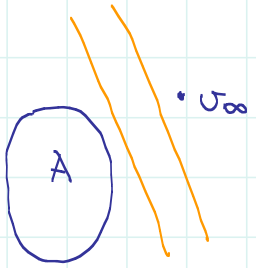
Then A is weakly closed.

Proof Assume that $A \ni v_n \longrightarrow v_\infty$

Claim: $v_\infty \in A$

Assume it is not the case.

By strong sep. with $B = \{v_\infty\}$ we obtain that



$$\begin{tikzcd} f(v_n) \arrow[d] & \le \delta_1 < \delta_2 \le f(v_\infty) \\ f(v_\infty) & \end{tikzcd}$$

Note that a weak separation would not be enough.

— o — o —

Istituzioni di Analisi — LECTURE 54

Note Title 29/11/2021

Lemma (PSEUDO-NORM ASSOCIATED TO A CONVEX SET)

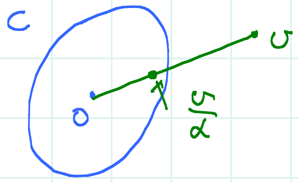
Let V be a normed space, and let $C \subseteq V$.

Let us assume that

(i) C is convex

(ii) $0 \in C$

(iii) C is open



Let us set

$$p(v) := \inf\left\{ \alpha > 0 : \frac{v}{\alpha} \in C \right\} \qquad \forall v \in V$$

Then

(1) p is a pseudo-norm on V

(2) there exists M such that $p(v) \leq M \|v\|_V$

(3) $C = \{ v \in V : p(v) < 1 \}$ (in some sense C is the unit ball wrt p)

Proof For every $v \in V$ let us set

$$A(v) := \left\{ \alpha > 0 : \frac{v}{\alpha} \in C \right\}$$

so that $p(v) = \inf A(v)$.

(1) $p(\lambda v) = \lambda p(v)$ for every $\lambda > 0$ because $A(\lambda v) = \lambda A(v)$

$p(v_1 + v_2) \leq p(v_1) + p(v_2)$ follows from

$$A(v_1 + v_2) \supseteq A(v_1) + A(v_2)$$

because

$$\begin{aligned} p(v_1+v_2) = \inf A(v_1+v_2) &\leq \inf \left[A(v_1) + A(v_2) \right] \\ &= \inf A(v_1) + \inf A(v_2) \\ &= p(v_1) + p(v_2) \end{aligned}$$

In order to prove the inclusion of sets, we consider $\alpha_1 \in A(v_1)$ and $\alpha_2 \in A(v_2)$ and we show that

$$\alpha_1 + \alpha_2 \in A(v_1 + v_2)$$

$$\frac{v_1+v_2}{\alpha_1+\alpha_2} = \frac{\alpha_1}{\alpha_1+\alpha_2} \underset{\in C}{\boxed{\frac{v_1}{\alpha_1}}} + \frac{\alpha_2}{\alpha_1+\alpha_2} \underset{\in C}{\boxed{\frac{v_2}{\alpha_2}}} \in C$$

Sum = 1

(2) Since C is open and $0 \in C$, there exists $R>0$ s.t.

$$\overline{B}_V(0,R) \subseteq C$$

Claim: we can take $M = \frac{1}{R}$. Indeed for every $v \in V$ we have that $\frac{1}{R}\|v\|_V \in A(v)$ because

$$\frac{v}{\frac{1}{R}\|v\|_V} = \text{vector with norm } R \in \overline{B}(0,R) \subseteq C$$

[Trivial also if $\|v\|_V = 0$]

(3) $\boxed{\subseteq}$ Let $v \in C$ then $1 \in A(v) \Rightarrow p(v) < 1$

(in general $A(v)$ is a nonempty open set and it is a right half-line)

$\boxed{\supseteq}$ Let $v \in V$ with $p(v) < 1$. Then $1 \in A(v)$ and therefore $\frac{v}{1} \in C \rightsquigarrow v \in C$.

—o—o—

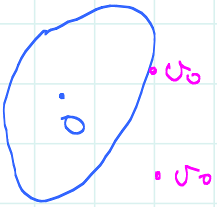
$\boxed{\text{Prop.}}$ Let $C \subseteq V$ and let $v_0 \in V$.

Assume that

(i) C is an OPEN convex set and $0 \in C$

(ii) $v_0 \notin C$

Then there exists $f \in V'$ that separates weakly C and v_0



$\boxed{\text{Proof}}$ We consider $E := \text{Span}(v_0)$ and the function

$$f : E \to \mathbb{R}$$

defined as

$$f(\alpha v_0) := \alpha \qquad \forall \alpha \in \mathbb{R}$$

We observe that f is linear and

$$\underset{\alpha}{\underset{\shortparallel}{f(\alpha v_0)}} \leq p(\alpha v_0) \qquad \forall \alpha \in \mathbb{R}$$

Indeed, the inequality is trivial when $\alpha \leq 0$ and if $\alpha > 0$ it is equivalent to $\quad 1 \leq p(v_0)$

and this is true because of (3) of the Lemma.

HB $\Rightarrow$ we can extend f to $f : V \to \mathbb{R}$.

We claim that this is the required separation.

Indeed

$$f(v) \underset{\substack{\uparrow \\ \text{HB}}}{\leq} p(v) \underset{\substack{\uparrow \\ (3)}}{<} 1 \underset{\substack{\uparrow \\ \text{defn}}}{=} f(v_0) \qquad \forall v \in C$$

We need to prove that f is continuous wrt $\|v\|_V$.

By statement (2) of the previous lemma we know that

$$f(v) \leq p(v) \leq M \|v\|_V \qquad \forall v \in V$$

This is ok if $f(v) \geq 0$. Otherwise

$$|f(v)| = -f(v) = f(-v) \leq p(-v) \leq M\|-v\|_V = M\|v\|_V$$

—o—o—

Prop. Again $C \subseteq V$ and $v_0 \in V$.

Assume that

(i) C is convex open and nonempty. (now 0 is not nec. in C)

(ii) $v_0 \notin C$

Then there exists a weak separation

Proof Take any $w_0 \in C$. Then consider

$$\hat{C} := C - w_0 \qquad \hat{v}_0 := v_0 - w_0$$

Apply the previous proposition to $\hat{v}_0$ and $\hat{C}$ and simplify $f(w_0)$

— o — o —

PROOF OF WEAK SEPARATION (1st geom. form of HB)

A, B convex $\quad A \cap B = \emptyset \quad A$ is open $\Rightarrow$ weak sep.

Introduce $C := A - B = \{a - b : a \in A, b \in B\}$

We observe that

→ C is convex (general fact)

→ C is open (union of translations of A)

→ $0 \notin C$ ($A \cap B = \emptyset$)

By previous prop. we can separate 0 from C, and obtain $f \in V'$ s.t.

$$\underset{f(a-b)}{\overset{\shortparallel}{f(c)}} < \underset{0}{\overset{\shortparallel}{f(0)}} \qquad \forall c \in C$$

namely $\quad f(a) - f(b) < 0 \qquad \forall a \in A \quad \forall b \in B$

$$f(a) < f(b)$$

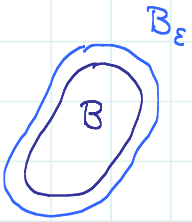
— o — o —

PROOF OF STRONG SEPARATION

A, B convex sets, $A \cap B$, A closed and B compact

⇝ strong separation

Idea: enlarge one of them a little bit



For every $\varepsilon > 0$ we set $B_\varepsilon := \bigcup_{b \in B} \underset{\substack{\uparrow \\ \text{open ball}}}{B(b, \varepsilon)}$

Claim: $A \cap B_\varepsilon = \emptyset$ if ε is small enough

Assume it is not the case. Then $A \cap B_{1/m} \neq \emptyset$ for every n

This means that there exist

$$\underset{\substack{\uparrow \\ \in A}}{a_n} = \underset{\substack{\uparrow \\ \in B}}{b_m} + \frac{1}{m} \underset{\substack{\uparrow \\ \|v_m\|_V \le 1}}{v_n}$$

Up to subsequences $b_m \longrightarrow \overset{\in B}{b_\infty}$ because B is compact.

But then also $a_n \longrightarrow \underset{\substack{\pitchfork \\ A}}{b_\infty}$ because A is closed

but $A \cap B = \emptyset$!!

Since B_ε is open, there exists $f \in V'$ such that

$$f(a) < f(b) \qquad \forall b \in B_\varepsilon$$

$$f(a) < f(b) + \varepsilon f(v) \qquad \forall b \in B \quad \forall v \text{ with } \|v\|_V < 1$$

Now set $\delta_1 := \sup \{ f(a) : a \in A \}$

$\delta_2 := \delta_1 + \varepsilon \|f\|_{V'}$

We have $f(a) \leq \delta_1 \quad \forall a \in A$

we claim that

$$f(b) \geq \delta_2 \qquad \forall b \in B$$

$$f(b) + \varepsilon f(v) \geq \delta_1 \qquad \forall b \in B \qquad \forall \|v\|_V < 1$$

$$\rightsquigarrow \qquad f(b) \geq \delta_1 - \varepsilon f(v) \qquad \forall b \in B \qquad \forall \|v\|_V < 1$$

If we consider the sup of the RHS among all v's in the ball we obtain exactly $\varepsilon \|f\|_{V'} + \delta_1$.

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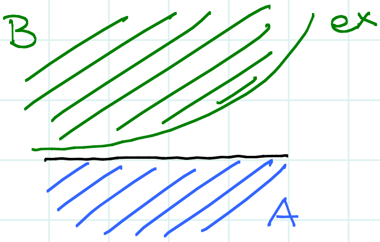
<u>Example</u> A and B closed is NOT enough for a strong separation



— o — o —

[Last example edited after video. Summary:

→ in finite dimension any two convex disjoint sets admit a weak sep. (nice exercise)

→ in infinite dimension the do exist two convex <u>closed</u> disjoint sets that do NOT admit a weak separation (hard exercise)]

Istituzioni di Analisi — LECTURE 55

Note Title 01/12/2021

SEQUENCE SPACES Notation $x = \{x_i\}_{i\geq 1}$ seq. of real numbers.

Definition

- We say that $x \in \ell^p$ with $p \in [1, +\infty)$ if

$$\|x\|_{\ell^p}^p := \sum_{i\geq 1}^{\infty} |x_i|^p < +\infty$$

- We say that $x \in \ell^\infty$ if

$$\|x\|_{\ell^\infty} := \sup\{ |x_i| : i \geq 1\} < +\infty$$

- we say that $x \in c$ if $\lim_{i\to+\infty} x_i$ exists
- we say that $x \in c_0$ if $\lim_{i\to+\infty} x_i = 0$
- we say that $x \in c_{00}$ if $x_i = 0$ eventually (for i large enough)

We use the norm ℓ^∞ also in c, c_0, c_{00}.

Rmk Of course $\boxed{c_{00} \subseteq c_0 \subseteq c \subseteq \ell^\infty}$

Rmk The spaces ℓ^p with $p \in [1, +\infty]$, c and c_0 are Banach spaces. The space c_{00} is NOT complete, and its completion is c_0

[Proofs are an exercise: in the case of ℓ^p one can imitate the elementary proof in the case ℓ^2, in the case of c and c_0 it is the usual result that limits commute with uniform convergence]

DUALITY PAIRING

Given two sequences x and y we define

$$[J(y)](x) := \sum_{i=1}^{\infty} x_i y_i$$

The series converges under suitable assumptions

— o — o —

Topological dual of ℓ^1

Brutal statement : $(\ell^1)' = \ell^\infty$ (the top. dual of ℓ^1 is ℓ^∞)

Theorem The duality pairing induces a map

$$J : \ell^\infty \to (\ell^1)'$$

which is

- → well-defined, linear, continuous
- → an ISOMETRY (and hence injective)
- → surjective.

Proof For every $y \in \ell^\infty$ we consider

$$x \to [J(y)](x) = \sum_{i=1}^{\infty} x_i y_i \qquad \forall x \in \ell^1$$

Step 1 $\displaystyle\sum_{i=1}^{\infty} |x_i y_i| \leq \|y\|_{\ell^\infty} \sum_{i=1}^{\infty} |x_i| = \|y\|_{\ell^\infty} \|x\|_{\ell^1}$

This proves that $J(y) \in (\ell^1)'$ because it is clearly linear.
The same inequality prove that

$$\|J(y)\|_{(\ell^1)'} \leq \|y\|_{\ell^\infty}$$

because

$$\| J(y) \|_{(\ell^1)'} := \sup \{ | [J(y)](x) | \, : \, x \in \ell^1 , \, \|x\|_{\ell^1} \leq 1 \}$$

$\boxed{\text{Step 2}}$ We claim that $\| J(y) \|_{(\ell^1)'} = \| y \|_{\ell^\infty}$

By definition of norm in ℓ^∞, for every $\varepsilon > 0$ there exists $k \geq 1$ such that

$$|y_k| \geq \| y \|_{\ell^\infty} - \varepsilon$$

Define

$$x_i := \begin{cases} \text{sign}(y_k) & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\| y \|_{\ell^\infty} - \varepsilon \leq |y_k| = y_k \cdot \text{sign}(y_k) = [J(y)](x) \leq \| J(y) \|_{(\ell^1)'} \underbrace{\|x\|_{\ell^1}}_{\overset{\shortparallel}{1}}$$

Since ε is arbitrary, this proves that $\| y \|_{\ell^\infty} \leq \| J(y) \|_{(\ell^1)'}$

$\boxed{\text{Step 3}}$ We claim that J is surjective, namely for every $L \in (\ell^1)'$ there exists $y \in \ell^\infty$ such that

$$\boxed{L(x) = [J(y)](x) \qquad \forall \, x \in \ell^1} \qquad (*)$$

In order to define y we set $y_i := L(e_i)$

$$e_i = 0, \dots, 0, \underset{\substack{\uparrow \\ \text{pos } i}}{1}, 0, \dots 0, \dots$$

Let us check that $y \in \ell^\infty$:

$$|y_i| \leq \| L \|_{(\ell^1)'} \cdot \underset{\overset{\shortparallel}{1}}{\| e_i \|_{\ell_1}} \qquad \forall \, i \geq 1$$

Let us check (*). Define

$$\hat{L}(x) := [J(y)](x) \qquad \forall x \in \ell^1$$

Observe that L and $\hat{L}$ are elements of $(\ell^1)'$, and coincide if $x = e_i$ $\rightsquigarrow$ they coincide on $\mathrm{Clos}(\mathrm{Span}(e_i))$, but this span is dense in ℓ^1

— o — o —

Theorem (The dual of ℓ^p is $\ell^{p'}$ where $\frac{1}{p}+\frac{1}{p'}=1$ if $p \in (1,+\infty)$)

The duality pairing defines a map $J : \ell^{p'} \longrightarrow (\ell^p)'$ with the usual three properties

Proof **Step 1** J is well-defined and continuous because

$$\Big|\underset{\ell^{p'}}{\underset{\uparrow}{[J(y)]}}\,(\underset{\ell^p}{\underset{\uparrow}{x}})\Big| \leq \sum_{i=1}^{\infty} \underset{\frac{1}{p}}{|x_i|} \cdot \underset{\frac{1}{p'}}{|y_i|} \leq \|y\|_{\ell^{p'}} \cdot \|x\|_{\ell^p}$$

If we take the sup of LHS among all $x \in \ell^p$ with $\|x\|_{\ell^p} \leq 1$ we obtain

$$\|J(y)\|_{(\ell^p)'} \leq \|y\|_{\ell^{p'}}$$

Step 2 We obtain the opposite inequality. Given $y \in \ell^{p'}$ define x by

$$\boxed{x_i := |y_i|^{p'-1} \operatorname{sign}(y_i)}$$

First claim : $x \in \ell^p$

$$\sum_{i=1}^{\infty} |x_i|^p = \sum_{i=1}^{\infty} |y_i|^{p'p-p} = \sum_{i=1}^{\infty} |y_i|^{p'} = \|y\|_{\ell^{p'}}^{p'}$$

$$p'p - p \overset{?}{=} p' \iff p'p = p + p' \iff 1 = \frac{1}{p'} + \frac{1}{p}$$

Second claim : $\| J(y) \|_{(\ell^p)'} = \| y \|_{\ell^{p'}}$

$$\| y \|_{\ell^{p'}}^{p'} = \sum_{i=1}^{\infty} |y_i|^{p'} = \sum_{i=1}^{\infty} x_i \, y_i = [J(y)](x)$$

$$\leq \| J(y) \|_{(\ell^p)'} \, \| x \|_{\ell^p}$$

$$= \| J(y) \|_{(\ell^p)'} \, \| y \|_{\ell^{p'}}^{p'/p}$$

If we simplify $\| y \|_{\ell^{p'}}^{p'/p}$ we obtain the required inequaty

$\boxed{\text{Step 3}}$ $J : \ell^{p'} \to (\ell^p)'$ is surjective

Take any $L \in (\ell^p)'$ and define $y_i := L(e_i)$ (vec. cond.)

We need to prove that $y \in \ell^{p'}$ and $J(y) = L$.

For the first fact we argue as in step 2

$$\sum_{i=1}^{N} |y_i|^{p'} = \sum_{i=1}^{N} \underbrace{|y_i|^{p'-1} \operatorname{sign}(y_i)}_{z_i} \underbrace{y_i}_{L(e_i)} = \sum_{i=1}^{N} z_i \, L(e_i)$$

$$= L\left(\sum_{i=1}^{N} z_i e_i \right) \leq \| L \|_{(\ell^p)'} \cdot \left\{ \sum_{i=1}^{N} |z_i|^p \right\}^{1/p}$$

$$= \| L \|_{(\ell^p)'} \cdot \left\{ \sum_{i=1}^{N} |y_i|^{p'} \right\}^{1/p}$$

If we simplify and we let $N \to +\infty$ we obtain that

$$\| y \|_{\ell^{p'}} \leq \| L \|_{(\ell^p)'} .$$

The conclusion follows as in the previous case : L and $J(y)$ are two elements of $(\ell^p)'$ and they coincide on e_i for every $i \geq 1$. Since $\boxed{p < +\infty}$, the span of all e_i's is dense in ℓ^p.

— o — o —

Istituzioni di Analisi — LECTURE 56

Note Title 01/12/2021

TOUR IN THE DUAL OF ℓ^∞

Let us consider the duality pairing $[J(y)](x) := \sum_{i=1}^{\infty} x_i y_i$

$\boxed{\text{Step 1}}$ It defines a map $J : \ell^1 \to (\ell^\infty)'$ that is well-defined, linear and continuous

$$|[J(y)](x)| \leq \sum_{i=1}^{\infty} |x_i| \cdot |y_i| \leq \|y\|_{\ell^1} \|x\|_{\ell^\infty}$$

which proves it is well-defined and $\|J(y)\|_{(\ell^\infty)'} \leq \|y\|_{\ell^1}$

$\boxed{\text{Step 2}}$ J is an ISOMETRY, and therefore injective

Given $y \in \ell^1$, define $x \in \ell^\infty$ with $x_i := \text{sign}(y_i)$. Then

$$\|y\|_{\ell^1} = \sum_{i=1}^{\infty} |y_i| = \sum_{i=1}^{\infty} x_i y_i = [J(y)](x) \leq \|J(y)\|_{(\ell^\infty)'} \cdot \underset{1}{\underset{\shortparallel}{\|x\|_{\ell^\infty}}}$$

[one should discuss $y = 0$ separately]

$\boxed{\text{Step 3}}$ Let's try to prove surjectivity

Given $L \in (\ell^\infty)'$, let us define y by $y_i := L(e_i)$ (nec. cond.)

Claim: $y \in \ell^1$ and $J(y) = L$.

$$\sum_{i=1}^{N} |y_i| = \sum_{i=1}^{N} \underbrace{\text{sign}(y_i)}_{z_i} \underset{\underset{L(e_i)}{\uparrow}}{y_i} = \sum_{i=1}^{N} z_i L(e_i) = L\left(\sum_{i=1}^{N} z_i e_i\right)$$

$$\leq \|L\|_{(\ell^\infty)'} \cdot \underset{\leq 1}{\left\| \sum_{i=1}^{N} z_i e_i \right\|_{\ell^\infty}}$$

So we have proved that $y \in \ell^1$.
As in the previous cases we know that L and $J(y)$ coincide in $\mathrm{Clos}(\mathrm{Span}(e_i))$, which is NOT ℓ^∞, BUT c_0.

— o — o —

Actually we have $\boxed{\text{almost}}$ proved the following

MISLEADING THEOREM The duality pairing defines a map

$$J : \ell^1 \to (c_0)'$$

with the usual three properties

We need to modify the argument in Step 2, because we need $x \in c_0$ and not in ℓ^∞.
Given $y \in \ell^1$ and $N \geq 1$ define

$$(x_N)_i := \begin{cases} \mathrm{sign}(y_i) & \text{if } i \leq N \\ 0 & \text{if } i \geq N+1 \end{cases}$$

Then

$$\sum_{i=1}^{N} |y_i| = \sum_{i=1}^{N} y_i (x_N)_i = [J(y)](x_N) \leq \|J(y)\|_{(c_0)'} \underbrace{\|x_N\|_{c_0}}_{\leq 1}$$

Letting $N \to +\infty$ we obtain $\|J(y)\|_{(c_0)'} \geq \|y\|_{\ell^1}$

— o — o —

Prop. $c_{00}' = c_0' = \ell^1$

Proof Consider any element of c_{00}'. It is a linear cont. funct.

$$L : c_{00} \longrightarrow \mathbb{R}$$

Cont. $\Rightarrow$ Lip $\Rightarrow$ unif. cont. $\Rightarrow$ it can be extended in a unique way to $\hat{L} : c_0 \to \mathbb{R}$ which remains Lip. and linear.

— o — o —

Very brutal statement : $(c)' = \ell^1$

Less brutal statement : $(c)' = \underset{\substack{\uparrow}}{\ell^1} \oplus_1 \mathbb{R} \sim \ell^1$

set of pairs (y,z) with $y \in \ell^1$ and $z \in \mathbb{R}$
with $\|(y,z)\| = \|y\|_{\ell^1} + |z|$
$(y,z) \rightsquigarrow (z, y_1, y_2, y_3, \dots)$

$\boxed{\text{Theorem}}$ The duality pairing

$$[J(\underset{\substack{\uparrow\\ \ell^1}}{y}, \underset{\substack{\uparrow\\ \mathbb{R}}}{y_\infty})](x) := \sum_{i=1}^{\infty} x_i y_i + y_\infty \cdot \underset{\substack{\uparrow\\ \lim\limits_{i\to+\infty} x_i}}{x_\infty}$$

defines a map $J : \ell^1 \oplus_1 \mathbb{R} \to c'$ with the usual three properties

$\boxed{\text{Proof}}$ $\boxed{\text{Step 1}}$ As usual we obtain $\|J(y, y_\infty)\|_{c'} \leq \|y\|_{\ell^1} + |y_\infty|$

$\boxed{\text{Step 2}}$ Given $y \in \ell^1 \oplus_1 \mathbb{R}$ define x_N as

$$(x_N)_i := \begin{cases} \operatorname{sign}(y_i) & \text{if } i \leq N \\ \operatorname{sign}(y_\infty) & \text{if } i \geq N+1 \end{cases}$$

With this choice as before we obtain the opposite inequality (please check the details)

$\boxed{\text{Step 3}}$ Given $L \in c'$ we need to find $(y, y_\infty) \in \ell^1 \oplus_1 \mathbb{R}$ such that $L = J(y, y_\infty)$

To this end we set

$$y_i := L(e_i)$$

$$y_\infty := L(\underset{\substack{\uparrow\\ (1,1,1,\dots)}}{e_\infty}) - \sum_{i=1}^{\infty} L(e_i)$$

As in the ℓ^∞ case we can prove that $y \in \ell^1$, and therefore also y_∞ is well-defined.

Finally we have to check that L and $J(y, y_\infty)$ coincide in c.

→ For sure they coincide in e_i

→ They coincide if $x = e_\infty$

We observe that e_i's and e_∞ span a dense subset of c.

— o — o —

$\boxed{\text{WHAT ABOUT THE DUAL OF } \ell^\infty}$ It contains $J(y)$ for every $y \in \ell^1$, and more precisely we proved that $J : \ell^1 \to (\ell^\infty)'$ is an injective ISOMETRY.

$\boxed{\text{BUT IT IS NOT SURJECTIVE}}$ Let us find $L \in (\ell^\infty)'$ which is NOT in the image of J.

Consider $L : c \to \mathbb{R}$ defined as

$$L(x) := \lim_{i \to +\infty} x_i$$

This map is linear and continuous wrt the norm of ℓ^∞

Thanks to HB we can extend L to some $\hat{L} : \ell^\infty \to \mathbb{R}$.

$\boxed{\text{Fact 1}}$ $\hat{L}$ cannot be represented as $J(y)$ for some $y \in \ell^1$.

Assume that $y_i \neq 0$ for some $i \geq 1$. If x and $\hat{x}$ coincide everywhere but not in position i, then $[J(y)](x) \neq [J(y)](\hat{x})$. On the other hand, both x and $\hat{x}$ can have the same limit at ∞, and therefore the same $\hat{L}$.

Alternative proof: consider $\hat{L}(e_i)$.

$\boxed{\text{Fact 2}}$ $\hat{L}$ is NOT the liminf or limsup (they are NOT linear)

$\boxed{\text{Fact 3}}$ For every $x \in \ell^\infty$ it turns out that

$$\liminf_{i \to +\infty} x_i \leq \hat{L}(x) \leq \limsup_{i \to +\infty} x_i$$

For example consider $x_0 = 7, 9, 7, 9, 7, 9, \ldots$

From the proof of HB we know that

$$L(z) - \|z - x_0\| \leq \hat{L}(x_0) \leq \|y + x_0\|_{\ell^\infty} - L(y) \qquad \forall y \in C \quad \forall z \in C$$

Consider $y = 0, 0, 0, \ldots \rightsquigarrow \hat{L}(x_0) \leq 9$

Consider $z = 9, 9, 9, 9 \ldots \rightsquigarrow \hat{L}(x_0) \geq 9 - 2 = 7$

$\boxed{\text{Fact 4}}$ $\forall \alpha \in [7, 9]$ there exists an extension $\hat{L}$ with

$$\hat{L}(x_0) = \alpha \qquad !!$$

— o — o —

Istituzioni di Analisi — LECTURE 57

Note Title 02/12/2021

Dual of L^p spaces

Setting : $(\mathbb{X}, \mathcal{M}, \mu)$ is a measure space, namely

- $\mathbb{X}$ is a set
- $\mathcal{M}$ is a σ-algebra of subsets
- $\mu : \mathcal{M} \to [0, +\infty]$ is a measure (countably additive on disjoint subsets)

$L^p(\mathbb{X}, \mathcal{M}, \mu)$ usual L^p space with $p \in [1, +\infty]$.

Model case : when the measure is finite $\mu(\mathbb{X}) < +\infty$

RADON - NIKODYM THEOREM Let μ be a $\boxed{\text{finite}}$ measure on $\mathbb{X}$.
Let $\nu : \mathcal{M} \to \underset{\uparrow\uparrow}{\mathbb{R}}$ be a $\boxed{\text{signed}}$ measure (count. additive)

Let us assume that ν is absolutely continuous wrt μ, namely

$$\forall A \in \mathcal{M} \qquad \mu(A) = 0 \quad \Rightarrow \quad \nu(A) = 0.$$

Then there exists $g \in L^1(\mathbb{X}, \mathcal{M}, \mu)$ such that

$$\nu(A) = \int_A g(x)\, d\mu \qquad \forall A \in \mathcal{M}.$$

DUALITY PAIRING BETWEEN FUNCTIONS

$$\boxed{[J(g)](f) := \int_{\mathbb{X}} g(x) f(x)\, d\mu}$$

well-defined under suitable assumptions on f and g.

Quick statement: the dual of L^p is $L^{p'}$ where $\frac{1}{p}+\frac{1}{p'}=1$

if $p \in [1, \underset{\uparrow\uparrow \text{ EXCLUDED}}{+\infty})$

In the sequel we assume that $\mu(X) < +\infty$.

$\boxed{\text{Fact 1}}$ For every $p \in [1,+\infty]$ the duality pairing defines a map

$$J : L^{p'} \longrightarrow (L^p)' \qquad\qquad g \longrightarrow J(g)$$

that is linear, well-defined, and continuous

$\boxed{\text{Proof}}$ Linearity is clear.

$$\Big|\big[\underset{\overset{\uparrow}{L^{p'}}}{J(g)}\big]\big(\underset{\overset{\uparrow}{L^p}}{f}\big)\Big| = \Big|\int_X f(x)g(x)\,d\mu\Big| \underset{\overset{\uparrow}{\text{Hölder ineq.}}}{\le} \|g\|_{L^{p'}} \cdot \|f\|_{L^p}$$

This proves that $J(g)$ is well-defined as an element of $(L^p)'$ and the norm satisfies

$$\underset{\overset{\uparrow}{\text{sup of LHS among all } f\text{'s in } L^p \text{ with } \|f\|_{L^p} \le 1}}{\|J(g)\|_{(L^p)'}} \le \|g\|_{L^{p'}}$$

Rmk We did not exploit that $\mu(X) < +\infty$ and the argument works even for $p = 1$ and $p = +\infty$.

$\boxed{\text{Fact 2}}$ For every $p \in [1,+\infty]$, the duality pairing is an isometry namely

$$\boxed{\|J(g)\|_{(L^p)'} = \|g\|_{L^{p'}}}$$

$\boxed{\text{Proof}}$ We distinguish 3 cases

$\boxed{p=+\infty}$ Given $g \in L^1$ we claim that $\|J(g)\|_{(L^\infty)'} \geq \|g\|_{L^1}$

We define

$$f(x) := \text{sign}(g(x)) \in L^\infty(X)$$

and observe that

$$\|g\|_{L^1} = \int_X |g(x)|dx = \int_X g(x)\,\text{sign}(g(x))$$

$$= \int_X g(x) f(x)\,dx$$

$$= [J(g)](f)$$

$$\leq \|J(g)\|_{(L^\infty)'} \underbrace{\|f\|_{L^\infty}}_{\leq 1}$$

$$\leq \|J(g)\|_{(L^\infty)'}$$

$\boxed{p=1}$ Given $g \in L^\infty(X)$ we claim that $\|J(g)\|_{(L^1)'} \geq \|g\|_{L^\infty}$

For every $\varepsilon > 0$ there exists a measurable set $A_\varepsilon \subseteq X$ s.t.

$$|g(x)| \geq \|g\|_{L^\infty} - \varepsilon \qquad \forall x \in A_\varepsilon \qquad \text{and} \qquad \text{meas}(A_\varepsilon) > 0$$

WLOG we can assume $g(x) \geq \|g\|_{L^\infty} - \varepsilon$ (the other case is analogous)

Now we observe that

$$(\|g\|_{L^\infty} - \varepsilon)\,\text{meas}(A_\varepsilon) \leq \int_{A_\varepsilon} g(x)\,dx = \int_X g(x)\,\mathbb{1}_{A_\varepsilon}(x)\,d\mu$$

$$= [J(g)](\mathbb{1}_{A_\varepsilon}) \leq \|J(g)\|_{(L^1)'} \cdot \|\mathbb{1}_{A_\varepsilon}\|_{L^1} = \|J(g)\|_{(L^1)'}\,\text{meas}(A_\varepsilon)$$

Now it is enough to simply meas (A_ε) and then let $\varepsilon \to 0^+$.

$\boxed{p \in (1,+\infty)}$ Given $g \in L^{p'}$ we claim that $\|J(g)\|_{(L^p)'} \geq \|g\|_{L^{p'}}$

We define $f(x) := |g(x)|^{p'-1} \operatorname{sign}(g(x))$

We need that $f \in L^p$ but, as in the case of sequences,

$$\int_X |f(x)|^p d\mu = \int_X |g(x)|^{p(p'-1)} d\mu \underset{\substack{\uparrow \\ p(p'-1)=p'}}{=} \int_X |g(x)|^{p'} d\mu$$

Now we observe that

$$\int_X |g(x)|^{p'} d\mu = \int_X g(x) f(x) d\mu = [J(g)](f)$$

$$\leq \|J(g)\|_{(L^p)'} \cdot \|f\|_{L^p}$$

$$\leq \|J(g)\|_{(L^p)'} \left\{ \int_X |g(x)|^{p'} d\mu \right\}^{\frac{1}{p}}$$

If we simplify we obtain the conclusion

<u>Rmk</u> Again we did NOT use that $\mu(X) < +\infty$ and the argument works also for $p=1$ and $p=+\infty$.

$\boxed{\text{Fact 3}}$ If $p \in [1, +\infty)$, then the duality pairing is surjective.
$\uparrow\uparrow$

Let $L : L^p \to \mathbb{R}$ be a continuous linear functional.
We need to find $g \in L^{p'}$ s.t.

$$\boxed{L(f) = [J(g)](f) \qquad \forall f \in L^p}$$

$\boxed{3.1}$ Let us set

$$\nu(A) := L(\underset{\nearrow}{\mathbb{1}_A}) \qquad \forall A \in \mathcal{M}$$

$\in L^p$ because meas$(A) < +\infty$

Claim: ν is a signed measure in $\mathbb{X}$ and ν is abs. cont. wrt μ

$\mu(A) = 0 \Rightarrow \nu(A) = 0 \rightsquigarrow$ trivial

$\nu(\emptyset) = 0 \rightsquigarrow$ trivial

The main point is countable additivity. Let $\{A_k\}_{k\geq 1}$ be a sequence of pairwise disjoint elements of $\mathcal{M}$.

Let

$$A_\infty := \bigcup_{i\geq 1} A_i .$$

Then we need that $\nu(A_\infty) = \sum_{i=1}^{\infty} \nu(A_i)$. But

$$\nu(A_\infty) = L(\mathbb{1}_{A_\infty})$$

$$= \lim_{m\to+\infty} L\left(\mathbb{1}_{\bigcup_{i=1}^{m} A_i}\right)$$

Here we need that $\mathbb{1}_{\bigcup_{i=1}^{m} A_i} \longrightarrow \mathbb{1}_{A_\infty}$ in $L^p(\mathbb{X})$

and this is true only if $p < +\infty$

$$\overset{A_i\text{'s disjoint}}{=} \lim_{m\to+\infty} L\left(\sum_{i=1}^{m} \mathbb{1}_{A_i}\right)$$

$$= \lim_{m\to+\infty} \sum_{i=1}^{m} L(\mathbb{1}_{A_i})$$

$$= \lim_{m\to+\infty} \sum_{i=1}^{m} \nu(A_i)$$

$$= \sum_{i=1}^{\infty} \nu(A_i)$$

$\boxed{3.2}$ By RN there exists $g \in L^1$ such that

$$\nu(A) = L(\mathbb{1}_A) = \int_A g(x)\,d\mu = \int_X g(x)\,\mathbb{1}_A(x)\,d\mu$$

The claim is that $g \in L^{p'}$.

At this point L and $J(g)$ are two elements of $(L^p)'$ that coincide on $\mathbb{1}_A$, therefore also on the closure of the span which is L^p if $p < +\infty$.

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Note Title 02/12/2021

Given $L \in (L^p)'$ we found $g \in L^1$ such that

$$L(\mathbb{1}_A) = \int_A g(x)\,d\mu = [J(g)](\mathbb{1}_A)$$

Claim: $g \in L^{p'}$ (which is better than L^1 because $\mu(X) < +\infty$)

$\boxed{p=1}$ We claim $\|g\|_{L^\infty} \leq \|L\|_{(L^1)'}$

Assume this is not the case. Then there exists A_ε with meas $(A_\varepsilon) > 0$ s.t.

$$|g(x)| \geq \|L\|_{(L^1)'} + \varepsilon \qquad \forall x \in A_\varepsilon$$

WLOG we can assume the same with $g(x)$ in the LHS instead of $|g(x)|$. Now we observe that

$$\begin{aligned}(\|L\|_{(L^1)'} + \varepsilon)\,\text{meas}(A_\varepsilon) &\leq \int_{A_\varepsilon} g(x)\,d\mu \\ &= [J(g)](\mathbb{1}_{A_\varepsilon}) \\ &= L(\mathbb{1}_{A_\varepsilon}) \\ &\leq \|L\|_{(L^1)'} \cdot \|\mathbb{1}_{A_\varepsilon}\|_{L^1} \\ &= \|L\|_{(L^1)'} \cdot \text{meas}(A_\varepsilon)\end{aligned}$$

If we simplify, we obtain a contradiction.

$\boxed{p \in (1,+\infty)}$ We know that L and $J(g)$ coincide on $\mathbb{1}_A$ and $g \in L^1$.

This implies that they coincide in $L^\infty(X)$

(here we use that the span of $\mathbb{1}_A$ is dense in $L^\infty(X)$)

Given $g(x)$ let us consider a truncation

$$g_n(x) := \begin{cases} n & \text{if } g(x) \geq n \\ g(x) & \text{if } -n \leq g(x) \leq n \\ -n & \text{if } g(x) \leq -n \end{cases}$$

and let us set $\quad f_n(x) := |g_n(x)|^{p'-1} \cdot \underbrace{\operatorname{sign}(g_n(x))}_{\operatorname{sign}(g(x))} \in L^\infty$

$$\int_X |g_n(x)|^{p'} d\mu = \int_X |g_n(x)|^{p'-1} \operatorname{sign}(g_n(x)) \cdot g_n(x) \, d\mu$$

$$= \int_X |g_n(x)|^{p'-1} \underbrace{\operatorname{sign}(g(x)) \; g_n(x)}_{\operatorname{sign}(g(x)) \cdot g(x)} d\mu$$

$$\leq \int_X f_n(x) \, g(x) \, d\mu$$

$$= [J(g)](f_n) \qquad \text{since } J(g) = L \text{ on } L^\infty$$

$$= L(f_n)$$

$$\leq \|L\|_{(L^p)'} \, \|f_n\|_{L^p}$$

$$= \|L\|_{(L^p)'} \left\{ \int_X |g_n(x)|^{p'} d\mu \right\}^{1/p}$$

If we simplify we obtain that

$$\|g_n\|_{L^{p'}} \leq \|L\|_{(L^p)'}$$

We let $n \to +\infty$ and obtain that $\|g\|_{L^{p'}} \leq \|L\|_{(L^p)'}$.

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Passing from $\mu(X) < +\infty$ and σ-finite measure, namely X is the increasing union of subsets X_k with $\mu(X_k) < +\infty$.

$\boxed{\text{Step 1}}$ Consider $L : L^p(X) \to \mathbb{R}$ linear and continuous

By restriction it defines

$$L_k : L^p(X_k) \to \mathbb{R}$$

By the previous result

$$L_k(f) = \int_{X_k} g_k(x) f(x)\, d\mu \qquad \forall f \in L^p(X_k)$$

$\boxed{\text{Step 2}}$ If $X_k \subseteq X_\ell$, then $g_k(x) = g_\ell(x)$ for every $x \in X_k$

(it is some kind of FLCV)

and therefore we can define

$$g(x) := g_k(x) \qquad \text{if} \quad x \in X_k$$

$\boxed{\text{Step 3}}$ We observe that

$$\|g\|_{L^{p'}(X_k)} = \|g_k\|_{L^{p'}(X_k)} = \|L_k\|_{(L^p(X_k))'}$$
$$\leq \|L\|_{(L^p(X))'}$$

and this implies that $g \in L^{p'}(X)$. We conclude that $J(g)$ and L coincide on $L^p(X)$.

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HILBERT SPACES

Short version : $(H)' = H$

Duality pairing $\quad [J(v)](w) := \langle v, w \rangle \qquad \forall (v,w) \in H^2$

Theorem The duality pairing defines a map

$$H \ni v \longrightarrow J(v) \in H'$$

which has the usual properties

- → well-defined, linear, continuous
- → isometry
- → surjective.

Proof of first two parts

$$|[J(v)](w)| = |\langle v, w \rangle| \le \|v\|_H \cdot \|w\|_H$$

This proves that $\quad \|J(v)\|_{H'} \le \|v\|_H$

In order to prove the opposite inequality we choose $w := v$ and observe that

$$\|v\|_H^2 = \langle v, w \rangle = [J(v)](w) \le \|J(v)\|_{H'} \cdot \|w\|_H = \|J(v)\|_{H'} \cdot \|v\|_H$$

If we simplify we conclude (consider also $v = 0$)

Proof #1 of surjectivity Assume that H is separable and let $\{e_n\}$ be a Hilbert basis (if H is not separable the same argument works with a general Hilbert basis $\{e_i\}_{i \in I}$)

Let $L : H \to \mathbb{R}$ be linear and continuous.

We need to find $v \in H$ such that

$$L(w) = \langle v, w \rangle \qquad \forall w \in H$$

Let us set

$$v := \sum_{k=1}^{\infty} \underbrace{L(e_k)}_{v_k} e_k$$

Assume that v is well-defined. Then $L(w)$ and $\langle v, w \rangle$ coincide in e_k for every $k \geq 1$, and therefore in H.

For every $n \geq 1$ observe that

$$\sum_{k=1}^{n} v_k^2 = \sum_{k=1}^{n} v_k L(e_k) = L\left(\sum_{k=1}^{n} v_k e_k\right)$$

$$\leq \|L\|_{H'} \cdot \left\| \sum_{k=1}^{n} v_k e_k \right\|$$

$$= \|L\|_{H'} \cdot \left\{ \sum_{k=1}^{n} v_k^2 \right\}^{1/2}$$

We simplify and we let $n \to +\infty$.

$\boxed{\text{Proof \#2 of surjectivity}}$ Let $K := \operatorname{Ker}(L)$. If $K = H$, then $v = 0$ represents L.

Assume that $K \neq H$. Let $v_0 \neq 0$ be any vector in $K^\perp$ (here we use that $H = K \oplus K^\perp$)

Every $w \in H$ can be written as $w = \alpha v_0 + w_1$ with $\alpha \in \mathbb{R}$ and $w_1 \in \operatorname{Ker}(L)$. What is α?

$$\langle w, v_0 \rangle = \alpha \|v_0\|^2 + \underset{\overset{\parallel}{0}}{\langle w_1, v_0 \rangle} \quad \rightsquigarrow \quad \alpha = \frac{\langle v_0, w \rangle}{\|v_0\|^2}$$

Observe that

$$L(w) = L(\alpha v_0 + w_1) = \alpha L(v_0) + \underbrace{L(w_1)}_{=0}$$

$$= \alpha L(v_0)$$

$$= \frac{\langle v_0, w \rangle}{\| v_0 \|^2} L(v_0)$$

$$= \langle \boxed{\frac{L(v_0)}{\| v_0 \|^2} v_0}, w \rangle$$

$\uparrow$ vector v that represents L.

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[Added after video: when we write $w = \alpha v_0 + w_1$ we need to know that $K^\perp = \mathrm{Span}(v_0)$, namely that $K^\perp$ has dimension 1. This is true because, if v_1 and v_2 are vectors in $K^\perp$, then $L(v_2)\, v_1 - L(v_1)\, v_2 \in K^\perp \cap K$, and hence it is 0, and therefore v_1 and v_2 are linearly dependent.

Alternative proof: it turns out that $w = \alpha v_0 + w_1$ if we choose $\alpha := \frac{L(w)}{L(v_0)}$

(just check that the difference $w - \alpha v_0 \in \mathrm{Ker}$]

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Note Title 06/12/2021

REFLEXIVE SPACES

Defn Let V be a normed space. The CANONICAL INJECTION is the map

$$J : V \longrightarrow (V')'$$

defined by

Duality pairing between V and V'

$$\boxed{[J(v)](f) := f(v) \qquad \forall v \in V \quad \forall f \in V'}$$

Prop. J is a linear ISOMETRY (and in particular injective).

Proof. Linear is trivial. We claim that

$$\|J(v)\|_{(V')'} = \|v\|_V \qquad \forall v \in V$$

$\boxed{\leq}$ $\quad |[J(v)](f)| = |f(v)| \leq \|v\|_V \cdot \|f\|_{V'}$

Consider the sup of LHS as f varies in $B_{V'}(0,1)$.

$\boxed{\geq}$ For every $v \in V$ there exists $f \in V'$ with $\|f\|_{V'} \leq 1$ such that

$$\|v\|_V = |[J(v)](f)| \leq \|J(v)\|_{(V')'} \cdot \underset{1}{\underset{\shortparallel}{\|f\|_{V'}}}$$

This f is the aligned functional.

— ○ — ○ —

Defn The spaced V is called $\boxed{\text{REFLEXIVE}}$ if J is surjective

Rmk If V is reflexive, then V is isometric and isomorphic to $(V')'$, and in particular is a Banach space.

Achtung! The are weird examples of spaces where V is isometric and isomorphic to $(V')'$, but the isomorphis is NOT the canonical injection. These spaces are NOT reflexive.

Theorem (weak compactness of balls in reflexive spaces)
Let V be a normed spaces, and let $\{v_n\}$ be a sequence in V. Let us assume that

(i) $\{v_n\}$ is bounded

(ii) V is reflexive

(iii) V' is separable (the price to pay in order to use sequences)

Then there exists

$$v_{n_k} \longrightarrow v_\infty \quad \text{weakly in } V.$$

$\boxed{\text{Proof}}$ Usual argument

$\boxed{\text{Step 1}}$ Let us satisfy a countable dense subset $\{f_n\}$ of V'. More precisely, there exists v_{n_k} such that

$$\lim_{k\to+\infty} f_i(v_{n_k}) = \text{something} \quad \text{for every } i \in \mathbb{N}$$

[Usual diagonal argument after observing that

$$|f_i(v_n)| \le \underset{\substack{\uparrow \\ \text{fixed}}}{\|f_i\|_{V'}} \cdot \underset{\substack{\uparrow \\ \text{bounded}}}{\|v_n\|}$$

$\boxed{\text{Step 2}}$ We show that

$$\lim_{k\to+\infty} f(\underset{\uparrow}{v_{n_k}}) = \text{something} \qquad \forall f \in V'$$

the SAME subsequence that
satisfies $\{f_n\}_{n\geq 1}$

[We use that $\{f(v_{n_k})\}$ is a Cauchy sequence]

$\boxed{\text{Step 3}}$ Let $L(f)$ denote the limit that we found in Step 2.
It is possible to see that

- L is linear
- $L : V' \to \mathbb{R}$ is continuous

$$|L(f)| = \lim_{k\to+\infty} |f(v_{n_k})| \leq \|f\|_{V'} \cdot \underbrace{\|v_{n_k}\|_V}_{\text{bounded}}$$

This means that $L \in (V')'$ and therefore $L = J(v_\infty)$ for some $v_\infty \in V$.
This proves that

$$\lim_{k\to+\infty} f(v_{n_k}) = L(f) = [J(v_\infty)](f) = f(v_\infty) \qquad ☺$$

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<u>Corollary</u> Assume that V is a reflexive space with V' separable.
Then for every $f \in V'$ it turns out that

$$\|f\|_{V'} = \underset{\substack{\uparrow \\ \text{and not only sup}}}{\max} \{ |f(v)| : v \in V \text{ and } \|v\|_V \leq 1 \}$$

<u>Proof</u> Let $\{v_n\}$ be a sup-izing sequence. It admits a weakly conv. subsequence $v_{n_k} \rightharpoonup v_\infty$, and finally

$$f(v_\infty) \underset{\substack{\uparrow \\ \text{weak conv.}}}{=} \lim_{k\to+\infty} f(v_{n_k}) \underset{\substack{\uparrow \\ \text{definition of } v_n}}{=} \|f\|_{V'}$$

Added after video: proof without assuming V' to be separable. Consider the aligned functional of f in $(V')'$. It is $J(\text{something})$

Rmk V separable $\not\Rightarrow$ V' separable

[For example L^1 and ℓ^1 are separable, but L^∞ and ℓ^∞ are not separable]

Prop. (The converse is true : V' separable $\Rightarrow$ V is separable)
We assume that V is a normed space.

$\boxed{\text{Proof}}$ $\boxed{\text{Step 1}}$ V' separable $\Rightarrow$ unit sphere of V' is separable
(any subset of a separable metric space is separable, because a metric space is sep. $\Leftrightarrow$ it admits a countable basic)

Let $\{f_n\}_{\geq 1}$ be a countable dense subset in this sphere.
Since $\|f_n\|_{V'} = 1$, there exists $v_n \in V$ such that

$$\|v_n\|_V \leq 1 \qquad \text{and} \qquad f(v_n) \geq \underset{\uparrow}{\frac{1}{2}}$$

any constant < 1 is good

Let $W := \text{Clos}(\text{Span}(v_1, \dots, v_n, \dots))$. We claim that $W = V$.
[If YES, then the FINITE lin. comb. of $v_1, \dots, v_n, \dots$ with coeff. in $\mathbb{Q}$ are a countable dense subset of V].

$\boxed{\text{Step 2}}$ We prove the claim. Assume by contrad. it is false.
Then there exists $z \in V \setminus W$ and $\|z\|_V = 1$.
Now there exists $g \in V'$ such that

$$g(z) = 1$$
$$g(w) = 0 \qquad \forall w \in W$$

[Existence follow from Hahn-Banach]

Added after video : this is not that simple.
Please check the last page of the lecture.

Note that the extension satisfies $\|g\|_{V'} = 1$

But now $g \in$ unit sphere in V' and therefore there exists $n \geq 1$ such

$$\|g - f_n\|_{V'} \leq \frac{1}{4}$$

On the other hand

$$\frac{1}{2} \leq f_n(v_n) = f_n(v_n) - \underset{\overset{\shortparallel}{0} \text{ because } v_n \in W}{g(v_n)} + g(v_n)$$

$$\leq \underbrace{\|f_n - g\|_{V'}}_{\leq \frac{1}{4}} \cdot \underbrace{\|v_n\|_V}_{\leq 1}$$

$$\leq \frac{1}{4}.$$

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THE PARADOX OF HILBERT TRIPLES

Consider $H := L^2((0,1))$ and $V := H^1((0,1))$.
They are Hilbert spaces and $V \subseteq H$.
Consider $f \in H'$. So $f : H \to \mathbb{R}$ is linear and cont. in H.
The restriction of f to V is linear and cont. wrt to V because

$$|f(v)| \leq \text{const} \cdot \|v\|_H \leq \text{const} \cdot \|v\|_V$$

This means that there exist a map

$$H' \longrightarrow V'$$

which is injective (if $f(v) = 0$ for every $v \in V$, then $f(v) = 0$ for every $v \in H$ because V is dense).
In other words

$$V \subseteq H \underset{\substack{\uparrow}}{=} H' \subseteq V' \underset{\substack{\uparrow}}{=} V$$

dual of Hilbert spaces

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[This part was added after video]

$\boxed{\text{General fact}}$ V normed space, $W \subseteq V$ CLOSED subspace with $W \neq V$. Then there exists $g \in V'$ such that $\|g\|_{V'} = 1$ and $g(w) = 0$ for every $w \in W$.

Proof Let us assume that $B'(v_0, r_0) \subseteq V \setminus W$. It is enough to find $g \in V'$ with $g(v_0) = 1$ and $g(w) = 0$ for every $w \in W$. This function g does not satisfy necessarily $\|g\|_{V'} = 1$, but a suitable multiple does.

Consider $E := W \oplus \mathbb{R} v_0$, and define $g : E \to \mathbb{R}$ by

$$g(w + \alpha v_0) = \alpha \qquad \forall \alpha \in \mathbb{R} \quad \forall w \in W.$$

Claim: this function satisfies

$$g(w + \alpha v_0) \leq \frac{1}{r_0} \|w + \alpha v_0\| \qquad \forall \alpha \in \mathbb{R} \quad \forall w \in W.$$

and therefore HB provides an extension with the same property.

Indeed (wlog $\alpha > 0$):

$$g(w + \alpha v_0) = \alpha = \alpha \frac{r_0}{r_0} \leq \frac{1}{r_0} \alpha \operatorname{dist}\Big(v_0, \overbrace{-\frac{w}{\alpha}}^{\text{this is in } W\text{, and hence not in } B(v_0, r_0)}\Big)$$

$$= \frac{1}{r_0} \alpha \left\| v_0 + \frac{w}{\alpha} \right\|$$

$$= \frac{1}{r_0} \| \alpha v_0 + w \|$$

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Istituzioni di Analisi — LECTURE 60

Note Title 06/12/2021

Weak convergence in L^p spaces

Balls in L^p with $p \in (1, +\infty)$ (endpoints excluded) are weakly compact, at least for classical measure spaces.

Proof 1 L^p is reflexive for p in this range

Proof 2 L^p is the dual of $L^{p'}$ where $\frac{1}{p'} + \frac{1}{p} = 1$. The ball in L^p coincides with the ball of $(L^{p'})'$ which is weakly * compact.

Finally we observe that weak * convergence in $(L^{p'})'$ coincides with weak convergence in L^p

$$\{f_n\} \subseteq L^p \qquad \int f_n g \to \int f_\infty g \qquad \forall g \in L^{p'} \qquad (*)$$

Proof 3 Let us work by hands. Take a sequence $\{f_n\} \subseteq L^p$ bounded. Want to prove $(*)$.

Step 1 Up to subsequences $(*)$ is true for every $g \in \underset{\substack{\uparrow \\ \text{countable dense subset}}}{D} \subseteq L^{p'}$

Step 2 Up to the same subsequence

$$\int f_n g \to L(g) \qquad \forall g \in L^{p'}$$

Step 3 We know that $L(g) \in (L^{p'})'$. Now we need to know that $(L^{p'})' = L^p$ and this requires the representation theorems.

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$\boxed{\text{LAX-MILGRAM}}$ Simple version.

For every $f \in L^2((0,1))$ there exists a unique $u \in H^1_0((0,1))$ such that

$$\ddot{u} = f$$

in the sense that

$$\int_0^1 \dot{u}\,\dot{\varphi}\,dx = -\int_0^1 f\varphi \qquad \forall \varphi \in \underset{\substack{\uparrow \\ \text{and also in } H^1_0((0,1))}}{C^\infty_c}((0,1)) \qquad (*)$$

$\boxed{\text{Proof 1}}$ Variational formulation

$$\rightsquigarrow \min\left\{ \int_0^1 \left[\frac{1}{2}\dot{u}^2 + f(x)\,u\right]dx \;:\; u \in H^1_0((0,1))\right\}$$

Direct method $\Rightarrow$ existence ELE in the 1st int. form $\rightarrow$ u satisfies $(*)$.

$\boxed{\text{Proof 2}}$ Define $F(x) := \int_0^x f(t)\,dt$ and

$$\hat{F}(x) := \int_0^x F(t)\,dt + ax + b$$

then adjust the constants a and b so that $\hat{F} \in H^1_0((0,1))$.

$\boxed{\text{Proof 3}}$ À la LAX-MILGRAM.

Consider $H := H^1_0((0,1))$. Consider

$$\langle\langle u, v \rangle\rangle_H := \int_0^1 \dot{u}\,\dot{v}\,dx$$

This is a scalar product:

- → bilinear is trivial
- → positive definite ok due to boundary conditions
- → completeness is an exercise. If $\{u_n\}$ is a Cauchy seq. in H then $\{\dot{u}_n\}$ is a Cauchy seq. in $L^2((0,1))$ and

$\{u_n\}$ is bounded in $L^\infty((0,1))$. ← again we use DBC
At this point $\dot{u}_n \longrightarrow V_\infty$ strongly in L^2 and $u_n \longrightarrow u_\infty$ uniformly (at least up to subsequences).
At this point $V_\infty = \dot{u}_\infty$ and by the sub-sub lemma the whole sequence u_n converges.

Now consider $L : H \longrightarrow \mathbb{R}$ defined by

$$L(v) := -\int_0^1 f(x)\, v(x)\, dx$$

Clearly L is linear. Is L continuous in H. We need

$$|L(v)| \leq \|f\|_{L^2((0,1))} \cdot \|v\|_{L^2((0,1))}$$

$$\underset{\substack{\uparrow \\ \text{Poincaré}}}{\leq} \|f\|_{L^2((0,1))} \cdot \text{const} \cdot \underbrace{\|\dot{v}\|_{L^2((0,1))}}_{\|v\|_H}$$

Therefore $L \in H'$ and any element of H' can be represented as a scalar product, namely there exists $u \in H$ s.t.

$$L(v) = \langle\langle u, v \rangle\rangle_H$$

$$-\int_0^1 f v \overset{\shortparallel}{} = \int_0^1 \dot{u}\dot{v} \qquad \forall v \in H_0^1((0,1))$$

as required

— o — o —

<u>Example</u> Strange elements in $[L^\infty((-1,1))]'$

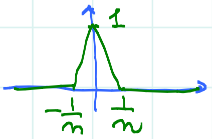
POINTWISE VALUES

$$\underset{\substack{\uparrow\\ \text{vector subspace of } L^\infty((-1,1))}}{C^0([-1,1])} \ni u(x) \longrightarrow \underset{\substack{\shortparallel\\ L(u)}}{u(0)} \in \mathbb{R}$$

The linear map defined above is cont. wrt the norm of L^∞.
HB $\Rightarrow$ we can extend L to a lin. cont. function on $L^\infty((-1,1))$!

It is possible to see that there is NO $g \in L^1((-1,1))$ s.t.

$$\int_{-1}^{1} g(x)\, u(x)\, dx = u(0) \qquad \forall u \in C^1([-1,1])$$

[Consider $u(x)$ as in the figure ]

<u>Rmk</u> If we consider the standard proof that works for $p<+\infty$ we define, given $L \in (L^\infty)'$, the measure

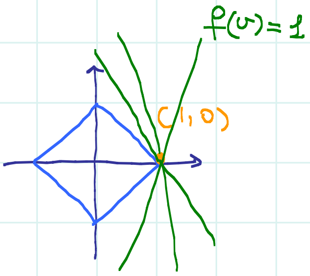
$$\nu(A) := L(\mathbb{1}_A) \qquad \forall A \text{ measurable}$$

It is possible to check that ν is FINITELY ADDITIVE but NOT countably additive. This is NOT enough for Radon - Nikodym.

<u>Example</u> Non-uniqueness of the aligned functional.

Consider $V = \mathbb{R}^2$ with the 1-norm

$$\|(x,y)\|_1 := |x| + |y|$$



Consider $v = (1,0)$. Which are the aligned functionals of v?

They are of the form $f(x,y) = ax + by$

What is $\|f\|_{V'}$? We have to compute

$$\underset{\overset{\uparrow}{\text{sup}}}{\max}\{\,|f(x,y)| : \|(x,y)\|_1 \le 1\} = \max\{|a|, |b|\}$$

$$\begin{aligned}|f(x,y)| &\le |a|\cdot|x| + |b|\cdot|y| \\ &\le \max\{|a|,|b|\}\cdot \underset{\|(x,y)\|_1}{\underset{\shortparallel}{(|x|+|y|)}}\end{aligned}$$

Given a and b, we can choose x and y in such a way that we have equalities.

So we need that

→ $\max\{|a|, |b|\} \le 1$ $\qquad \|f\|_{V'} \le 1$

→ $a = 1$ $\qquad f(1,0) = \|(1,0)\|_V = 1$

There are infinitely many possibilities! $\quad f(x,y) = x + by$ with $|b| \le 1$

— o — o —

Istituzioni di Analisi — LECTURE 61

Note Title 09/12/2021

BAIRE SPACES

Prop Let X be a topological space.
The following properties are equivalent

(i) If $\{A_i\}_{i\geq 1}$ is a sequence of OPEN DENSE subsets, then

$$\bigcap_{i\geq 1}^{\infty} A_i \quad \text{is DENSE}$$

(ii) If $\{C_i\}_{i\geq 1}$ is a sequence of CLOSED subsets with $\text{Int}(C_i) = \emptyset$ then

$$\bigcup_{i=1}^{\infty} C_i \quad \text{has EMPTY interior}$$

(iii) If $\{C_i\}_{i\geq 1}$ is a seq. of closed subsets such that

$$\text{Int}\left(\bigcup_{i=1}^{\infty} C_i\right) \neq \emptyset$$

then there exists $i_0 \geq 1$ s.t. $\text{Int}(C_{i_0}) \neq \emptyset$

Proof (i) $\Leftrightarrow$ (ii) consider the complement set
(ii) $\Leftrightarrow$ (iii) counterpositive ($A \Rightarrow B \iff \text{NOT}B \Rightarrow \text{NOT } A$)

— o — o —

Defn A Baire space is a top. space that satisfies any of the equivalent statements.

— o — o —

Further notations A subset $Y \subseteq X$ is called

- **NOWHERE DENSE** if $\text{Int}(\text{Clos}(Y)) = \emptyset$ (very very small from the topological point of view)
- **MEAGER** or **FIRST CATEGORY** if it is the countable union of nowhere dense subsets (small from the top. point of view)
- **RESIDUAL** if $X \setminus Y$ is meager.

Rmk In a Baire space a meager set has empty interior

Rmk In any space the countable union of meager sets is again meager

—o—o—

THREE MAIN CLASSES OF BAIRE SPACES

① COMPLETE METRIC SPACES
② LOCALLY COMPACT TOP. SPACES (maybe HAUSSDORF)
③ OPEN SUBSETS OF BAIRE SPACE

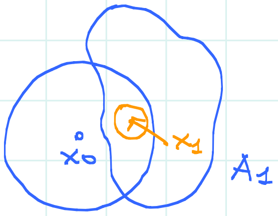
Theorem (Baire Category theorem) If X is a complete metric space, then X is a Baire space.

Proof We use the characterization (i). So let $\{A_i\}_{i \geq 1}$ be a sequence of dense open sets.
We prove that the intersection is dense, namely every ball $B(x,r)$ intersects the intersection.

Idea: we construct a sequence of balls $B(x_n, r_n)$ such that $B(x_0, r_0) = B(x, r)$ and then

$$\boxed{\overline{B}(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n) \cap A_{n+1}}$$

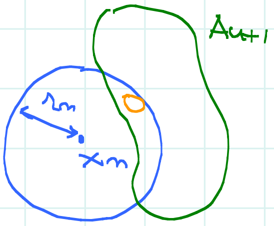
and $r_n \to 0$ as $n \to +\infty$.



The construction is inductive.

Consider

$$\bar{B}(x_1, r_1) \subseteq \underbrace{B(x_0, r_0) \cap A_1}_{\substack{\text{nonempty open set because} \\ A_1 \text{ is open and dense}}}$$



If $x_1, \dots, x_n$ have been defined, then consider

$$\bar{B}(x_{n+1}, r_{n+1}) \subseteq \underbrace{B(x_n, r_n) \cap A_{n+1}}_{\text{open set } \neq \emptyset}$$

wlog we can assume that $r_n \leq \frac{1}{n}$.

The sequence $\{x_n\}$ is a Cauchy sequence because if $n \geq n_0$ and $m \geq n_0$, then

$$\text{dist}(x_n, x_m) \leq \underbrace{\text{dist}(x_n, x_{n_0})}_{\leq r_{n_0}} + \underbrace{\text{dist}(x_m, x_{n_0})}_{\leq r_{n_0}}$$

Therefore $x_n \to x_\infty$. But $x_\infty \in \bar{B}(x_n, r_n)$ for every $n \geq 1$ and therefore

$$x_\infty \in \bigcap_{n\geq 1}^{\infty} A_n \cap \bar{B}(x_0, r_0)$$

This proves that the intersection is dense

—o —o —

<u>Rmk</u> The proof of case ② is analogous, just define a sequence of set $\{U_n\}$ s.t.

$$\underbrace{\text{Clos}(U_{n+1})}_{\text{compact}} \subseteq U_n \cap A_{n+1}$$

<u>Rmk</u> The idea for ③ is the following. Let $\Upsilon \subseteq \mathbb{X}$ be an open set. Let $\{A_i\}_{i\geq 1}$ be a seq. of open dense subsets of Υ. Then

$$\hat{A}_i := A_i \cup (\mathbb{X} \setminus \mathrm{Clos}(\Upsilon))$$

are open dense subsets of $\mathbb{X}$, and therefore

$$\bigcap_{i=1}^{\infty} \hat{A}_i = \bigcap_{i=1}^{\infty} A_i \cup (\mathbb{X} \setminus \mathrm{Clos}(\Upsilon))$$

is dense in $\mathbb{X}$, but this is possible only if the first intersection is dense in Υ.

— o — o —

<u>Interesting exercise</u>
→ Prove that a closed subset of a Baire space is a Baire space
→ Find a counterexample (it is false)
→ Realize that the restricted topology is always difficult to understand.

<u>Rmk</u>
→ Any open subset of a Banach space is a Baire space (open set in Baire space)
→ Any closed subset of a Banach space is a Baire space (it is again a complete metric space)
→ Any closed subset of a loc. comp. top. space is a Baire space (it is again a loc. comp. top. space)
→ $\mathbb{R}, \underset{\uparrow\uparrow}{\mathbb{Z}}, \underset{\uparrow\uparrow}{\mathbb{N}}$ are Baire spaces (complete metric spaces)

→ $\mathbb{Q}$ is not a Baire spaces
→ $\mathbb{R} \setminus \mathbb{Q}$ are a Baire space
} exercises

BAIRE POINT 1

Let $f : (0,+\infty) \longrightarrow \mathbb{R}$ be a continuous function.
Let us assume that for every $x > 0$ there exists

$$\lim_{n\to+\infty} f(nx) = \ell \in \mathbb{R} \quad \text{(always the same)}$$

Then $\displaystyle\lim_{x\to+\infty} f(x) = \ell$

$\boxed{\text{Proof}}$ For every $\varepsilon > 0$ we know that

$$|f(nx)-\ell| \leq \varepsilon \qquad \text{for } n \ \underbrace{\text{LARGE ENOUGH}}_{\text{QUALITATIVE}}$$

Let us introduce

$$C_k := \{x > 0 : \ |f(nx)-\ell| \leq \varepsilon \quad \underbrace{\forall n \geq k}_{\text{QUANTITATIVE}}\}$$

$$= \bigcap_{n\geq k} \underbrace{\{x>0 : \ |f(nx)-\ell| \leq \varepsilon\}}_{\text{closed subset of } (0,+\infty)}$$

By assumption $\displaystyle\bigcup_{k=1}^{\infty} C_k = (0,+\infty)$

Baire lemma $\Longrightarrow \exists\, k_0$ s.t. C_{k_0} has nonempty interior, namely $(a,b) \subseteq C_{k_0}$ for some interval (a,b), namely

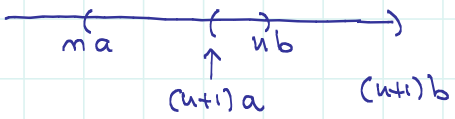
$$|f(nx)-\ell| \leq \varepsilon \qquad \forall n \geq k_0 \quad \forall x \in (a,b)$$

or equivalently

$$|f(y)-\ell| \leq \varepsilon \qquad \forall y \in (na, nb) \qquad n \geq k_0$$

We observe that

$$|f(y)-\ell| \le \varepsilon \qquad \forall y \in \underbrace{\bigcup_{n \geq k_0} (na, nb)}_{\text{this union contains a half-line } (A_0, +\infty)}$$



The intervals start to overlap as soon as

$$(n+1)a < nb \quad \rightsquigarrow \quad a \underset{\uparrow \text{ true for } n \text{ large enough}}{<} n(b-a)$$

<u>Nice variant</u> Consider the case when ℓ might depend on x.

Istituzioni di Analisi — LECTURE 62

Note Title 09/12/2021

$\boxed{\text{BAIRE POINT 2}}$ Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous in x if and only if $x \in \mathbb{R} \setminus \mathbb{Q}$?

Rmk There DOES exist a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous exactly in $\mathbb{Q}$.

$\boxed{\text{Defn}}$ Let X be a top. space. A subset $Y \subseteq X$ is called

- an F_σ set if it is the countable union of closed subsets
  fermée ↑ ↖ summ = union
- a G_δ set if it is the countable intersection of open sets
  ↑ GEBIET

$\boxed{\text{Lemma 1}}$ Let X be a metric space, and let $f : X \to Y$ be a function
another metric space
Then the set of discontinuity points of f is a F_σ-set.

$\boxed{\text{Proof}}$ Let us define

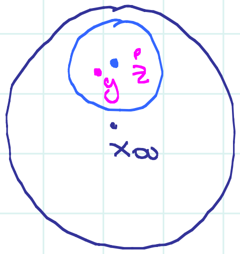
$$D_\varepsilon := \{ x \in X : \forall r>0 \ \exists (y,z) \in B(x,r) \text{ s.t. } d_Y(f(y), f(z)) \geq \varepsilon \}$$

It is possible to check that $\bigcup_{\varepsilon>0} D_\varepsilon$ is exactly the set of discontinuity points of f

$$\bigcup_{\varepsilon>0} D_\varepsilon = \bigcup_{m=1}^{\infty} D_{1/m}$$

More important, D_ε is a closed set for every $\varepsilon > 0$. Indeed, if $x_n \to x_\infty$ in X, then consider any ball $B(x_\infty, r)$

For n large enough



$$B(x_n, \tfrac{r}{2}) \subseteq B(x_\infty, r)$$
$$\uparrow$$

contains two points y and z with

$$d_Y(f(y), f(z)) \geq \varepsilon.$$

— o — o —

Lemma 2 $\mathbb{R} \setminus \mathbb{Q}$ is not a F_σ subset of $\mathbb{R}$.

Proof Assume that

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} \underset{\uparrow \text{ closed subsets}}{C_n}$$

Then $\mathrm{Int}(C_n) \subseteq \mathrm{Int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$ for every $n \geq 1$.

Now observe that

$$\underbrace{\mathbb{R} = \bigcup_{n=1}^{\infty} C_n \cup \bigcup_{q \in \mathbb{Q}} \{q\}}$$

we have written $\mathbb{R}$ as a countable union of closed subsets with empty interior.

— o — o —

BAIRE POINT 3 Let $f : \mathbb{R}^2 \to \mathbb{R}$. Let us assume that

(i) f is separately continuous in the two variables, namely

$\forall y \in \mathbb{R} \quad x \to f(x,y)$ is cont. wrt x

$\forall x \in \mathbb{R} \quad y \to f(x,y)$ is cont. wrt y

(i) there exists $D \subseteq \mathbb{R}^2$ dense s.t.

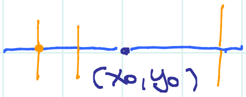
$$f(x,y) = 0 \qquad \forall (x,y) \in D$$

Then $f(x,y) = 0$ for every $(x,y) \in \mathbb{R}^2$.

Rmk This is simple if $D = \mathbb{Q}^2$, but in general it might happen that the intersection of D with every line $/\!/$ to the axes contains at most point.

$\boxed{\text{Proof}}$ Let us assume it is not true. wlog $f(x_0,y_0)>0$

$\exists\, \delta>0$ s.t. $f(x,y_0)>0$ for every $x\in(x_0-\delta,\, x_0+\delta)$



For every such x, there exists $r(x)$ such that

$$f(x,y)>0 \qquad \forall\, y\in[y_0-r(x),\, y_0+r(x)]$$

We need $r(x)$ to be in some sense UNIFORM wrt x.
Let us make it QUANTITATIVE.
Assume that $f(x_0,y_0)\geqslant 4a$.
Assume that

$$f(x,y_0)\geq 2a \qquad \forall\, x\in[x_0-\delta,\, x_0+\delta]$$

Define

$$C_k := \Big\{x\in[x_0-\delta,x_0+\delta] : f(x,y)\geq a \ \underbrace{\forall y\in[y_0-\tfrac{1}{k},\, y_0+\tfrac{1}{k}]}_{\text{QUANTITATIVE}}\Big\}$$

The union of C_k's is $[x_0-\delta,\, x_0+\delta]$. They are closed. Indeed assume that

$$C_k \ni x_n \longrightarrow x_\infty$$

Then for every $y\in[y_0-\tfrac{1}{k},\, y_0+\tfrac{1}{k}]$ it turns out that

$$f(x_\infty,y) \underset{\substack{\uparrow\\ \text{cont. wrt } x}}{=} \lim_{n\to+\infty} \underbrace{f(x_n,y)}_{\geq a} \geq a$$

Baire lemma $\Rightarrow \exists\, k_0\geq 1$ s.t. $C_{k_0}\supseteq$ interval (c,d) and $(c,d)\times(y_0-\tfrac{1}{k_0},\, y_0+\tfrac{1}{k_0})$ intersects D.

BAIRE POINT 4

Theorem (Hilbert version) Let H be a Hilbert space.
Let us assume that $v_n \longrightarrow v_\infty$ weakly in H.
Then $\{v_n\}$ is bounded

Proof For every $v \in H$ we know that $\langle v_n, v\rangle \to \langle v_\infty, v\rangle$ and in particular

$$|\langle v_n, v\rangle| \text{ is bounded}$$

QUALITATIVE

Define

$$C_k := \{ v \in H : |\langle v_m, v\rangle| \leq k \quad \forall m \in \mathbb{N} \}$$

$$= \bigcap_{m=1}^{\infty} \underbrace{\{ v \in H : |\langle v_m, v\rangle| \leq k \}}_{\text{closed set}}$$

Again H is the union of the closed sets C_k, and therefore there exists

$$\overline{B}(w_0, r_0) \subseteq C_{k_0}$$

Claim:

$$|\langle v_n, w\rangle| \leq \frac{2k_0}{r_0} \qquad \forall n \geq 1 \quad \forall w \in \overline{B}(0,1)$$

Indeed
$$w = \frac{1}{r_0} \underbrace{(r_0 w + w_0)}_{\in \overline{B}(w_0, r_0)} - \frac{1}{r_0} w_0$$

and therefore

$$|\langle v_n, w\rangle| \leq \frac{1}{r_0} \underbrace{|\langle v_n, r_0 w + w_0\rangle|}_{\leq k_0} + \frac{1}{r_0} \underbrace{|\langle v_n, w_0\rangle|}_{\leq k_0}$$

which proves the claim.

Now it is enough to set $\omega := \frac{v_n}{\|v_n\|} \in \overline{B}(0,1)$ and obtain

$$\|v_n\| \leq \frac{2k_0}{r_0} \qquad \forall n \geq 1$$

— o — o —

Rmk The same idea works in normed spaces, namely

$v_n \longrightarrow v_\infty$ weakly in V $\Rightarrow$ $\{v_n\}$ is bounded

$\boxed{\text{Proof}}$ ...

$$C_k := \{ f \in V' : |f(v_n)| \leq k \}$$

... $V' = \bigcup_{k=1}^{\infty} C_k$ and C_k is closed ...

$$\exists \; \overline{B}_{V'}(f_0, r_0) \subseteq C_{k_0}$$

Claim : $|f(v_n)| \leq \frac{2k_0}{r_0} \qquad \forall n \geq 1 \qquad \forall f \in \overline{B}_{V'}(0,1)$

... $$f = \frac{1}{r_0} \underbrace{(r_0 f + f_0)}_{B_{V'}(f_0, r_0)} - \frac{1}{r_0} \underset{\sim\sim}{f_0}$$

...

Conclude by taking f = aligned functional of v_n.

— o — o —

Istituzioni di Analisi – LECTURE 63

Note Title 13/12/2021

$\boxed{\text{BAIRE POINT 5}}$ There do NOT exist a Banach space with COUNTABLE ALGEBRAIC basis.

$\boxed{\text{Proof}}$ Let us assume that $v_1, \dots, v_n, \dots$ is a countable basis.

Define

$$C_k := \text{Span}(v_1, \dots, v_k) \leftarrow \text{Finite dim. vector subspace } (\Rightarrow \text{closed})$$

Again $\underset{\substack{\uparrow \\ \text{whole space}}}{V} = \bigcup_{k=1}^{\infty} C_k$

Baire category $\Rightarrow$ there exists C_{k_0} with $\text{Int}(C_{k_0}) \neq \emptyset$, namely $B_V(x_0, r_0) \subseteq C_{k_0}$.

This is not possible because

$$\underset{\substack{\uparrow \\ \in C_{k_0}}}{x_0} + \varepsilon v_{k_0+1}$$

for ε small would be in C_{k_0}, which is impossible.

— o — o —

Lemma Let V be a normed space, and let $W \subseteq V$ be a subspace with finite dimension.
Then W is closed.

$\boxed{\text{Proof \#1}}$ Let $w_1, \dots, w_m$ be a basis of W.

On W we define two norms

→ norm # 1 : restriction of the norm of V

→ norm # 2 : write $w \in W$ as $c_1 w_1 + \dots + c_m w_m$ and define

$$\|w\| = (c_1^2 + \dots + c_m^2)^{1/2}$$

General fact: any two norms on a finite dim. vector space are equivalent. This is enough to conclude because if $v_n \to v_\infty$ with $v_n \in W$, then $\{v_n\}$ is a Cauchy seq.

wrt to the first norm, therefore it is a Cauchy seq. with respect to the second norm, but W is complete wrt norm #2, and therefore it is complete wrt norm # 1.

Proof # 2 Assume that $v_k \to v_\infty$ is a sequence with $v_k \in W$ for every $k \geq 1$. Claim: $v_\infty \in W$.
For every $k \geq 1$ write

$$v_k = c_1^k w_1 + \dots + c_n^k w_n$$

We hope that

$$c_i^k \longrightarrow c_i^\infty \qquad \text{as } k \to +\infty,$$

at least up to subsequences. This is trivial if $\{c_i^k\}_{k\geq 1}$ are bounded for every $i = 1, \dots, n$. Assume it is not the case.
wlog assume that $|c_1^k| \geq |c_i^k|$ for every k and i and $|c_1^k| \to +\infty$
Then

$$\underset{\substack{\downarrow\\ 0}}{\boxed{\frac{1}{c_1^k}\, v_k}} = w_1 + \underset{\substack{\downarrow\\ \alpha_2}}{\boxed{\frac{c_2^k}{c_1^k}}}\, w_2 + \dots + \underset{\substack{\downarrow\\ \alpha_n}}{\boxed{\frac{c_n^k}{c_1^k}}}\, w_n$$

(v_k is bounded and $|c_1^k| \to +\infty$)

(up to subsequences because they are bounded)

and in this way we have a nontrivial lin. comb. of $w_1, \dots, w_n$ which is 0.

* * *

BAIRE POINT 6

Theorem There exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is NOT differentiable at every $x \in \mathbb{R}$.

Proof Consider V = space of continuous and bounded functions with the uniform norm (L^∞).
This is a complete metric space.

Claim: the set of elements of V that are diff. in at least one point are a countable union of closed sets with empty interior.
This implies that there are "many" counterexamples.

$$\exists\, x_0 \in \mathbb{R} \quad \text{s.t.} \quad \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) \in \mathbb{R}$$

QUALITATIVE!

quantitative position

quantitative limit

$$C_k := \{ f \in V : \exists\, x_0 \in [-k,k] \ \forall h \in [-\tfrac{1}{k}, \tfrac{1}{k}] \quad |f(x_0+h) - f(x_0)| \leq k\,|h| \}$$

quantitative bound on the derivative

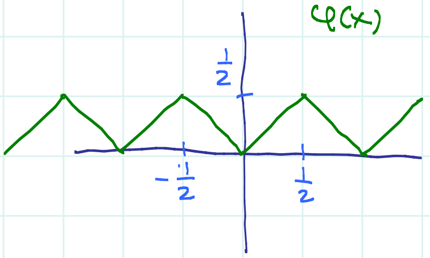
The set of NON counterexamples is $\bigcup_{k=1}^{\infty} C_k$.

Claim: $\mathrm{Int}(C_k) = \emptyset$.

Assume it is NOT empty, namely $\exists\, B_V(f_0, r_0) \subseteq C_{k_0}$

Step 1 We can assume, up to reducing r_0, that f_0 is rather smooth, for example Lip. cont., or even of class C^{22}.
(just regularize f_0)

Step 2 Consider $\varphi(x)$ as in the figure and observe that, if we set



$$\varphi_n(x) := \frac{1}{n}\, \varphi(n^2 x)$$

then for every $x \in \mathbb{R}$ it turns out that

$$\lim_{h \to 0^+} \frac{\varphi_n(x+h) - \varphi_n(x)}{h} = \pm n \quad \text{(depending on } x\text{)}$$

Define $$f_n(x) = \underset{\substack{\uparrow \\ \text{center of} \\ \text{the ball}}}{f_0(x)} + \underset{\substack{\uparrow \\ \text{small in } V \text{ if } n \\ \text{is large enough}}}{\varphi_n(x)}$$

$\Rightarrow$ $f_n \in C_{k_0}$ if n is large enough.

On the other hand

$$\underset{\text{tends to } \pm\infty}{\boxed{\frac{\varphi_n(x_0+h) - \varphi_n(x_0)}{h}}} = \underset{\substack{\text{Bounded by } k_0 \\ \text{if } h \in [-\frac{1}{k_0}, \frac{1}{k_0}]}}{\boxed{\frac{f_n(x_0+h) - f_n(x_0)}{h}}} - \underset{\substack{\text{Bounded by the Lip} \\ \text{const of } f_0}}{\boxed{\frac{f_0(x_0+h) - f_n(x_0)}{h}}}$$

We had to check that C_k are closed sets wrt uniform conv.

Assume that $f_n \to f_\infty$ unif.

Assume that

$$|f_n(\underset{\substack{\uparrow \\ \text{depends on } n}}{x_n}+h) - f_n(x_n)| \leq K|h| \qquad \forall h \in [-\tfrac{1}{k}, \tfrac{1}{k}]$$

Since $x_n \in [-k, k]$, up to subsequences $x_n \to x_\infty \in [-k, k]$ and due to unif. conv. we can pass to the limit (double limit) and obtain

$$|f_\infty(x_\infty + h) - f_\infty(x_\infty)| \leq k|h| \qquad \forall h \in [-\tfrac{1}{k}, \tfrac{1}{k}]$$

— o — o —

BAIRE POINT 7 — BANACH – STEINHAUS THEOREM

Setting: V is a Banach space, W is a normed space, I is an index set.

For every $i \in I$, we consider a linear map

$$L_i : V \to W$$

that is also continuous

$\boxed{\text{Statement 1}}$ (Equivalence)

The following two facts are equivalent

(1) (Qualitative boundedness)

$$\forall v \in V \qquad \sup_{i \in I} \| L_i(v) \|_W < +\infty$$

(2) (Quantitative boundedness)

$$\exists M \in \mathbb{R} \text{ s.t. } \qquad \| L_i(v) \|_W \leq M \| v \|_V$$

(this is equivalent to $\sup_{i \in I} \| L_i \|_{\mathcal{L}(V,W)} < +\infty$)

$\boxed{\text{Statement 2}}$ (Alternative)

The following two facts are alternative.

(2) (Quantitative boundedness)

(1) There exists a G_δ dense subset $A \subseteq V$ such that

$$\sup_{i \in I} \| L_i(v) \|_W = +\infty \qquad \forall v \in A$$

$\boxed{\text{Corollary}}$ Let $L_n : V \longrightarrow W$ be a sequence of lin. cont. operators.

Assume that $L_n(v) \longrightarrow L_\infty(v)$ for every $v \in V$.

Then L_∞ is linear and continuous.

Proof Pointwise conv. ⇒ qualitative boundedness ⇒ quantitative boundedness ⇒ equi - lip. continuity.

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Proof of BS, equivalence version (2) => (1) is trivial, so we prove the converse

Hyp: $\forall v \in V \quad \sup_{i \in I} \|L_i(v)\|_W < +\infty$

Th: $\forall v \in V \quad \|L_i(v)\|_W \le \underset{\text{indep. of } i}{M} \|v\|_V$

$$C_k := \{ v \in V : \ \forall i \in I \ \ \|L_i(v)\| \le k \}$$

$$= \bigcap_{i \in I} \underbrace{\{ v \in V : \ \|L_i(v)\| \le k \}}_{\text{closed set because } L_i \text{ is cont.}}$$

By assumption $\bigcup_{k=1}^{\infty} C_k = V$. Baire lemma $\Rightarrow \exists \, \overline{B}_r(v_0, r_0) \subseteq C_{k_0}$

Claim: $\overline{B}_V(0,1) \subseteq C_{\frac{2k_0}{r_0}}$

Consider $v \in \overline{B}_V(0,1)$ and write it as

$$v = \frac{1}{r_0} \underbrace{(v_0 + r_0 v)}_{\in \overline{B}_V(v_0, r_0)} - \frac{1}{r_0} \underbrace{v_0}$$

and therefore

$$\|L_i v\|_W \le \frac{1}{r_0} \underbrace{\|L_i(v_0 + r_0 v)\|_W}_{\le k_0} + \frac{1}{r_0} \underbrace{\|L_i v_0\|_W}_{\le k_0} \le \boxed{\frac{2k_0}{r_0}} \ M$$

We conclude because $\|L_i\|_{\mathcal{L}(V,W)}$ is the sup of LHS for $v \in \overline{B}_V(0,1)$.

—o—o—

Proof of BS, alternative version Assume that (2) is false, where

(2) is $\exists\ \underset{\substack{\uparrow\\ G_\delta \text{ dense}\\ \text{subset}}}{A} \subseteq V \quad \forall v \in A \quad \sup_{i\in I} \| L_i(v) \|_W = +\infty$

We observe that

$$\{ v \in V : \sup_{i\in I} \|L_i(v)\|_W = +\infty \} = \bigcap_{k=1}^{\infty} \underbrace{\{ v \in V : \sup_{i\in I} \|L_i(v)\|_W > k \}}_{\text{open set} \quad A_k}$$

This implies that the LHS is always a G_δ set. If A_k is dense for every k, then the LHS is dense.

If (2) is false, then A_{k_0} is NOT dense for some $k_0 \geq 1$, and therefore $\underbrace{V \setminus A_{k_0}}_{C_{k_0} \text{ in the previous proof}}$ has nonempty interior

— o — o —

BAIRE POINT 7-bis Failure of pointwise conv. of Fourier series of continuous functions.

Light version Let V be the space of 2π-periodic continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the sup norm. Then there exists $f \in V$ s.t. the Fourier series of f does not converge in $x = 0$.

Strong version There exists a G_δ dense subset $\mathcal{Y} \subseteq V$ such that for every $f \in \mathcal{Y}$ the F. series of f does not $\underset{\substack{\uparrow\\ \text{in the sense that}\\ \text{partial sums are}\\ \text{NOT bounded}}}{\text{converge}}$ in a G_δ dense set of points (depending on f)

$\boxed{\text{Proof of light version}}$ Let us define

$$S^f_n(x) := \sum_{k=0}^{n} \underset{\uparrow}{\alpha_k} \cos(kx) + \sum_{k=1}^{n} \underset{\uparrow}{\beta_k} \sin(kx)$$

formulae for α_k and β_k

Consider the sequence of operators $L_n : V \longrightarrow \mathbb{R}$ defined as

$$L_n(f) := S^f_n(0) = \sum_{k=0}^{n} \alpha_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx + \frac{1}{\pi}\sum_{k=1}^{n}\int_0^{2\pi} f(x)\cos(kx)\,dx$$

If the series converges in $x=0$ for every $f \in V$, then L_n is qualitatively bounded.
This implies a quantitative bound, namely

$$|L_n(f)| \leq M \, \|f\|_{L^\infty}$$

On the other hand, L_n can be represented as

$$L_n(f) = \int_0^{2\pi} \underset{\uparrow}{D_n(t)}\, f(t)\,dt$$

Dirichlet kernel

where $$D_n(t) := \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{t}{2}\right)}$$

Now we know that

$$|L_n(f)| \leq \|f\|_{L^\infty} \, \|D_n\|_{L^1}$$

Claim: we have equality and $\|D_n\|_{L^1} \to +\infty$ as $n \to +\infty$.

In order to show $=$ we would like to choose $f(t) = \operatorname{sign}(D_n(t))$ but we need $f(t)$ to be continuous. To fix the problem, we approximate $\operatorname{sign}(D_n(f))$ with continuous functions and we pass to the limit by dominated convergence.

$\boxed{\text{GENERAL FACT}}$ If $g \in L^1((a,b))$, then the norm of the operator

$$L(f) := \int_a^b f(t)\, g(t)\, dt$$

is $\|g\|_{L^1((a,b))}$ both if the domain is $L^\infty((a,b))$, and if the domain is $C^0([a,b])$.

The computation of $\|D_n\|_{L^1}$ is an exercise

$$\int_0^{2\pi} \frac{|\sin(n+\frac{1}{2})t|}{\sin\frac{t}{2}}\, dt \underset{\substack{\uparrow \\ \sin\frac{t}{2} \leq \frac{t}{2}}}{\geq} 2\int_0^{2\pi} \frac{|\sin(n+\frac{1}{2})t|}{t}\, dt$$

$$\underset{\substack{\uparrow \\ (n+\frac{1}{2})t = y \\ (n+\frac{1}{2})dt = dy}}{=} 2\int_0^{2n\pi+\pi} \frac{|\sin(y)|}{y}\, dy$$

$$\downarrow \quad 2\int_0^{+\infty} \frac{|\sin(y)|}{y}\, dy = +\infty$$

— o — o —

$\boxed{\text{Proof of strong version}}$ From BS in alternative version we know that there exists a dense G_δ set in V such that

$$|S_n^f(0)|$$

is unbounded for every f in this set.

In the same way we obtain a dense G_δ set $\mathcal{Y}$ in V such that

$$|S_n^f(x)| \text{ is unbounded for every } x \in \mathbb{Q}$$

(Here we use that countable int. of dense G_δ sets is again a dense G_δ set)

So for every $f \in \mathcal{Y}$ and $x \in \mathbb{Q}$ $|S_n^f(x)|$ is unbounded.

Is it unbounded for more values of x? YES! It is unbounded on a G_δ set of points x (which is dense because it contains $\mathbb{Q}$)

$\boxed{\text{GENERAL FACT}}$ Let $\varphi_n : (a,b) \to \mathbb{R}$ be cont. functions. Then

$$\{ x \in (a,b) : \sup_{n \geq 1} |\varphi_n(x)| = +\infty \}$$

is a G_δ set

Indeed it can be written as

$$\bigcap_{k=1}^{\infty} \underbrace{\{ x \in (a,b) : \sup_{n \geq 1} |\varphi_n(x)| > k \}}_{\text{open set}}$$

Alternative proof: the sup of cont. functions is LSC and the set where it is $+\infty$ is the intersection of the sets where it is $> k$, and these are open sets because the complement where it is $\leq k$ is closed.

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BAIRE POINT 8 — Open mapping theorem and related issues

General problem Given $f : V \to W$, and given $w \in W$, solve the equation

$$f(v) = w$$

Defn Setting as above.

(1) A QUALITATIVE SOLVER is a function $S : W \to V$ such that

[Added after video]

$$\boxed{f(S(w)) = w \qquad \forall w \in W}$$

$S(w)$ is a possible solution of $f(v) = w$

(2) A QUANTITATIVE SOLVER is a solver that satisfies

$$\boxed{\exists M \in \mathbb{R} \quad \text{s.t.} \quad \|S(w)\|_V \leq M \|w\|_W \quad \forall w \in W}$$

(3) A quantitative solver on a dense subset is a function

$\hat{S} : \hat{D} \to V$, where $\hat{D}$ is dense in W and

$$f(\hat{S}(w)) = w \qquad \forall w \in \hat{D}$$

$$\exists \hat{M} \in \mathbb{R} \text{ s.t. } \quad \|\hat{S}(w)\|_V \leq \hat{M} \|w\|_W \qquad \forall w \in \hat{D}.$$

Questions Existence of

[Q1] a solver

[Q2] a linear solver

[Q3] a quantitative solver

[Q4] a linear quantitative solver

$\boxed{\text{Answer to Q1}}$ Let $f : \underset{\text{set}}{\underset{\uparrow}{V}} \longrightarrow \underset{\text{set}}{\underset{\uparrow}{W}}$ be a function

Then a solver exists if and only if f is surjective

$\boxed{\text{Proof}}$ Well-know set theoretic fact (S is a one-sided inverse of f).

$\boxed{\text{Answer to Q2}}$ Let $f : \underset{\text{vector}}{\underset{\uparrow}{V}} \longrightarrow \underset{\text{spaces}}{\underset{\uparrow}{W}}$ be a linear map.

Then a linear solver exists if and only if f is surjective.

$\boxed{\text{Proof}}$ → Choose an algebraic basis $\{w_i\}_{i\in I}$ of W

→ For every $i \in I$, choose $v_i \in V$ s.t. $f(v_i) = w_i$

→ Define $S(w_i) = v_i$

→ Extend S to W by linearity

→ Check that it works.

$\boxed{\text{Answer to Q3}}$ Is contained in three statements.

$\boxed{\text{Prop 1}}$ (Characterization of quantitative solvers)

Let $f : \underset{\text{NORMED}}{\underset{\uparrow}{V}} \longrightarrow \underset{\text{SPACES}}{\underset{\uparrow}{W}}$ be a linear map (we do NOT need continuity)

Then the following are equivalent.

(1) Existence of a quantitative solver (not nec. linear)

(2) f is OPEN (namely f (open subset of V) = open in W)

(3) $\exists R > 0$ s.t. $f(B_V(0,R)) \supseteq B_W(0,1)$. (exchange stat.)

<u>Rmk.</u> What we actually prove is that

$$(1) \iff (3)$$

$$(2) \iff (3)$$

$\boxed{\text{Prop 2}}$ (Weaker characterization)

Assume $f : \underset{\text{BANACH}}{\underset{\uparrow}{V}} \longrightarrow \underset{\text{NORMED}}{\underset{\uparrow}{W}}$ linear and $\boxed{\text{continuous}}$

Then the following are equivalent.

(1) Existence of a quantitative solver

(2) " " " " " on a dense subset

(3) $\exists\, R > 0$ s.t. $\underset{\uparrow\uparrow\uparrow}{\mathrm{Clos}_W} \left(f\left(B_V(0,R)\right)\right) \supseteq B_W(0,1)$

<u>Rmk</u> What we actually prove is that

$(1) \Longleftrightarrow (2)$ ((2) is the exchange statement)

$(2) \Longleftrightarrow (3)$

$\boxed{\text{Theorem}}$ (Open mapping thm)

Assume that $f : \underset{\text{BANACH}}{\underset{\uparrow}{V}} \longrightarrow \underset{\text{BANACH}}{\underset{\uparrow}{W}}$ is linear and continuous

Then the following facts are equivalent.

(1) f is OPEN (which in turn is equivalent to the existence of a quantitative solver)

(2) f is surjective.

The open mapping has three corollaries.

$\boxed{\text{COROLLARY 1}}$ (Continuity of the inverse function)

Assume that $f : \underset{\text{BANACH}}{\underset{\uparrow}{V}} \rightarrow \underset{\text{BANACH}}{\underset{\uparrow}{W}}$ is linear, continuous and invertible

Then the inverse function is linear and continuous.

$\boxed{\text{Proof}}$ Linearity is a general fact.

The continuity of the inverse is equivalent to f being open.

$\boxed{\text{COROLLARY 2}}$ (Equivalence of norms)
Let V be a vector space, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms in V.
Assume that
(i) V is a Banach space wrt $\boxed{\text{both norms}}$
(ii) there exists $M \in \mathbb{R}$ s.t.

$$\|v\|_2 \le M \|v\|_1 \qquad \forall v \in V$$

Then

$$\exists\ \hat{M} \text{ s.t. } \quad \|v\|_1 \le \hat{M} \|v_2\| \qquad \forall v \in V$$

(and therefore the two norms are equivalent)

$\boxed{\text{Proof}}$ Consider $Id : V_1 \longrightarrow V_2$. This map is linear, cont. and invertible
because of (ii)
Therefore, also the inverse is continuous, and hence Lip. cont., ...

$\boxed{\text{COROLLARY 3}}$ (Closed graph theorem)
Assume that $f : \underset{\text{BANACH}}{\underset{\uparrow}{V}} \longrightarrow \underset{\text{BANACH}}{\underset{\uparrow}{W}}$ is linear

Then f is continuous if and only if the graph of f is a closed subset of $V \times W$.

$\boxed{\text{Proof}}$ The nontrivial implication is closed graph $\Rightarrow$ cont.
Consider two norms on V:

$$\|v\|_1 := \|v\|_V$$
$$\|v\|_2 := \|v\|_V + \|f(v)\|_W$$

It is clear that $\|v\|_1 \le \|v\|_2$ for every $v \in V$.
Claim: V is a Banach space wrt norm 2.
Assume $\{v_n\}$ is a Cauchy seq. wrt norm 2. This implies that $\{v_n\}$ is a Cauchy seq. wrt norm 1 and $\{f(v_n)\}$ is a Cauchy seq. in W.

Therefore $v_n \to v_\infty \in V$

$f(v_n) \to w_\infty \in W$

But $(v_\infty, w_\infty) \in$ graph of f (because graph is closed)

and therefore $w_\infty = f(v_\infty)$.

At this point $v_n \to v_\infty$ also wrt norm 2.

Now the two norms are equivalent, namely

$$\|v\|_V + \|f(v)\|_W \le \hat{M}\,\|v\|_V$$

which implies that f is continuous.

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Rmk If $f : \underset{\uparrow}{X} \to \underset{\uparrow}{Y}$ it is $\boxed{\text{NOT TRUE}}$ that

metric spaces

f is cont. $\Longleftrightarrow$ graph is closed

— o — o —

$\boxed{\text{Proof of Prop 1}}$

(1) $\Rightarrow$ (3) Claim $f(B_V(0,\underset{\uparrow}{M})) \supseteq B_W(0,1)$

constant of quantitative solver

If $w \in B_W(0,1)$, then take $v := S(w)$ so that $f(v) = w$

and $\|v\|_V = \|S(w)\|_V \le M\,\|w\|_W < M$

(3) $\Rightarrow$ (1). Take $w \in W$. Consider $\dfrac{w}{2\|w\|_W} \in B_W(0,1)$

$\Rightarrow \exists\ v \in B_V(0,R)$ s.t. $f(v) = \dfrac{w}{2\|w\|_W}$

Now define $S(w) := 2\,\|w\|_W \cdot v$ and observe that

$$\|S(w)\| \le \underbrace{2\,\|v\|_W}_{2R} \cdot \|w\|_W$$

(2) $\Rightarrow$ (3) If f is open, then $f(B_V(0,1)) \supseteq B_W(0,r_0)$
for some $r_0 > 0$
By linearity

$$f\Big(B_V\big(0,\underset{\uparrow R}{\tfrac{1}{r_0}}\big)\Big) \supseteq B_W(0,\underset{\uparrow}{1})$$

[Corrected after video]

(3) $\Rightarrow$ (2) If $f(B_V(0,R)) \supseteq B_W(0,1)$ then

$$\underset{\shortparallel}{f(B_V(x_0,r_0))} \supseteq B_W\Big(f(x_0),\frac{r_0}{R}\Big)$$

$$\begin{aligned} f(x_0 + r_0 B_V(0,1)) &= f(x_0) + \frac{r_0}{R} f(B_V(0,R)) \\ &\supseteq f(x_0) + \frac{r_0}{R} B_W(0,1) \\ &= B_W\Big(f(x_0),\frac{r_0}{R}\Big). \end{aligned}$$

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$\boxed{\text{Proof of open mapping thm}}$ (1) => (2) easy because open => (quantitative) solver => surjective

$\boxed{(2) \Rightarrow (1)}$ We prove that (3) of Prop 2 is satisfied, namely $\exists R > 0$ s.t. $\mathrm{Clos}_W(f(B_V(0,R))) \supseteq B_W(0,1)$.

Here we use that W is Banach. Consider

$$C_k := \mathrm{Clos}_W(f(B_V(0,k)))$$

Since f is surjective, then $\bigcup_{k=1}^{\infty} C_k = W$

Baire category => $\exists k_0$ s.t. $B_W(w_0, r_0) \subseteq C_{k_0}$

Claim: $B_W(0,1) \subseteq C_{\underbrace{\boxed{\frac{2k_0}{r_0}}}_{= R}}$

Take any $w \in B_W(0,1)$ and write

$$w = \frac{1}{r_0} \underbrace{(w_0 + r_0 w)}_{B_W(w_0, r_0)} - \frac{1}{r_0} \underbrace{w_0}_{B_W(w_0, r_0)}$$

By assumption $\exists\ f(v_m) \longrightarrow w_0 + r_0 w \qquad v_m \in B_V(0, k_0)$

$\exists\ f(\hat{v}_m) \longrightarrow w_0 \qquad \hat{v}_m \in B_V(0, k_0)$

Now observe

$$f\Big(\underbrace{\frac{1}{r_0} v_m - \frac{1}{r_0} \hat{v}_n}_{\in B_V(0, \frac{2k_0}{r_0})}\Big) \longrightarrow w$$

— o — o —

Prop 2 (1) ⇒ (2) Trivial

(2) ⇒ (3) Let $\hat{D}$ the set where the quantitative solver is defined. Let $w \in B_W(0,1)$ and let $\underset{\cap\, \hat{D}}{w_n} \to w$

For n large enough $\|w_n\| < 1$. Consider $v_n := \hat{S}(w_n)$

By definition $f(v_n) = w_n \to w$ and

$$\|v_n\|_V = \|\hat{S}(w_n)\|_V \le \hat{M}\, \|w_n\|_W \le \hat{M} = R$$

Therefore $\operatorname{Clos}(f(B_V(0,R))) \supseteq B_W(0,1)$.

(3) ⇒ (2) Observe that $\operatorname{Clos}(f(B_V(0,R))) \supseteq \bar{B}_W(0,1)$.

Observe also that (3) implies that the image of is dense in W. Let $\hat{D}$ denote the image.

Now we define a solver in $\hat{D}$. Let $w \in \hat{D}$. Then

$$\frac{w}{\|w\|_W} \in \text{Image of } f \cap \bar{B}_W(0,1)$$

$\Rightarrow \exists\, v \in B_V(0,R)$ s.t. $f(v) = \dfrac{w}{\|w\|_W}$. Now define

$\hat{S}(w) := \|w\|_W \cdot v$ and observe that it works with

$$\|\hat{S}(w)\|_V = \underbrace{\|v\|_V}_{\le R} \cdot \|w\|_W$$

(2) ⇒ (1) relies on a Lemma (UP TO NOW WE DID NOT USE THAT V IS BANACH AND THAT f IS CONTINUOUS)

[Added after video: NORMED is enough]

Lemma Let W be a **Banach** space, and let $D \subseteq W$ be a dense subset. Then for every $\underset{\uparrow\, w \neq 0}{w \in W}$ there exists $\{w_n\} \subseteq D$ such that

$$\sum_{i=1}^{\infty} w_i = w \qquad \text{and} \qquad \sum_{i=1}^{\infty} \|w_i\|_W \le 2\, \|w\|_W$$

Rmk We can replace 2 in the last inequality with any constant > 1.

$\boxed{\text{Proof}}$ Choose $w_1 \in D$ s.t. $\| w - w_1 \|_W \le \frac{1}{8} \| w \|_W$

Observe that

$$\| w_1 \|_W \le \| w \|_W + \| w_1 - w \|_W \le \| w \|_W + \underset{\uparrow \text{generous}}{\frac{1}{2}} \| w \|_W$$

Choose $w_2 \in D$ s.t $\| w - w_1 - w_2 \|_W \le \frac{1}{16} \| w \|_W$

Observe that

$$\| w_2 \|_W \le \underset{\le \frac{1}{8} \| w \|_W}{\| w - w_1 \|_W} + \underbrace{\| w_2 + w_1 - w \|_W}_{\le \frac{1}{16} \| w \|_W} \le \frac{1}{4} \| w \|_W$$

... and so on ... at step k we choose $w_k \in D$ s.t.

$$\| w - w_1 - \dots - w_k \|_W \le \frac{1}{2^{\underset{\uparrow\uparrow}{k+2}}} \| w \|_W \qquad (*)$$

and observe that

$$\| w_k \|_W \le \underbrace{\| w - w_1 - \dots - w_{k-1} \|_W}_{\le \frac{1}{2^{k+1}} \| w \|_W} + \underbrace{\| w_k + w_{k-1} + \dots + w_1 - w \|_W}_{\le \frac{1}{2^{k+2}} \| w \|_W} \le \frac{1}{2^k} \| w \|_W$$

Observe that $(*)$ implies that $w = \sum_{i=1}^{\infty} w_i$ and

$$\sum_{i=1}^{\infty} \| w_i \| = \| w_1 \|_W + \sum_{i=2}^{\infty} \| w_i \|$$

$$\le \| w \|_W + \underbrace{\frac{1}{2} \| w \|_W + \| w \|_W \sum_{i=2}^{\infty} \frac{1}{2^i}}_{\| w \|_W}$$

— o — o —

Proof of (2) ⇒ (1) Let $D \subseteq W$ be the dense subset where the quantitative solver is defined.

Goal: extend S to the whole space W.

Take any $w \in W$. Consider $\{w_n\} \subseteq D$ as in the lemma.

Consider $v_n := \hat{S}(w_n)$ and define

$$v := \sum_{n=1}^{\infty} \hat{S}(w_n)$$

Is it well defined? Yes, because

$$\sum_{n=1}^{\infty} \|\hat{S}(w_n)\|_V \le \hat{M} \sum_{n=1}^{\infty} \|w_n\|_W \le \hat{M} \cdot 2\|w\|_W$$

(Absolute conv. ⇒ conv. because V is Banach)

$$f(v) = f\left(\sum_{n=1}^{\infty} \hat{S}(w_n)\right) \underset{\substack{\uparrow \\ f \text{ IS CONTINUOUS}}}{=} \sum_{n=1}^{\infty} f(\hat{S}(w_n)) = \sum_{n=1}^{\infty} w_n = w.$$

— o — o —

ANSWER TO Q4

Theorem Assume $f : \underset{\substack{\uparrow \\ \text{HILBERT}}}{H} \to \underset{\substack{\uparrow \\ \text{BANACH}}}{W}$ is linear and continuous.

Assume f is surjective.

Then a quantitative linear solver exists (and therefore also continuous)

Proof Let $K := \ker(f)$. Observe that $H = K \underset{\substack{\uparrow \\ \text{orthogonal}}}{\oplus} K^{\perp}$

Simple facts: $K^{\perp}$ is closed (and therefore a Hilbert space)

$f : K^{\perp} \to W$ is surjective and injective

Given $w \in W$ there exists $v \in H$ s.t. $f(v) = w$. But

$v = \underset{\underset{K}{\cap}}{v_1} + \underset{\underset{K^\perp}{\cap}}{v_2}$ and observe that

$$w = f(v) = \underset{\underset{0}{\shortparallel}}{f(v_1)} + f(v_2) = f(v_2)$$

Therefore $f : K^\perp \to W$ is invertible and linear and cont. between Banach spaces.
The inverse is the required solver.

<u>Rmk</u> The same proof works for $f : \underset{\underset{\text{BANACH}}{\uparrow}}{V} \to W$

provided that $K := \ker(f)$ has a TOPOLOGICAL COMPLEMENT in V, namely

$$V = K \oplus \underset{\underset{\text{CLOSED vector subspace}}{\uparrow}}{M}$$

$\boxed{\text{BAD NEWS}}$ If every closed subspace $K \leq V$ has a top. complement, then the norm of V is equivalent to a norm generated by a scalar product.

— o — o —

[CREDITS TO TERENCE TAO FOR THESE TWO LECTURES].

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Note Title 18/12/2021

BAIRE POINT 9 **Pointwise limit of cont. functions**

Theorem Let $\mathbb{X}$ be a COMPLETE metric $\overset{\text{space}}{\smile}$ (Baire space is probably enough)
Let $\mathbb{Y}$ be a metric space
Let $f_n : \mathbb{X} \to \mathbb{Y}$ be a sequence of cont. function.
Let $f_\infty : \mathbb{X} \to \mathbb{Y}$ be the pointwise limit, namely

$$f_n(x) \to f_\infty(x) \qquad \forall x \in \mathbb{X}.$$

Then f_∞ is continuous in a G_δ dense set, namely the set of disc. points of f_∞ is an F_σ-set with empty interior.

$\boxed{\text{Proof}}$ (The proof of the three nested open sets)

By a general fact the discontinuity points of f_∞ are

$$\bigcup_{\varepsilon>0} D_\varepsilon = \bigcup_{k=1}^{\infty} D_{1/k}$$

where

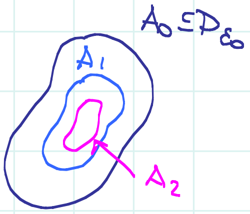
$$D_\varepsilon := \{x \in \mathbb{X} : \forall r>0 \ \exists (y,z) \in B(x,r) \text{ s.t. } d_{\mathbb{Y}}(f_\infty(y), f_\infty(z)) \geq \varepsilon\}$$

↑closed sets

Claim: $\text{Int}(D_\varepsilon) = \emptyset$ for every $\varepsilon > 0$.

Assume by contradiction that it is not the case.

Then there exists $\underset{\substack{\uparrow \\ \text{open set} \neq \emptyset}}{A_0} \subseteq D_{\varepsilon_0}$



We observe that A_0 is a Baire space.

For every $k \geq 1$ we consider

$$C_k := \{x \in A_0 : d_{\mathbb{Y}}(f_n(x), f_m(x)) \leq \tfrac{1}{4}\varepsilon_0 \quad \forall n \geq k, \forall m \geq k\}$$

$$= \bigcap_{\substack{n \geq k \\ m \geq k}} \underbrace{\{x \in A_0 : d_{\mathbb{Y}}(f_n(x), f_m(x)) \leq \tfrac{1}{4}\varepsilon_0\}}_{} \nwarrow \text{closed sets}$$

[Question : why we do not define

$$C_k := \{x \in A_0 : d_Y(f_n(x), f_\infty(x)) \leq \frac{1}{4}\varepsilon \quad \forall n \geq k\} \quad]$$

We observe that C_k is a closed set for every $k \geq 1$ and $\bigcup_{k=1}^{\infty} C_k = A_0$ and in addition

$$d_Y(f_n(x), f_\infty(x)) \leq \frac{1}{4}\varepsilon_0 \qquad \forall n \geq k \quad \forall x \in C_k$$

$\uparrow$ pass to the limit as $m \to +\infty$

Baire category $\Rightarrow \exists A_1 \subseteq C_{k_1}$

Now consider f_{k_1}. It is a cont. function, and therefore

$\exists A_2 \subseteq A_1$ s.t. the oscillation of $f_{k_1}(x)$ inside A_2 is $\leq \frac{1}{4}\varepsilon_0$

Now for every y and z in A_2 we have that

$$\begin{aligned} d_Y(f_\infty(y), f_\infty(z)) &\leq d_Y(f_\infty(y), f_{k_1}(y)) \quad \leq \frac{1}{4}\varepsilon_0 \\ &+ d_Y(f_{k_1}(y), f_{k_1}(z)) \quad \leq \frac{1}{4}\varepsilon_0 \\ &+ d_Y(f_{k_1}(z), f_\infty(z)) \quad \leq \frac{1}{4}\varepsilon_0 \\ &\leq \frac{3}{4}\varepsilon_0 \end{aligned}$$

but on the other we know that LHS $\geq \varepsilon_0$.

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Corollary Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at each $x \in \mathbb{R}$. Then $f'(x)$ is continuous in a G_δ dense set.

$\boxed{\text{Proof}}$ Just observe that $f'(x)$ is the pointwise limit of

$$\underset{\uparrow h = 1/m}{f_h(x)} := \frac{f(x+h) - f(x)}{h} \quad \nwarrow \text{continuous function}$$

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UNBOUNDED OPERATORS (in HILBERT SPACES)

Def Let H be a Hilbert space. An **unbounded operator** is a linear function

$$A : D(A) \longrightarrow H$$

↑ vector subspace of H, called the DOMAIN of A

Achtung! The term is misleading because UNBOUNDED does not imply NOT BOUNDED.

For example, the operator $Ax \equiv 0$ defined on a subspace of H, or even with $D(A) = H$, is an unbounded operator ☺

In other words:

UNBOUNDED = we do not ask a priori a bound.

Def. (Multiplication operator)

Let us assume that H is separable. We say that A is a MULTIPLICATION OPERATOR if there exists

→ a Hilbert basis $\{e_n\}$

→ a sequence of real numbers $\{\lambda_n\}$

such that

$$A e_n = \lambda_n e_n$$

In words, these operators are the infinite dim. versions of diagonal matrices

The operator is extended by linearity, so if

$$u = \sum_{n \geq 1}^{\infty} u_n e_n$$

then
$$Au = \sum_{n=1}^{\infty} \lambda_n u_n e_n$$

The series converges if and only if $\sum_{n=1}^{\infty} \lambda_n^2 u_n^2 < +\infty$ and therefore

$$D(A) := \left\{ u \in H : \sum_{n=1}^{\infty} \lambda_n^2 u_n^2 < +\infty \right\}$$

If $\{\lambda_n\}$ is bounded, then $D(A) = H$

If $\{\lambda_n\}$ is unbounded, then $D(A)$ is strictly contained in H.

$\boxed{\text{STANDARD EXAMPLE}}$ Let $A : H \to H$ be as in the spectral thm.

- → linear
- → continuous and symmetric
- → compact.

Assume that A is injective, namely 0 is not an eigenvalue. Then both A and A^{-1} are multiplication operators.

Because of the spectral theorem
$$A e_n = \underset{\uparrow \text{ bounded and } \lambda_n \to 0}{\lambda_n} e_n$$

$$A^{-1} e_n = \underset{\substack{\uparrow \\ \text{unbounded sequence}}}{\frac{1}{\lambda_n}} e_n$$

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POWERS OF MULTIPLICATION OPERATORS

Given a mult. op. $A : D(A) \longrightarrow H$, we can consider the power

$$A^s : D(A^s) \longrightarrow H$$

defined by $\boxed{A^s e_n = \lambda_n^s e_n}$

with domain

$$D(A^s) := \left\{ u \in H : \sum_{n=1}^{\infty} \lambda_n^{2s} u_n^2 < +\infty \right\}$$

What about s ? → If $s \in \mathbb{N} \setminus \{0\}$, then no restriction
→ If $s \in \mathbb{Z}$, it is enough that $\lambda_n \neq 0$
→ If $s \in \mathbb{R}$, we have to restrict to $\lambda_n > 0$
(positive operators)

Rmk We have the classical composition rule, namely

$$A^s (A^r u) = A^{s+r} u \qquad \forall u \in \text{suitable domain}$$

Def If $f : \mathbb{R} \to \mathbb{R}$ is any function, we can define $f(A)$ as the unbounded operator such that

$$f(A) e_n = f(\lambda_n) e_n \qquad \forall n \geq 1$$

With this rule PRODUCT OF FUNCTIONS becomes COMPOSITION OF OPERATORS, namely

$$\boxed{g(f(A)) = (g \cdot f)(A)}$$

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THE EXAMPLE **DIRICHLET LAPLACIAN**

$$\ddot{u} = f \qquad\qquad \Delta u = f$$

$u(0) = u(\pi) = 0$ $\qquad$ $u_{|\partial\Omega} = 0$

$(a,b) = (0,\pi)$ $\qquad$ $\Omega \subseteq \mathbb{R}^d$ bounded open set with reasonable boundary (for example a ball)

Let us consider the operator

$$I : L^2((a,b)) \to L^2((a,b)) \qquad\qquad I : L^2(\Omega) \to L^2(\Omega)$$

$$f \longrightarrow \underset{\uparrow\text{ unique solution}}{u} \qquad\qquad \underset{\text{Inverter of the Laplacian}}{\overset{\uparrow}{}} \quad f \to u$$

Step 1 The operator I is well-defined (existence and uniqueness)

First way Variational formulation

$$\min\left\{ \int_\Omega \left(\frac{1}{2}|\nabla u(x)|^2 + f(x)\,u\right)dx : u \in \underset{\uparrow\text{DBC}}{H^1_0}(\Omega)\right\}$$

Standard direct method applies and produces a unique solution u which is also in $H^2(\Omega)$.

Second way Lax – Milgram (because the equ. is linear)

$$-\int_\Omega \langle \nabla u(x), \nabla\varphi(x)\rangle\,dx = \int_\Omega f(x)\,\varphi(x)\,dx \qquad \forall \varphi \in C^\infty_c(\Omega)$$

$\boxed{\text{Third way}}$ Only in dimension one we set

$$F(x) := \int_0^x f(t)\,dt \qquad \hat{F}(x) := \int_0^x F(t)\,dt$$

$$u(x) = \hat{F}(x) + \underset{\uparrow}{a}x + \underset{\uparrow}{b}$$

suitable values for DBC

Linearity is always trivial.

$\boxed{\text{Step 2}}$ We claim that I is symmetric

$\boxed{\text{Proof}}$ In dim 1 $\quad \ddot{u} = f \qquad \ddot{v} = g$

$\qquad\qquad\qquad\quad I(f) = u \qquad I(g) = v$

$$< I(f), g > = \int_0^\pi u g \,dx = \int_0^\pi u \ddot{v} \underset{\substack{\uparrow \\ \text{DBC} \\ \text{on } u}}{=} -\int_0^\pi \dot{u}\dot{v} \underset{\substack{\uparrow \\ \text{DBC} \\ \text{on } v}}{=} \int_0^\pi \ddot{u} v$$

$$= \int_0^\pi f v = < f, I(g) >$$

In higher dimension is the same

$$\int_\Omega u g \,dx = \int_\Omega u \Delta v = -\int_\Omega < \nabla u, \nabla v > = \int_\Omega \Delta u \cdot v = \int f v$$

$\boxed{\text{Step 3}}$ We claim that I is compact.

Informally: an operator is COMPACT if "it improves the INPUT is such a way that the OUTPUT is better than the input"

$\boxed{\text{Proof in dim 1}}$ Let $\{f_n\} \subseteq L^2((0,\pi))$ be any sequence.

Consider u_n the corresponding solutions.

Consider $\dot{u}_n = v_n$.

Then $\|\dot{v}_n\|_{L^2}$ is bounded. By Rolle's theorem, there exists $x_n \in (0,\pi)$ s.t. $\dot{u}_n(x_n) = v_n(x_n) = 0$.

In the usual way we can apply Ascoli - Arzelà to v_n and therefore

$$v_n \longrightarrow v_\infty \qquad \text{uniformly}$$

Now consider $\{u_n\}$. It is a sequence of Lipschitz functions (because of the L^∞ bound on v_n), and again $u_n(0) = 0$.

Again by AA

$$u_n \longrightarrow u_\infty \qquad \text{uniformly} \quad \text{(and in particular in } L^2((0,\pi))\text{)}$$

Of course $v_\infty - \dot{u}_\infty$.

(All convergences are up to subsequences)

Rmk $I : L^2((0,\pi)) \longrightarrow H^1((0,\pi))$ Is it compact?

(Yes because we proved also unif. conv. of derivatives)

$$I : L^2((0,\pi)) \longrightarrow H^2((0,\pi)) \quad \text{Is it compact ?}$$

(No: indeed compactness would imply that

$$\begin{array}{lcl} \{f_n\} \subseteq L^2 \text{ bounded} & \Longrightarrow & u_{n_k} \longrightarrow u_\infty \text{ in } H^2 \\ & & \quad \Downarrow \\ & & \ddot{u}_{n_k} \longrightarrow \ddot{u}_\infty \text{ in } L^2 \\ & & \,\, \shortparallel \\ & & f_{n_k} \end{array}$$

and this in not true for every f_n.

Proof in dim d $\Delta u_n = f_n$ with $\{f_n\}$ bounded in L^2

As in the regularity theory we obtain bounds on

$$\|\nabla u_n\|_{L^2} \leq M' \qquad \|D^2 u_n\|_{L^2} \leq M''$$

↑all second derivatives of u_n

From Poincaré inequality we obtain also

$$\| u_n \|_{L^2} \leq M'''$$

At this point we can apply compact embeddings and deduce that

$$u_{n_k} \longrightarrow u_\infty \text{ in } L^2 \qquad \nabla u_{n_k} \longrightarrow \nabla u_\infty \text{ in } L^2$$

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At the end of the day

$$\boxed{I \text{ IS A COMPACT OPERATOR}}$$

and we can apply the spectral theorem.
By uniqueness I is also injective, and therefore

$$\boxed{\begin{array}{l}\text{THE INVERSE OF } I \text{ IS AN UNBOUNDED MULT. OPERATOR} \\ \text{CALLED THE DIRICHLET LAPLACIAN}\end{array}}$$

$$A : \underset{\uparrow \, H^2(\Omega) \cap \underset{\uparrow \text{DBC}}{H_0^1(\Omega)}}{D(A)} \longrightarrow \underset{\uparrow \, L^2(\Omega)}{H}$$

$\boxed{\text{Special case in } (0,\pi)}$ Let us find eigenvalues / eigenvectors

$$\ddot{u} = \lambda u$$

First of all, $\lambda < 0$:

$$u\ddot{u} = \lambda u^2$$

$$\int u\ddot{u} = \lambda \int u^2$$

$$\overset{\shortparallel}{-\int \dot{u}^2}$$

$$\ddot{u} = -\alpha u \quad \text{with } \alpha > 0$$

$$u(x) = c_1 \cos(\sqrt{\alpha}\, x) + c_2 \sin(\sqrt{\alpha}\, x)$$

we apply DBC $\rightsquigarrow$ $c_1 = 1$ $\rightsquigarrow$ $\sqrt{\alpha} = n$ $\rightsquigarrow$ $\alpha = n^2$

Eigenvalues : $-n^2$ with n positive integer

Eigenvectors : $\sin(nx)$ up to a multiplicative constant if we want them to be orthonormal

Rmk These are the eigenvalues of the second derivative with DBC.

The same problem with NBC produces an operator with eigenvalues $-n^2$ and eigenvectors $\cos(nx)$.

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$\boxed{-\text{DIRICHLET LAPLACIAN}}$ $-\ddot{u}$ $(0,\pi)$ $H^2 \cap H^1_0$

eigenvalues $\underset{\overset{\uparrow}{m \geq 1}}{m^2}$ eigenvectors $\sin(mx)$ (up to a mult. const.)

$\boxed{-\text{NEUMANN LAPLACIAN}}$ H^2 with NBC

eigenvalues $\underset{\overset{\uparrow}{m \geq 0}}{m^2}$ eigenvectors $\cos(mx)$ (up to const.)

↑ including the case $m=0$

$\boxed{-\text{PERIODIC LAPLACIAN}}$ Interval $(0,2\pi)$ H^2 periodic $\nearrow u$ $\searrow \ddot{u}$

eigenvalues $\underset{\overset{\uparrow}{m \geq 0}}{m^2}$ eigenvectors $\underbrace{\sin(mx), \cos(mx)}$

all eigenspaces with $m \geq 1$ are of dim 2

($m=0 \rightsquigarrow \dim = 1$)

<u>Rmk</u> The theory is similar for derivatives of order $2k$ with k integers

(even is important in order to obtain a symmetric operator $\rightsquigarrow$ even # of int. by parts)

<u>Rmk</u> All these operators are CLOSED, namely

$$\left.\begin{array}{l} u_n \to u_\infty \\ A u_n \to v_\infty \end{array}\right\} \Rightarrow v_\infty = A u_\infty$$

This in weaker than continuity because of the second assumption

From now on we focus on the Dirichlet case

Prop. If $A = -$ dir. Lap. $\Rightarrow D(A^{1/2}) = H_0^1((0,\pi))$

$\boxed{\text{Proof.}}$ $\boxed{\Rightarrow}$ Let us write $u(x) = \sum_{n=1}^{\infty} u_n \sin(nx)$

Assumption $\sum_{n=1}^{\infty} n^2 u_n^2 < +\infty$

[In abstract: $D(A^s) = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2s} u_n^2 < +\infty \right\}$
$\lambda_n = n^2$]

$S_m(x) : \sum_{k=1}^{m} u_k \sin(kx)$ $\qquad S_m(x) \to u(x)$ in $L^2((0,\pi))$

$S_m'(x) = \sum_{k=1} k u_k \cos(kx)$

Now $\cos(kx)$ is (up to a constant) an orthogonal system in L^2

Therefore $\sum_{k=1}^{\infty} k u_k \cos(kx)$ converges in $L^2((0,\pi))$ if and only if $\sum_{k=1}^{\infty} k^2 u_k^2 < +\infty$ and this is the assumption.

$\rightsquigarrow$ $S_m'(x) \to v(x)$ in $L^2((0,\pi))$ and as always $v(x) = \dot{u}(x)$

This proves that $u \in H^1((0,\pi))$. Moreover $S_m(x) \to u(x)$ uniformly in $[0,\pi]$ and since $S_m(x) = 0$ in $x=0$ and $x=\pi$ this is true also for $u(x)$, namely $u \in H_0^1$.
Here we exploited Ascoli - Arzelà.

$\boxed{\Leftarrow}$ Assumption: $u \in H_0^1((0,\pi))$ Claim: $\sum_{n=1}^{\infty} n^2 u_n^2 < +\infty$

$$u_n := \int_0^\pi u(x)\sin(nx)\,dx \qquad \text{(up to constants)}$$

$$= \underbrace{\left[-\frac{1}{n}\cos(nx)\,u(x)\right]_{x=0}^{x=\pi}}_{\overset{\shortparallel}{0}} + \frac{1}{n}\underbrace{\int_0^\pi \underbrace{\dot{u}(x)}_{v(x)}\cos(nx)\,dx}_{\substack{\text{component of } v(x) \text{ wrt} \\ \text{the orthogonal system } \cos(nx) \\ =: v_n}}$$

We have proved that

$$\sum_{n=1}^{\infty} n^2 u_n^2 = \sum_{n=1}^{\infty} v_n^2 < +\infty \quad \text{because } v(x) \in L^2$$

$\boxed{\text{Important point}}$ What is $A^{1/2}u$?

If $u = \sum_{n=1}^{\infty} u_n \sin(nx)$ then $A^{1/2}u = \sum_{n=1}^{\infty} \underset{\lambda_n^{1/2}}{\overset{\uparrow}{n}}\, u_n \sin(nx)$

We observe that $\|A^{1/2}u\|_{L^2}^2 = \sum_{n=1}^{\infty} n^2 u_n^2$

On the other hand

$$\dot{u} = \sum_{n=1}^{\infty} n\, u_n \cos(nx) \quad \text{and} \quad \|\dot{u}\|_{L^2}^2 = \sum_{n=1}^{\infty} n^2 u_n^2$$

So it is true that $\|A^{1/2}u\|_{L^2} = \|\dot{u}\|_{L^2}$

but it is not true that $A^{1/2}u = \dot{u}$

FRACTIONAL SOBOLEV SPACES

$p=2$

$$\underset{+\ \mathrm{DBC}}{\overset{}{H^s}}((0,\pi)) = D(A^{s/2})$$

$\boxed{\text{Prop.}}$ If $u \in H^s((0,\pi))$ with $s > \frac{1}{2}$, then u is continuous and satisfies DBC

Brutal mode: $u \in W^{m,p}$ is continuous when $\underset{s}{m}\,\underset{2}{p} > \underset{1}{d}$

$\boxed{\text{Proof}}$ $$u(x) = \sum_{n=1}^{\infty} u_n \sin(nx)$$

$$u \in H^s \iff \sum_{n=1}^{\infty} \underset{\lambda_m^{2\frac{s}{2}}}{n^{2s}}\, u_n^2 < +\infty$$

Sufficient condition for u cont. is that $\sum_{n=1}^{\infty} |u_n| < +\infty$

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \underbrace{n^s |u_n|}_{\frac{1}{2}} \underbrace{\frac{1}{n^s}}_{\frac{1}{2}}$$

$$\leq \left\{ \sum_{n=1}^{\infty} n^{2s} u_n^2 \right\}^{1/2} \underbrace{\left\{ \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \right\}^{1/2}}_{\text{converges iff } 2s>1}$$

Rmk We expect $u(x)$ to be holder continuous with a suitable exponent.

We expect that for $s \leq \frac{1}{2}$ u is not even in L^∞, and therefore DBC are "lost"

Example For $s<\frac{1}{2}$ we expect that $u(x)=\frac{1}{x^a}$ is in H^s for some a.

Heuristics $\frac{1}{x^a}\in L^2((0,\pi)) \iff 2a<1 \iff a<\frac{1}{2}$

we expect s-deriv. to be $\sim \frac{1}{x^{a+s}}$

and therefore $\frac{1}{x^a}\in H^s \iff 2a+2s<1$

$$u_n := \int_0^{\pi} \frac{1}{x^a}\sin(nx)\,dx \underset{\uparrow}{=} \frac{1}{n}\int_0^{n\pi}\frac{n^a}{y^a}\sin(y)\,dy$$

$nx=y$

$x=\frac{y}{n}$

$dx=\frac{1}{n}dy$

$$= \frac{n^a}{n}\boxed{\int_0^{n\pi}\frac{\sin y}{y^a}\,dy}$$

convergent

$$u_n \sim \frac{1}{n^{1-a}}$$

$$u\in H^2 \iff \sum n^{2s}u_n^2<+\infty \iff \sum \frac{n^{2s}}{n^{2-2a}}<+\infty$$

$$\iff 2-2a-2s>1$$

$$\iff 2a+2s<1$$

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Note Title 20/12/2021

Prop. Let us consider the Dirichlet Laplacian in a reasonable domain $\Omega \subseteq \mathbb{R}^d$.

Then

- → eigenvectors are of class C^∞
- → Their gradients are orthogonal in a suitable sense.

Proof. Eigenvectors are solutions to $\Delta u = \lambda u$

Elliptic regularity: $u \in L^2 \Rightarrow u \in H^2 \Rightarrow \text{RHS} \in H^2 \Rightarrow u \in H^4$

Bootstrap $\Rightarrow u \in H^k$ for every $k \in \mathbb{N}$

$\Rightarrow u \in C^\infty$ because weak deriv. are actually true derivatives when k is large enough.

Assume now that $\Delta u = \lambda u$

$\Delta v = \mu v$

Claim:

$$\boxed{\int_\Omega \langle \nabla u, \nabla v \rangle \, dx = 0}$$

This means that $\nabla u \perp \nabla v$ as elements of $L^2(\Omega, \mathbb{R}^d)$

Indeed if we integrate by parts:

$$\int_\Omega \langle \nabla u, \nabla v \rangle dx = -\int_\Omega u \Delta v \, dx = -\mu \underbrace{\int_\Omega u v \, dx}_{=0}$$

$= 0$ because eigenv. with diff. eigenvalues are orthogonal

— ○ — ○ —

Let us consider $(0,\pi)\times(0,\pi) \subseteq \mathbb{R}^2$

Which are eigenvectors / eigenvalues of − Dirichlet Laplacian

We know that $u(x,y) = \sin(nx)\sin(my)$

We observe that

$$\Delta u(x,y) = -(n^2+m^2)\,u(x,y)$$

So all these functions are eigenvectors with eigenvalue n^2+m^2

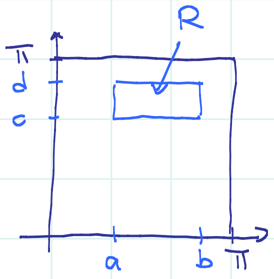
These are the only ones.

It is enough to show that they are a complete system, namely the span is dense in L^2.

There are at least two proofs.

→ Overkill : STONE − WEIERSTRASS ⇝ it is dense in cont. funct., which in turn are dense in L^2.

→ Elementary proof : it is enough to show that we can approximate some special functions that in turn span a dense subset of L^2



It is easy to approx $\mathbb{1}_R$

$g_n(x) \to \mathbb{1}_{(a,b)}(x)$

$f_n(y) \to \mathbb{1}_{(c,d)}(y)$

$$f_n(x)\cdot g_n(y) \longrightarrow \mathbb{1}_R(x,y)$$

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Now we know that

$$u \in H^1_0((0,\pi)\times(0,\pi)) \iff u = \sum_{n,m} u_{n,m}\sin(nx)\sin(my)$$

$$\text{with } \sum_{n,m}(n^2+m^2)\,u_{n,m}^2 < +\infty$$

$\boxed{\text{Theorem}}$ To every $u \in H^1_0((0,\pi)\times(0,\pi))$ let us associate the trace for $y = \frac{\pi}{2}$.

Then

$$u(x, \tfrac{\pi}{2}) \in H^{1/2}((0,\pi))$$

More important: the trace is surjective with values in $H^{1/2}$.

$\boxed{\text{Proof}}$ Let us set $g(x) := u(x, \frac{\pi}{2})$.

$\boxed{u \in H^1_0 \Rightarrow g \in H^{1/2}}$ We observe that

$$g(x) = \sum_{n,m} u_{n,m} \sin(nx) \underbrace{\sin(m\tfrac{\pi}{2})}_{\substack{(-1)^k \text{ if } m=2k+1 \\ 0 \quad \text{if } m=2k}}$$

$$= \sum_{n,k} u_{n,2k+1} (-1)^k \sin(nx)$$

$$= \sum_n g_n \sin(nx) \quad \text{where } g_n := \sum_k (-1)^k u_{n,2k+1}$$

Now $g \in H^{1/2} \Leftrightarrow \sum_n n g_n^2 < +\infty$

Observe that $g_n \le \left(\sum_k u^2_{n,2k+1}(n^2+(2k+1)^2)\right)^{1/2} \left(\sum_k \frac{1}{n^2+(2k+1)^2}\right)^{1/2}$

so that

$$n g_n^2 \le \underbrace{\boxed{n \sum_k \frac{1}{n^2+(2k+1)^2}}}_{\le M} \cdot \sum_k u^2_{n,2k+1}(n^2+(2k+1)^2)$$

$$\sim n \int_0^{+\infty} \frac{1}{n^2+x^2}\,dx$$

and therefore

$$\sum_n n g_n^2 \le M \sum_n \sum_k u_{n,2k+1}^2 \left(n^2+(2k+1)^2\right) < +\infty$$

$$\boxed{\text{For every } g \in H^{1/2}, \text{ there exists } u \in H^1 \text{ s.t. } g(x) = u(x, \tfrac{\pi}{2})}$$

We are given $g(x) = \sum_n g_n \sin(nx)$ with $\sum_n n g_n^2 < +\infty$

and we want to define $u_{n,m}$ such that

$$\sum_{n,m} u_{n,m}^2 (n^2+m^2) < +\infty \quad \rightsquigarrow u \in H^1 \qquad (*)$$

$$g_n = \sum_k (-1)^k u_{n,2k+1} \qquad (**)$$

A good choice is

$$u_{n,2k+1} := (-1)^k g_n \frac{1}{n^2+(2k+1)^2} \frac{1}{\left(\sum_k \frac{1}{n^2+(2k+1)^2}\right)}$$

In this way (**) is almost trivial. Let us check (*)

$$\sum_{n,k} u_{n,2k+1}^2 \left(n^2+(2k+1)^2\right)$$

$$= \sum_{n,k} g_n^2 \frac{1}{(n^2+(2k+1)^2)^{\cancel{2}}} \frac{1}{\left(\sum_k \dots\right)^2} \cancel{\left(n^2+(2k+1)^2\right)}$$

$$= \sum_n g_n^2 \frac{1}{\left(\sum_k \dots\right)^{\cancel{2}}} \cancel{\sum_{k=1}^{\infty} \frac{1}{n^2+(2k+1)^2}} = \sum_n g_n^2 \frac{1}{\underbrace{\left(\sum_k \dots\right)}_{\sim \frac{1}{n}}}$$

$$= \sum_n n g_n^2 < +\infty$$

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WHAT WOULD REMAIN TO DO

1. Evolution problems $u_t = \Delta u + \sin u$

   $u_{tt} = \Delta u + \sin u$

2. Bounded Variation Functions (extension of $W^{1,1}(\Omega)$ where things work)

3. Dual of C^0_c $\rightsquigarrow$ dual is a suitable space of measures

4. Spectral theorem for unbounded operators (and also for continuous operators that are not compact)

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