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Istituzioni di Analisi Matematica

Printout of lectures

(Volume 2)

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Istituzioni di Analisi — LECTURE 25

Note Title 25/10/2021

WEAK DERIVATIVES

Setting :
- → $d \geq 1$ (dimension)
- → $\Omega \subseteq \mathbb{R}^d$ open set (any open set)
- → $m \geq 1$ (order of derivation)
- → $p \in [1, +\infty]$ (endpoints included)

Multi-index : $\alpha \in \mathbb{N}^d$ $\quad \alpha = (\alpha_1, \dots, \alpha_d)$

$|\alpha| := \alpha_1 + \dots + \alpha_d$

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_d^{\alpha_d}}$$

W-weak derivatives Given $u \in L^1_{loc}(\Omega)$ and $v \in L^1_{loc}(\Omega)$ we say that v is $D^\alpha u$ in the W-sense if

$$\boxed{\int_\Omega u(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v(x) \varphi(x)\, dx}$$

for every test function $\varphi \in C^\infty_c(\Omega)$.

Rmk

(1) u and $v \in L^1_{loc}(\Omega)$ ⟹ the integrals are well-defined

(2) $v(x)$ plays the role of $D^\alpha u(x)$ in the integration by parts formula.

Easy properties

(1) Uniqueness $D^\alpha u$, if it exists, in unique

$$\int_\Omega v_1 \varphi \, dx = (-1)^\alpha \int_\Omega u \, D^\alpha \varphi \, dx = \int_\Omega v_2 \varphi \, dx$$

$$\Rightarrow \int_\Omega (v_1 - v_2) \varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty \; ; \text{ FLCV } \Rightarrow v_1 = v_2 \text{ alm. ev.}$$

(2) Linearity $D^\alpha (u_1 + u_2) = D^\alpha u_1 + D^\alpha u_2$

(3) Classical compatibility If $u \in C^{|\alpha|}(\Omega)$ then $D^\alpha u$ is what we expect

Less easy properties

(4) "Restriction" If $B \subseteq A$ are open sets, and $v = D^\alpha u$ in A in W-sense, then $v|_B$ is the W-$D^\alpha u|_B$

(5) "Locality". Assume that $u \in L^1(A \cup B)$.
Assume that $V_A = D^\alpha u$ in A
$V_B = D^\alpha u$ in B
$V_A = V_B$ in $A \cap B$

Then u admits $D^\alpha u$ in $A \cup B$ and it is what we expect.

(6) Assume that $u \in L^1_{loc}(\Omega)$ and $\dfrac{\partial u}{\partial x_i} = 0$ in W-sense for every $i = 1, \dots, d$ (all 1-st order derivatives)

Then $u(x)$ is locally constant in Ω.

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Stability when passing to limit

Assume $\{u_n\}$ and $\{v_n\}$ are seq. in $L^1_{loc}(\Omega)$.

Assume that

(i) $v_n = D^\alpha u_n$ for every $n \geq 1$

(ii) $u_n \longrightarrow u_\infty$ in $L^1_{loc}(\Omega)$

(iii) $v_n \longrightarrow v_\infty$ in $L^1_{loc}(\Omega)$

} it is enough that $u_n \to u_\infty$ and $v_n \to v_\infty$ weakly in $L^1(\Omega')$ for every $\Omega' \subset\subset \Omega$

Then

$$v_\infty = D^\alpha u_\infty$$

Proof

$$\int_\Omega u_n(x) D^\alpha \varphi(x)\,dx = (-1)^{|\alpha|} \int_\Omega v_n(x)\varphi(x)\,dx$$

$$\downarrow \qquad\qquad \downarrow$$

$$\int_\Omega u_\infty(x) D^\alpha \varphi(x)\,dx = (-1)^{\alpha} \int_\Omega v_\infty(x)\varphi(x)\,dx$$

(it is enough that $\Omega' \supseteq$ support of φ)

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H-weak derivatives Again v and u in $L^1_{loc}(\Omega)$.

We say that v is $D^\alpha u$ in H-sense if there exists a sequence

$$\{u_n\} \subseteq C^\infty_c(\Omega) \qquad (\text{or } C^\infty(\Omega) \text{ or } C^{|\alpha|}(\Omega))$$

such that

(i) $u_n \longrightarrow u$ in $L^1_{loc}(\Omega)$

(ii) $D^\alpha u_n \longrightarrow v$ in $L^1_{loc}(\Omega)$

↑ true derivatives

Easy proposition $H \Rightarrow W$

If $v = D^\alpha u$ in H-sense, then $v = D^\alpha u$ in W-sense.

Theorem (Approximation 0) ($W \Rightarrow H$ for weak derivatives)
If $v = D^\alpha u$ in W-sense, then $v = D^\alpha u$ in H-sense, namely there exists $\{u_n\} \subseteq C^\infty_c(\Omega)$ such that ...

Sobolev spaces, def. W

$$\boxed{W^{m,p}(\Omega) := \{ u \in L^p(\Omega) : \underset{\uparrow \text{W-weak derivatives}}{D^\alpha u} \in L^p(\Omega) \quad \forall |\alpha| \leq m \}}$$

Sobolev spaces, def. H

$H^{m,p}(\Omega) :=$ exactly the same, with H-weak derivatives

There is another different possibility

$$\|u\|_{\Omega,m,p} := \sum_{\underset{\text{also } (0,\dots,0) \text{ is included}}{\overset{\uparrow}{|\alpha| \leq m}}} \|D^\alpha u\|_{L^p(\Omega)} \qquad \text{FULL NORM}$$

$$C^{m,p}_{\infty}(\Omega) := \{ u \in C^m_{\infty}(\Omega) : \|u\|_{\Omega,m,p} < +\infty \}$$

It is not difficult to see that $\|u\|_{\Omega,m,p}$ is actually a norm on $C^{m,p}(\Omega)$.

At this point we can define $H^{m,p}(\Omega)$ as the abstract $\boxed{\text{completion}}$ of $C^{m,p}(\Omega)$.

Easy theorem $H^{m,p}(\Omega) \subseteq W^{m,p}(\Omega)$

$\boxed{\text{Difficult theorem}}$ (Approx. results for Sobolev functions)

Assume that Ω satisfies . . .

Assume that $u \in W^{m,p}(\Omega)$ with $\boxed{p < +\infty}$

Then there exists $\{u_n\} \subseteq \dots$

such that

$$u_n \to u \quad \text{in } L^p(\Omega)$$

and

$$D^\alpha u_n \to D^\alpha u \quad \text{in some sense}$$

for every $1 \le |\alpha| \le m$.

LOW COST	MEYERS-SERRIN $H=W$ (1964)	DE LUXE
any Ω	any Ω	$\partial\Omega$ "regular"
$u_n \in C^\infty_c(\mathbb{R}^d)$	$u_n \in C^\infty(\Omega)$	$u_n \in C^\infty_c(\mathbb{R}^d)$
$D^\alpha u_n \to D^\alpha u$ in $L^p(\Omega')$ for every $\Omega' \subset\subset \Omega$	$D^\alpha u_n \to D^\alpha u$ in $L^p(\Omega)$	$D^\alpha u_n \to D^\alpha u$ in $L^p(\Omega)$
"u_n is good"	u_n is rather good	u_n is good
conv. is "bad"	conv. in good	conv. is good
Proof is easy	Proof is long	Ω is NOT any open set.

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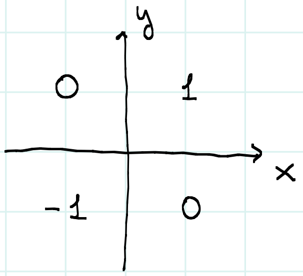
<u>Example</u> Consider $u(x,y)$ with 3 values as in the figure.



Then → $u_{xy} = 0$ in W-sense

→ u_x and u_y do NOT exist.

(at least in W sense)

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Istituzioni di Analisi — LECTURE 26

Note Title 25/10/2021

ALGEBRAIC THEOREMS

→ Product
→ composition 1 (sobolev inside)
→ Absolute value, max / min, truncation
→ Composition 2 (Sobolev outside)

Theorem (Product) Let us assume that

$$u \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \qquad v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$$

Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\boxed{D_{x_i}(uv) = D_{x_i} u \cdot v + u \cdot D_{x_i} v}$$

Rmk We need to assume u and v in L^∞ because otherwise it is not clear that $uv \in L^p$

(in dim 1 it is automatic that $W^{1,p} \Rightarrow L^\infty$, at least on intervals)

General strategy
- Take $u_m \to u$ and $v_m \to v$.
- Observe that the formula is true for every $m \in \mathbb{N}$
- Pass to the limit

For every $m \geq 1$ it is true that

$$\int_\Omega u_m v_m D_{x_i}\varphi \, dx = - \int_\Omega (u_m D_{x_i} v_m + D_{x_i} u_m \cdot v) \varphi \, dx$$

We observe that u_n and v_n can be taken to be equi $- L^\infty$.
We observe that it is enough to assume that derivatives converge in $L^p(\Omega')$ for a suitable $\Omega' \subset\subset \Omega$ that contains the support of φ.
Consequence: the low cost approx. result is enough.
In addition, L^p conv. $\Rightarrow$ a.e. convergence up to subsequences with L^p domination.
At this point we can pass to the limit in the integrals (pointwise conv. + domination).
This proves the result if $p < +\infty$.

$\boxed{\text{Case } p = +\infty}$ In this case we know that

$$\begin{matrix} u_n \to u \\ v_n \to v \end{matrix} \quad \text{in } L^{2021}(\Omega) \qquad \begin{matrix} D_{x_i} u_n \to D_{x_i} u \\ D_{x_i} v_n \to D_{x_i} v \end{matrix} \quad \text{in } L^{2021}(\Omega')$$

and therefore

$$u_n D_{x_i} v_n + v_n D_{x_i} u_n \to u D_{x_i} v + v D_{x_i} u \quad \text{in } L^{2021}(\Omega')$$

Moreover we know from the approx. theorem that $u_n, v_n, D_{x_i} u_n, D_{x_i} v_n$ are equi-bounded in $L^\infty(\Omega)$
This implies that $D_{x_i}(uv)$ is in L^∞.

<u>General fact</u> If $w_n \to w$ in $L^{2021}(\Omega)$ and $\|w_n\|_{L^\infty(\Omega)} \leq M$
then $\|w\|_{L^\infty(\Omega)} \leq M$.

(Because L^{2021} convergence is pointwise conv. up to subseq.).

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Theorem (Composition with Sobolev inside)

Let us assume that $u \in W^{1,p}(\Omega)$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that

(i) φ is of class C^1

(ii) $\varphi(0) = 0$ (not needed if Ω is bounded)

(iii) $|\varphi'(x)| \leq L$ for every $x \in \mathbb{R}$.

Then $\varphi(u) \in W^{1,p}(\Omega)$ and

$$\boxed{D_{x_i}[\varphi(u)] = \varphi'(u(x))\, D_{x_i} u(x)}$$

Proof Let us approximate $u_n \to u$ in the low-cost way. For every $n \geq 1$ and every $\psi \in C_c^\infty(\Omega)$

$$\int_\Omega \varphi(u_n(x)) D_{x_i} \psi(x)\, dx = -\int_\Omega \varphi'(u_n(x))\, D_{x_i} u_n(x)\, \psi(x)\, dx$$

Up to subsequences, the conv. is pointwise almost everywhere. We just need dominations. To this end, we observe that

$$\underset{\leq L}{|\varphi'(u_n(x))|} \cdot \underset{\uparrow\ L^p \text{ domination } \Rightarrow L^1 \text{ domination}}{|D_{x_i} u_n(x)|}$$

As for the LHS, we observe that

$$|\varphi(u_n(x))| = |\varphi(u_n(x)) - \varphi(0)| \underset{\substack{\uparrow \\ \varphi \text{ is} \\ L\text{-lip. cont.}}}{\leq} L\, \underset{\uparrow\ L^p \text{ domination}}{|u_n(x)|}$$

This is ok if $p < +\infty$. Otherwise we argue as before

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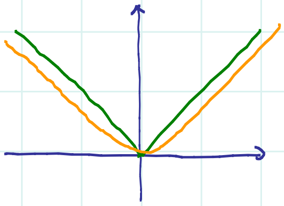
Theorem (Absolute value) Let us assume that $u \in W^{1,p}(\Omega)$. Then $|u| \in W^{1,p}(\Omega)$ and

$$\boxed{D_{x_i} |u| = \operatorname{sign}(u(x)) \cdot D_{x_i} u(x)}$$

Proof We approximate the absolute value by means of

$$\varphi_m(\sigma) = \sqrt{\sigma^2 + \frac{1}{m^2}} - \frac{1}{m}$$



We observe that $\varphi_m(\sigma) \to |\sigma|$ unif. in $\mathbb{R}$ and

$$\varphi_m'(\sigma) \to \begin{cases} +1 & \text{if } \sigma > 0 \\ -1 & \text{if } \sigma < 0 \end{cases}$$

Consider $u_m(x) := \varphi_m(u(x))$. Due to the previous thm (observe that $\varphi_m(0) = 0$ and $|\varphi_m'(\sigma)| \leq 1$) we know that $u_m \in W^{1,p}(\Omega)$ and

$$u_m'(x) = \varphi_m'(u(x)) D_{x_i} u(x)$$

We claim that $u_m(x) \to |u(x)|$ and $u_m'(x) \to \operatorname{sign}(u(x)) D_{x_i} u(x)$ in $L^p(\Omega)$.
This follows from the properties of φ_m and φ_m'.
Both cases: dominated pointwise convergence.

Rmk $\max\{a, b\} = \dfrac{a + b + |a-b|}{2}$ and analogously for min.

Therefore: if u and v are in $W^{1,p}$, then $\max\{u, v\}$ and $\min\{u, v\}$ are in $W^{1,p}$ and derivatives are what we expect.

$\rightsquigarrow$ truncation arguments.

Theorem (Composition – Sobolev outside)

Let A and B be open sets in $\mathbb{R}^d$. Let $u \in W^{1,p}(B)$.
Let

$$\Phi : A \to B$$

be a function such that

(i) Φ is of class C^1

(ii) Φ is invertible

(iii) J_Φ and $J_{\Phi^{-1}}$ are bounded (in A and B, respectively).

Consider the function

$$v(x) := u(\Phi(x)) \qquad x \in A$$

Then $v \in W^{1,p}(A)$ and

$$\frac{\partial v}{\partial x_i}(x) = \sum_{j=1}^{d} \frac{\partial u}{\partial y_j}(\Phi(x)) \cdot \frac{\partial \Phi_j}{\partial x_i}(x)$$

or even better

$$\boxed{\nabla v(x) = \nabla u(\Phi(x)) \cdot J_\Phi(x)}$$

(Assumption (i) and (ii) and (iii) provide the needed dominations).

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Istituzioni di Analisi — LECTURE 27

Note Title 27/10/2021

MOLLIFIERS AND CONVOLUTION

<u>Def</u> (Mollifier) A mollifier is a function $\rho : \mathbb{R}^d \to \mathbb{R}$ such that

(i) some regularity, for example $\rho \in C_c^\infty(\mathbb{R}^d)$

(ii) $\rho(x) \geq 0$ for every $x \in \mathbb{R}^d$

(iii) $\rho(x) = 0$ if $\|x\| \geq 1$

(iv) integral condition

$$\int_{\underset{\uparrow B(0,1)}{\mathbb{R}^d}} \rho(x)\,dx = 1$$

<u>Def</u> (convolution) Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a mollifier

Let $u \in L^1_{loc}(\mathbb{R}^d)$

Let $\varepsilon > 0$ be a real number

For every $x \in \mathbb{R}^d$ we define

$$\boxed{(u * \rho_\varepsilon)(x) := \int_{\underset{\underset{B(0,1)}{\uparrow}}{\mathbb{R}^d}} u(x+\varepsilon y)\,\rho(y)\,dy \underset{\underset{\begin{array}{c} z = x+\varepsilon y \\ y = \frac{z-x}{\varepsilon} \\ dy = \frac{1}{\varepsilon^d}dz \end{array}}{\uparrow}}{=} \frac{1}{\varepsilon^d}\int_{\mathbb{R}^d} u(z)\,\rho\left(\frac{z-x}{\varepsilon}\right)dz}$$

$\nwarrow B(x,\varepsilon)$

If we assume just that $u \in L^1(\Omega)$ with $\Omega \subseteq \mathbb{R}^d$ open, then we can consider

$$\hat{u} * \rho_\varepsilon \quad \leftarrow \text{defined for every } x \in \mathbb{R}^d$$

where

$$\hat{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{\text{Prop}}$ Properties of the convolution
Let us assume that ρ, u, ε are as in the def. of convolution.
Then the following statements are true

(1) $u * \rho_\varepsilon$ is of class C^∞ (if $\rho \in C^k$, then $u * \rho_\varepsilon \in C^k$ and with some extra work C^{k+1})

(2) $u \longrightarrow u * \rho_\varepsilon$ is linear

(3) Assume that support $(u) \subseteq \Omega$. Then

$$\operatorname{supp}(u * \rho_\varepsilon) \subseteq \operatorname{Clos}\Big(\bigcup_{x \in \Omega} B(x, \varepsilon)\Big)$$

(4) If $u \in L^p(\mathbb{R}^d)$, then $u * \rho_\varepsilon \in L^p(\mathbb{R}^d)$ and

$$\| u * \rho_\varepsilon \|_{L^p(\mathbb{R}^d)} \leq \| u \|_{L^p(\mathbb{R}^d)}$$

This is true for every $p \in [1, +\infty]$ ↖ included

(5) If $u \in L^p(\mathbb{R}^d)$ for some $p \in [1, +\infty)$ ↖ excluded then

$$u * \rho_\varepsilon \longrightarrow u \quad \text{in } L^p(\mathbb{R}^d)$$

Rmk No hope that (5) is true with $p = +\infty$
(consider $\mathbb{1}_{(a,b)}$ in $\mathbb{R}$).

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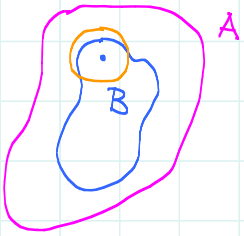
Exercise Prove the properties.

Proposition (The convolution commutes with w-weak derivatives)

Let $B \subset\subset A \subseteq \mathbb{R}^d$ be open sets.

Assume $u \in L^1(A)$ and $\underset{\uparrow \text{ w- weak derivative}}{D^\alpha u} = v \in L^1(A)$



Assume that

$$0 < \varepsilon < \underbrace{\operatorname{dist}(B, \partial A)}_{\text{No cond. if } A = \mathbb{R}^d}$$

and therefore $B(x,\varepsilon) \subset\subset A$ for every $x \in B$.

Then
$$\boxed{D^\alpha[\hat{u} * \rho_\varepsilon](x) = (\hat{v} * \rho_\varepsilon)(x) \qquad \forall x \in B}$$

Proof $(\hat{u} * \rho_\varepsilon)(x) = \frac{1}{\varepsilon^d}\int_{\mathbb{R}^d} \hat{u}(z)\, \rho\left(\frac{z-x}{\varepsilon}\right) dz$

All $\hat{u}(z)$ are actually $u(z)$

$$D^\alpha(\hat{u} * \rho_\varepsilon)(x) = \frac{1}{\varepsilon^d}\int_{\substack{\mathbb{R}^d \\ \uparrow B(x,\varepsilon)}} \hat{u}(z)\, D^\alpha_x\left[\rho\left(\frac{z-x}{\varepsilon}\right)\right] dz$$

$$\underset{\substack{\nearrow \\ \text{chain rule}}}{=} \frac{1}{\varepsilon^d}\int_{\substack{\mathbb{R}^d \\ \uparrow B(x,\varepsilon)}} \hat{u}(z)\, [D^\alpha \rho]\left(\frac{z-x}{\varepsilon}\right) \frac{(-1)^{|\alpha|}}{\varepsilon^{|\alpha|}}\, dz$$

$$\underset{\substack{\nearrow \\ \text{chain rule}}}{=} \frac{(-1)^{|\alpha|}}{\varepsilon^d}\int_{\substack{\mathbb{R}^d \\ \uparrow \\ B(x,\varepsilon) \subseteq A}} \hat{u}(z)\, D^\alpha_z \underbrace{\left[\rho\left(\frac{z-x}{\varepsilon}\right)\right]}_{\substack{\text{class } C^\infty_c(A) \\ \neq 0 \text{ only in } B(x,\varepsilon) \subset\subset A}} dz$$

$$\underset{\substack{\nearrow \\ v = D^\alpha u \text{ in } A}}{=} \frac{1}{\varepsilon^d}\int_{\substack{\mathbb{R}^d \\ \uparrow B(x,\varepsilon) \subseteq A \text{ and } v = \hat{v} \text{ in } A}} v(z)\, \rho\left(\frac{z-x}{\varepsilon}\right) dz$$

$$= (\hat{v} * \rho_\varepsilon)(x).$$

Provided that $x \in B$

$\boxed{\text{Low-cost approx thm}}$ Assume $u \in W^{m,p}(\Omega)$. $(p < +\infty)$

Then there exists $\{u_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ s.t.

$$u_n \to u \quad \text{in } L^p(\Omega)$$

$$D^\alpha u_n \to D^\alpha u \quad \text{in } L^p(\Omega') \text{ for every } \Omega' \subset\subset \Omega$$

and every $1 \leq |\alpha| \leq m$

Proof Let us consider $u_\varepsilon := \hat{u} * \rho_\varepsilon$

Then $u_\varepsilon \in C^\infty(\mathbb{R}^d)$ (also $C_c^\infty(\mathbb{R}^d)$ if Ω is bounded)

Moreover

$$u_\varepsilon \to \hat{u} \quad \text{in } L^p(\mathbb{R}^d)$$

and therefore

$$u_\varepsilon \to \hat{u} = u \quad \text{in } L^p(\Omega)$$

Finally $\widehat{(D^\alpha u)} * \rho_\varepsilon \to \widehat{D^\alpha u}$ in $L^p(\mathbb{R}^d)$

and therefore

$$(D^\alpha u) * \rho_\varepsilon \to D^\alpha u \quad \text{in } L^p(\Omega')$$

$\parallel$ ← true only in Ω' and if ε is small enough

$D^\alpha(u * \rho_\varepsilon)$

It remains to modify u_ε in order to have the compact support.

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Theorem (W ⇒ H for weak derivatives)

Let $u \in L^1_{loc}(\Omega)$ and let $v := D^\alpha u \in L^1_{loc}(\Omega)$.

Then there exists $\{u_n\} \subseteq C_c^\infty(\Omega)$ such that

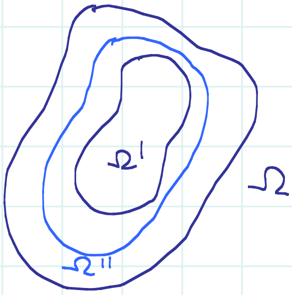
$u_n \to u$ in $L^1_{loc}(\Omega)$ (α is fixed)

$D^\alpha u_n \to v$ in $L^1_{loc}(\Omega)$

$\boxed{\text{Proof}}$ Let us fix $\Omega' \subset\subset \Omega$ and let us satisfy Ω'

Let us choose $\Omega' \subset\subset \Omega'' \subset\subset \Omega$

(prove that it DOES exist)



Define $u_\varepsilon := \underset{\substack{\uparrow \\ \text{extension to } 0 \\ \text{outside } \Omega''}}{\hat{u}} * \rho_\varepsilon$

As in the previous proof we can show that

$$u_\varepsilon \to u \quad \text{in } L^p(\Omega'') \text{ and also in } L^p(\Omega')$$

$$D^\alpha u_\varepsilon \to D^\alpha u \quad \text{in } L^p(\Omega')$$

How to satisfy any Ω'. For every $k \geq 1$ consider

$$\Omega_k := \{ x \in \Omega : \|x\| < k \text{ and } \operatorname{dist}(x, \partial\Omega) > \tfrac{1}{k} \}$$

It is easy to see that $\bigcup_{k\geq 1} \Omega_k = \Omega$ and for every $\Omega' \subset\subset \Omega$

it turns out that $\Omega' \subset\subset \Omega_k$ eventually.

For every $k \geq 1$ we know how to satisfy Ω_k, in particular there exists v_k s.t.

$$\|u - v_k\|_{L^p(\Omega_k)} \leq \frac{1}{k}$$

$$\|D^\alpha u - D^\alpha v_k\|_{L^p(\Omega_k)} \leq \frac{1}{k}$$

it is enough that it goes to 0

I claim that $\{v_k\}$ satisfies any Ω'. Indeed eventually it is true that

$$\|u - v_k\|_{L^p(\Omega')} \underset{\substack{\uparrow \\ \Omega' \subset\subset \Omega_k}}{\leq} \|u - v_k\|_{L^p(\Omega_k)} \leq \frac{1}{k} \quad \text{and the same for derivatives}$$

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Istituzioni di Analisi — LECTURE 28

Note Title 27/10/2021

$\boxed{\text{Prop}}$ (Sobolev · smooth = Sobolev)

Assume that $u \in W^{m,p}(\Omega)$ and $\varphi \in C^\infty_c(\Omega)$.

Then $\varphi u \in W^{m,p}(\Omega)$.

<u>Rmk</u> There is a "nice" formula for derivatives (general Leibniz rule)

$$D^\alpha [\varphi u] = \sum_{\beta \leq \alpha} \underset{\uparrow \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_d}{\beta_d}}{\binom{\alpha}{\beta}} D^\beta \varphi \, D^{\alpha-\beta} u$$

$\boxed{\text{Prop}}$ (Type A partition of the unity)

Let $A \subseteq \mathbb{R}^d$ be an open set, and let $\{A_n\}$ be a family of open sets such that

(i) $A_n \subset\subset A$ for every $n \geq 1$

(ii) it is a covering of A, namely

$$\bigcup_{n\geq 1} A_n = A$$

(iii) the covering is LOCALLY FINITE, namely for every compact set $K \subseteq A$ the set

$$\{ n \geq 1 : A_n \cap K \neq \emptyset \} \quad \text{is finite}$$

Then there exists $\{\varphi_n\}$ such that

(1) $\varphi_n \in C^\infty_c(A_n)$

(2) $\varphi_n(x) \geq 0$ for every $n \geq 1$ and every $x \in A_n$ ($x \in \mathbb{R}^d$)

(3)
$$\sum_{n=1}^\infty \varphi_n(x) = 1 \qquad \forall x \in A$$

↑ for every $x \in A$ this is a finite sum because of (iii)

$\boxed{\text{Theorem}}$ (H = W by Meyers – Serrin)

Assume that $u \in W^{m,p}(\Omega)$ with $\boxed{p < +\infty}$

Then there exists $\{u_n\} \subseteq C^{\infty}(\Omega)$ such that

↑ without c

$$u_n \to u \text{ in } L^p(\Omega)$$
$$D^{\alpha} u_n \to D^{\alpha} u \text{ in } L^p(\Omega) \text{ for every } |\alpha| \leq m.$$

$\boxed{\text{Proof}}$ The result is now trivial if $\text{supp}(u) \subset\subset \Omega$.
In this case it is enough to consider the low cost result with $\text{supp}(u) \subset\subset \Omega' \subset\subset \Omega$.

Let us consider now the general case. The goal is the following

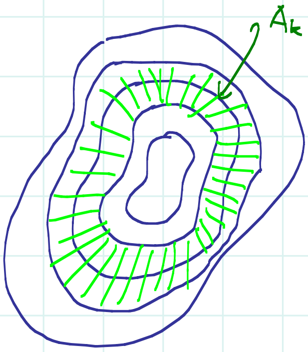
$$\forall \varepsilon > 0 \quad \exists \, v_\varepsilon \in C^{\infty}(\Omega) \quad \text{s.t.} \quad \underset{\uparrow \text{ full norm}}{\| u - v_\varepsilon \|_{m,p,\Omega}} \leq \varepsilon$$

In order to construct v_ε, we consider Ω_k as before

$$\Omega_k := \{ x \in \Omega : \| x \| < k, \ \text{dist}(x, \partial\Omega) > \tfrac{1}{k} \} \qquad k \geq 1$$

We set $\Omega_0 := \emptyset$ and

$$A_k := \Omega_{k+1} \setminus \text{clos}(\Omega_{k-1}) \qquad \forall \, k \geq 1$$



We observe that $\{A_k\}$ is an open covering of A as in the part. of unity of type A.
(easy check)
Let $\{\psi_k\}$ denote the corresponding part. of the unity, and let us set

$$u_k := \psi_k \cdot u$$

Properties of u_k.

- $\text{supp}(u_k) \subset\subset A_k$
- $u_k \in W^{m,p}(\Omega)$ because it is Sobolev · smooth
-

$$\sum_{k=1}^{\infty} u_k(x) = u(x) \qquad \text{because} \quad \sum \psi_k(x) = 1$$

↑ this is a finite sum for every fixed $x \in \Omega$

For every $k \geq 1$, there exists $v_k \in C_c^\infty(A_k)$ such that

$$\| u_k - v_k \|_{m,p,\Omega} = \| u_k - v_k \|_{m,p,A_k} \leq \frac{\varepsilon}{2^k}$$

(it is enough to choose $v_k = u_k * \rho_\delta$ for δ small enough)

Finally define

$$v_\varepsilon(x) := \sum_{k=1}^{\infty} v_k(x) \qquad \forall x \in \Omega$$

↑ finite sum for every fixed $x \in \Omega$

We claim that it works. Indeed

$$\| u - v_\varepsilon \|_{m,p,\Omega} = \Big\| \sum_{k=1}^{\infty} u_k - v_k \Big\|_{m,p,\Omega}$$

$$\leq \sum_{k=1}^{\infty} \| u_k - v_k \|_{m,p,\Omega}$$

$$\leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

Why is $v_\varepsilon \in C^\infty(\Omega)$? Because in every fixed BALL $B(x,r) \subset\subset \Omega$ the sum IS FINITE, and therefore $v_\varepsilon \in C^\infty(B(x,r))$ and C^∞ is a local property

— o — o —

$\boxed{\text{Proof of smooth. Sobolev}}$ Induction on m

$\boxed{m=1}$ Claim: $\psi u \in W^{1,p}(\Omega)$ and $D_{x_i}[\psi u] = D_{x_i}\psi \cdot u + \psi \cdot D_{x_i} u$

$$\int_\Omega [\psi u] D_{x_i} \underset{\substack{\uparrow \\ \text{test function}}}{\varphi}\, dx = \int_\Omega u\, [\psi D_{x_i}\varphi]\, dx$$

$$= \int_\Omega u \{ D_{x_i}[\psi\varphi] - D_{x_i}\psi \cdot \varphi \}\, dx$$

$$= \int_\Omega u\, D_{x_i}[\underbrace{\psi\varphi}_{\text{new test function}}]\, dx - \int_\Omega u\, D_{x_i}\psi\, \varphi\, dx$$

$$= -\int_\Omega D_{x_i}u \cdot \psi\varphi\, dx - \int_\Omega u\, D_{x_i}\psi\, \varphi\, dx$$

$$= -\int_\Omega \{ D_{x_i}u \cdot \psi + u \cdot D_{x_i}\psi \} \varphi\, dx \qquad ☺$$

$\boxed{m \Rightarrow m+1}$ Let us consider α with $|\alpha| = m+1$. Let us write

$$\alpha = \beta + (0, \dots, 0, \underset{\substack{\uparrow \\ \text{pos. } i}}{1}, 0, \dots, 0)$$

Now

$$D^\alpha[\psi u] = D^\beta D_{x_i}[\psi u] = D^\beta \Big(\underset{\substack{\uparrow \\ \text{smooth}}}{\underline{D_{x_i}\psi}} \cdot \underset{\substack{\uparrow \\ W^{m,p}}}{u} + \underset{\substack{\uparrow \\ \text{smooth}}}{\psi} \underset{\substack{\uparrow \\ W^{m,p}}}{D_{x_i} u} \Big)$$

so the result follows by the inductive hypothesis because $|\beta| = m$.

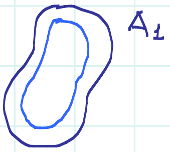
— o — o —

$\boxed{\text{Lemma}}$ Let $\{A_k\}$ be an open covering of A as in the Prop. Then there exists $\{\hat{A}_k\}$ a new covering as in the Prop. with

$$\hat{A}_k \subset\subset A_k \qquad \forall\, k \geq 1$$

"We can take it smaller"

$\boxed{\text{Proof}}$ I want to replace A_1 by



$$A_{1,m} := \{ x \in A_1 \ : \ \text{dist}\,(x, \partial A_1) > \tfrac{1}{m} \}$$

Claim: if m is large enough, I can replace A_1 by $\hat{A}_1 := A_{1,m}$. I have to check that

$$A_{1,m} \cup A_2 \cup A_3 \cup \dots \quad = A$$

To this end, it is enough to check that

$$\underbrace{A_{1,m} \cup A_2 \cup A_3 \cup \dots}_{B_m} \ \supseteq \ \text{Clos}\,(A_1)$$

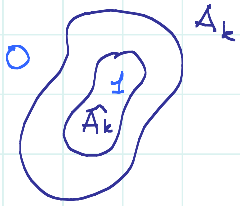
$\uparrow$ clos is not really needed

Just observe that B_m is an open covering of $\text{Clos}\,(A_1)$, which is a compact set. Therefore the conclusion follows because B_m's are increasing.

In the same way, I can define $\hat{A}_2, \hat{A}_3, \dots$

We need to check that $\bigcup_{k \geq 1} \hat{A}_k = A$. Let $x \in A$. Then x belongs to a finite number of A_k, so if x "survives" after the first k steps, then it survives forever.

Proof of part. of unity



Let us consider $\{\hat{A}_k\}$ as in the Lemma.
For each $k \geq 1$ we can find $\vartheta_k(x)$ s.t

- $\vartheta_k \in C_c^\infty(A_k)$
- $0 \leq \vartheta_k \leq 1$ in A_k
- $\vartheta_k(x) = 1$ for every $x \in \hat{A}_k$.

Define $\quad S(x) : \sum_{k \geq 1} \vartheta_k(x) \quad$ (>0 in A because $\{\hat{A}_k\}$ is a covering)

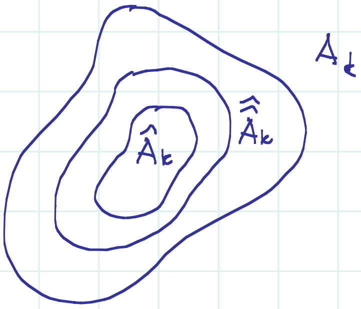
and finally $\quad \psi_k(x) := \dfrac{\vartheta_k(x)}{S(x)}$

Note that $S(x) \in C^\infty$ because the sum is locally finite

How to find $\vartheta_k(x)$?
Take

$$\hat{A}_k \subset\subset \hat{\hat{A}}_k \subset\subset A_k$$

and then define



$$\vartheta_k = \mathbb{1}_{\hat{\hat{A}}_k} * \rho_\varepsilon$$ with ε small enough

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Istituzioni di Analisi — LECTURE 29

Note Title 28/10/2021

SOBOLEV EMBEDDINGS

BASIC CASE : WHOLE $\mathbb{R}^d$, $m=1$

<u>Theorem</u> Let us assume that $u \in W^{1,p}(\mathbb{R}^d)$. $\frac{1}{p} - \frac{1}{p_*} = \frac{1}{d}$

$\boxed{\text{Case } p<d}$ Then $u \in L^{p_*}(\mathbb{R}^d)$ where $p_* := \frac{pd}{d-p}$ $(>p)$ and

$$\boxed{\|u\|_{L^{p_*}(\mathbb{R}^d)} \leq c(d,p)\, \|\nabla u\|_{L^p(\mathbb{R}^d)}}$$

PURE ESTIMATE: only ∇u in RHS

As a consequence, $u \in L^q(\mathbb{R}^d)$ for every $q \in [p, p_*]$ and

$$\boxed{\|u\|_{L^q(\mathbb{R}^d)} \leq c(d,p,q)\, \|u\|_{1,p,\mathbb{R}^d}}$$

IMPURE ESTIMATE: full norm in RHS

$\boxed{\text{Case } p=d}$ Then $u \in L^q(\mathbb{R}^d)$ for every $q \in [p, +\infty)$ and

Excluded

$$\boxed{\|u\|_{L^q(\mathbb{R}^d)} \leq c(d,p,q)\, \|u\|_{1,p,\mathbb{R}^d}}$$

equal

$\boxed{\text{Case } p>d}$ Then $u \in L^\infty(\mathbb{R}^d)$ and

$$\boxed{\|u\|_{L^\infty(\mathbb{R}^d)} \leq c(d,p)\, \|u\|_{1,p,\mathbb{R}^d}}$$

In addition, u is $\boxed{1-\frac{d}{p}}$ $\in (0,1]$ Hölder continuous and

$$\boxed{|u(y)-u(x)| \leq \underbrace{c(d,p)\, \|\nabla u\|_{L^p(\mathbb{R}^d)}}_{\text{Hölder constant}} |y-x|^{1-\frac{d}{p}}}$$

PURE ESTIMATE $\forall\, x,y$ in $\mathbb{R}^d$

ITERATED BASIC CASE : WHOLE $\mathbb{R}^d$, ANY m

Theorem Assume that $u \in W^{m,p}(\mathbb{R}^d)$.

$\frac{1}{p} - \frac{1}{p_*} = \frac{m}{d}$

Case 1: $mp<d$ In this case $u \in L^{p_*(m)}(\mathbb{R}^d)$ where and

$$p_*(m) = \frac{dp}{d-pm}$$

$$\boxed{\|u\|_{L^{p_*(m)}(\mathbb{R}^d)} \le c(d,p,m) \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)}}$$

As a consequence, $u \in L^q(\mathbb{R}^d)$ for all $q \in [p, p_*(m)]$ and

$$\boxed{\|u\|_{L^q(\mathbb{R}^d)} \le c(d,p,q,m) \underbrace{\sum_{|\alpha|\le m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)}}_{\|u\|_{m,p,\mathbb{R}^d}}}$$

Case 2 : $mp=d$ In this case $u \in L^q(\mathbb{R}^d)$ for every $q \in [p,+\infty)$ with impure estimate

Case 3 : $mp>d$ First of all, $u \in L^\infty(\mathbb{R}^d)$ with impure estimate. Moreover, let k be the smallest integer such that $kp>d$. Then $u \in C^{m-k}(\mathbb{R}^d)$ and derivatives of order $m-k$ are Hölder cont. with Hölder-exponent $1-\frac{d}{kp}$ and

$$\boxed{|D^\beta u(x) - D^\beta u(y)| \le c(d,m,p) \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)} \, |x-y|^{1-\frac{d}{kp}}}$$

for every x and y in $\mathbb{R}^d$.

Proof Induction using previous statement.

Lecture 29

MORE GENERAL CASE : $m=1$ and $\Omega \subseteq \mathbb{R}^d$ open set

Theorem Let us assume that Ω is smooth enough and let $u \in W^{1,p}(\Omega)$.

$\frac{pd}{d-p}$

Case $p<d$ Then $u \in L^q(\Omega)$ for every $q \in [p, p_*]$ and

$$\boxed{\|u\|_{L^q(\Omega)} \leq c(d,p,q,\Omega)\, \|u\|_{1,p,\Omega}}$$

always the full norm, even if $q = p_*$

Case $p=d$ $u \in L^q(\Omega)$ for every $q \in [p, +\infty)$ with impure est.

Case $p>d$ $u \in L^\infty(\Omega)$ and $1-\frac{d}{p}$ Hölder continuous, with impure estimates.

COMPETELY GENERAL CASE

Assume again that Ω is smooth enough. Assume $u \in W^{m,p}(\Omega)$. Then conclusions are the same as in the $\mathbb{R}^d$ case, but without pure estimates.

All constants DO depend also on Ω.

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EXTENSION PROBLEM

Def Let $\Omega \subseteq \mathbb{R}^d$ be an open set.

① A $\boxed{(m,p)\text{-extender}}$ is a map

$$E_{m,p} : W^{m,p}(\Omega) \longrightarrow W^{m,p}(\mathbb{R}^d)$$

such that

(i) $[E_{m,p}(u)](x) = u(x)$ for almost every $x \in \Omega$

(ii) $E_{m,p}$ is linear

(iii) we have a "norm control"

$$\| E_{m,p} u \|_{m,p,\mathbb{R}^d} \leq c(m,p,\Omega) \, \| u \|_{m,p,\Omega}$$

② A $\boxed{m\text{-extender}}$ is a map $E_m : L^1_{loc}(\Omega) \to L^1_{loc}(\mathbb{R}^d)$ such that the restriction of E_m to $W^{m,p}(\Omega)$ is a (m,p) extender for every $p \in [1,+\infty]$.

③ A $\boxed{\text{universal extender}}$ is a map $E_{univ} : L^1_{loc}(\Omega) \to L^1_{loc}(\mathbb{R}^d)$ which is good for every $m \geq 1$ and every $p \in [1,+\infty]$.

— o — o —

First application If Ω admits a $(1,p)$ extender, then Sobolev embeddings hold true in Ω.

Proof Let us set $\hat{u} := E_{1,p} u$. Then

$$\| u \|_{L^{p^*}(\Omega)} \underset{\substack{\uparrow \\ u = \hat{u} \\ \text{in } \Omega}}{=} \| \hat{u} \|_{L^{p^*}(\Omega)} \leq \| \hat{u} \|_{L^{p^*}(\mathbb{R}^d)} \overset{\substack{\text{Sobolev in } \mathbb{R}^d \\ \swarrow}}{\leq} c(d,p) \, \| \nabla \hat{u} \|_{L^p(\mathbb{R}^d)}$$

$$\underset{\substack{\uparrow \\ \text{property of the extender}}}{\leq} c(d,p,\Omega) \, \| u \|_{1,p,\Omega}$$

$\boxed{\text{Very important point}}$ If we want embeddings for $W^{m,p}(\Omega)$ it is enough to have $(1,p)$- extenders

$\boxed{\text{Second application}}$ If Ω admits an (m,p) extender, then the deluxe approx. holds true in $W^{m,p}(\Omega)$

$\boxed{\text{Proof}}$ Let $\hat{u} = E_{m,p}(u) \in W^{m,p}(\mathbb{R}^d)$.

Assume that we can approximate $\hat{u}$ in $\mathbb{R}^d$ in a deluxe way

$$\underset{\substack{\uparrow \\ \in C_c^\infty(\mathbb{R}^d)}}{D^\alpha \hat{u}_n} \longrightarrow D^\alpha \hat{u} \quad \text{in } L^p(\mathbb{R}^d) \text{ for every } |\alpha| \leq m$$

Then the converge is true a fortiori in $L^p(\Omega)$.

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Istituzioni di Analisi — LECTURE 30

Note Title 28/10/2021

GAGLIARDO INEQUALITY (1959) Special case of BRASCAMP-LIEB ineq.

Setting: we are in $\mathbb{R}^d$ and we consider d functions

$$\varphi_i \in C_c^0(\mathbb{R}^{d-1})$$

Let

$$P_i : \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1}$$

be the projections that "forgets the i-th component".

Define

$$\varphi(x) := \prod_{i=1}^{d} \varphi_i(P_i x) \qquad x \in \mathbb{R}^d$$

For example:

$$d=2 \qquad \varphi(x,y) = \varphi_1(y)\,\varphi_2(x)$$

$$d=3 \qquad \varphi(x,y,z) = \varphi_1(y,z)\,\varphi_2(x,z)\,\varphi_3(x,y)$$

(General BL ineq. $\varphi(x) := \prod_{i=1}^{k} \varphi_i(L_i x)$ $\quad L_i : \mathbb{R}^d \longrightarrow \mathbb{R}^{m_i}$ linear with $m_i < d$)

Theorem

$$\boxed{\|\varphi\|_{L^1(\mathbb{R}^d)} \leq \prod_{i=1}^{d} \|\varphi_i\|_{L^{d-1}(\mathbb{R}^{d-1})}}$$

Proof Induction on d. We can assume that $\varphi_i \geq 0$.

$\boxed{d=2}$ $\displaystyle\int_{\mathbb{R}^2} \varphi_1(y)\,\varphi_2(x)\,dx\,dy = \left(\int_{\mathbb{R}} \varphi_2(x)\,dx\right)\cdot\left(\int_{\mathbb{R}} \varphi_1(y)\,dy\right)$ ☺

$$\boxed{3=4} \quad \varphi(x,y,z,w) = a(y,z,w)\, b(x,z,w)\, c(x,y,w)\, d(x,y,z)$$

$$\int_{\mathbb{R}^4} \varphi(x,y,z,w)\, dx\, dy\, dz\, dw$$

$$= \int_{\mathbb{R}^3} \underset{\frac{1}{3}}{d(x,y,z)} \underbrace{\left\{ \int_{\mathbb{R}} a(y,z,w)\, b(x,z,w)\, c(x,y,w)\, dw \right\}}_{\underset{\frac{2}{3}}{\Phi(x,y,z)}} dx\, dy\, dz$$

$$\le \underset{\|d\|_{L^3(\mathbb{R}^3)}}{\underset{\shortparallel}{\left\{ \int_{\mathbb{R}^3} d(x,y,z)^3\, dx\, dy\, dz \right\}^{\frac{1}{3}}}} \left\{ \int_{\mathbb{R}^3} \Phi(x,y,z)^{3/2}\, dx\, dy\, dz \right\}^{2/3}$$

Let us estimate $\Phi(x,y,z)$:

$$\Phi(x,y,z) = \int_{\mathbb{R}} \underset{\frac{1}{3}}{a(y,z,w)}\, \underset{\frac{1}{3}}{b(x,z,w)}\, \underset{\frac{1}{3}}{c(x,y,w)}\, dw$$

$$\le \left\{ \int_{\mathbb{R}} a(y,z,w)^3\, dw \right\}^{1/3} \left\{ \int_{\mathbb{R}} b(\ldots)^3\, dw \right\}^{1/3} \left\{ \int_{\mathbb{R}} c(\ldots)^3\, dw \right\}^{1/3}$$

$$\Phi(x,y,z)^{3/2} \le \underbrace{\left\{ \int_{\mathbb{R}} a(y,z,w)^3\, dw \right\}^{1/2}}_{A(y,z)} \underbrace{\left\{ \int_{\mathbb{R}} \ldots \right\}^{1/2}}_{B(x,z)} \cdot \underbrace{\left\{ \int_{\mathbb{R}} \ldots \right\}^{1/2}}_{C(x,y)}$$

$$\int_{\mathbb{R}^3} \Phi(x,y,z)^{3/2}\, dx\, dy\, dz \le \int_{\mathbb{R}^3} A(y,z)\, B(x,z)\, C(x,y)\, dx\, dy\, dz$$

$$\underset{\text{inductive hyp} \nearrow}{\le} \left\{ \int_{\mathbb{R}^2} A(y,z)^2\, dy\, dz \right\}^{1/2} \left\{ \int_{\mathbb{R}^2} B \ldots \right\}^{1/2} \left\{ \int_{\mathbb{R}^2} C \ldots \right\}^{1/2}$$

Finally, let us compute any of the three terms

$$\int_{\mathbb{R}^2} A(y,z)^2\,dy\,dz = \int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}} a(y,z,w)^3\,dw\right\}dy\,dz$$

$$= \int_{\mathbb{R}^3} a(y,z,w)^3\,dy\,dz\,dw$$

and therefore

$$\left\{\int_{\mathbb{R}^2} A(y,z)^2\,dy\,dz\right\}^{1/2} = \|a\|^{3/2}_{L^3(\mathbb{R}^3)}$$

↙ cancelled by the 2/3 in the integral of Φ

— o — o —

$\boxed{d \Rightarrow d+1}$ We follow the same idea

Notation $x \in \mathbb{R}^{d-1}$, $\hat{x}_i$ = x without i-th comp.

$\hat{x}_{i,d+1}$ = x without comp. #i and #$d+1$

$$\int_{\mathbb{R}^{d+1}} \prod_{i=1}^{d+1} \varphi_i(\hat{x}_i)\,dx = \int_{\mathbb{R}^d} \varphi_{d+1}(\hat{x}_{d+1}) \underbrace{\left\{\int_{\mathbb{R}} \prod_{i=1}^{d} \varphi_i(\hat{x}_i)\,dx_{d+1}\right\}}_{\Phi(\hat{x}_{d+1})} d\hat{x}_{d+1}$$

$$\frac{1}{d} \qquad\qquad \frac{d-1}{d}$$

$$\le \left\{\int_{\mathbb{R}^d} \varphi_{d+1}(\hat{x}_{d+1})^d\,d\hat{x}_{d+1}\right\}^{1/d} \cdot \left\{\int_{\mathbb{R}^d} \Phi(\hat{x}_{d+1})^{\frac{d}{d-1}}\,d\hat{x}_{d+1}\right\}^{\frac{d-1}{d}}$$

$$= \|\varphi_{d+1}\|_{L^d(\mathbb{R}^d)}$$

$$\Phi(\hat{x}_{d+1}) = \int_{\mathbb{R}} \underset{\frac{1}{d}}{\varphi_1(\hat{x}_1)} \cdot \underset{\cdots}{\cdots} \cdot \underset{\frac{1}{d}}{\varphi_d(\hat{x}_d)}\,dx_{d+1}$$

$$\le \prod_{i=1}^{d} \left\{\int_{\mathbb{R}} \varphi_i(\hat{x}_i)^d\,dx_{d+1}\right\}^{\frac{1}{d}}$$

and therefore

$$\Phi(\hat{x}_{d+1})^{\frac{d}{d-1}} \le \prod_{i=1}^{d} \underbrace{\boxed{\left\{ \int_{\mathbb{R}} \varphi_i(\hat{x}_i)^d \, dx_{d+1} \right\}^{\frac{1}{d-1}}}}_{\Phi_i(\hat{x}_{i,d+1})}$$

$$\int_{\mathbb{R}^d} \Phi(\hat{x}_{d+1})^{\frac{d}{d-1}} \, d\hat{x}_{d+1} \le \int_{\mathbb{R}^d} \underbrace{\prod_{i=1}^{d} \Phi_i(\hat{x}_{i,d+1})}_{} \, d\hat{x}_{d+1}$$

function in $\mathbb{R}^d$ which is the product of d functions in $\mathbb{R}^{d-1}$
⇝ we can apply the inductive hyp

$$\le \prod_{i=1}^{d} \| \Phi_i(\hat{x}_{i,d+1}) \|_{L^{d-1}(\mathbb{R}^{d-1})}$$

Now it is enough to check that

$$\| \Phi_i(\hat{x}_{i,d+1}) \|_{L^{d-1}(\mathbb{R}^{d-1})} = \| \varphi_i \|_{L^d(\mathbb{R}^d)}^{\frac{d}{d-1}}$$

— o — o —

$\boxed{\text{Homothety argument}}$ If a pure estimate is true

$$\| u \|_{L^q(\mathbb{R}^d)} \le c(p,d) \, \| \nabla u \|_{L^p(\mathbb{R}^d)}$$

then necessarily $q = p_*$

Proof Assume the inequality is true for every $u \in C_c^\infty(\mathbb{R}^d)$. Then, for every $\lambda > 0$, it is true for $v(x) := u(\lambda x)$

$$\| v \|_{L^q(\mathbb{R}^d)} = \left\{ \int_{\mathbb{R}^d} u(\underset{\overset{\shortparallel}{y}}{\lambda x})^q \, dx \right\}^{1/q} = \left\{ \int_{\mathbb{R}^d} u(y)^q \, \frac{1}{\lambda^d} \, dy \right\}^{\frac{1}{q}}$$

$dy = \lambda^d dx$

$$= \boxed{\frac{1}{\lambda^{d/q}}} \, \| u \|_{L^q(\mathbb{R}^d)}$$

$\nabla v(x) = \lambda \nabla u(\lambda x)$ and therefore

$$\|\nabla v\|_{L^p(\mathbb{R}^d)} = \left\{ \int_{\mathbb{R}^d} \lambda^p \, \|\nabla u \underset{\overset{\shortparallel}{y}}{(\lambda x)}\|^p \, dx \right\}^{1/p}$$

$$= \boxed{\frac{\lambda}{\lambda^{d/p}}} \, \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

In particular, it has to be true that

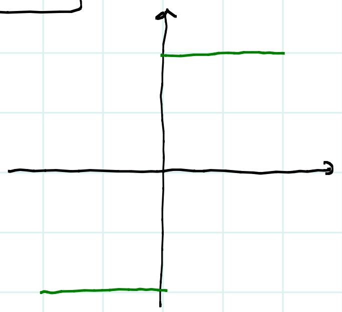
$$\frac{1}{\lambda^{d/q}} \|u\|_{L^q(\mathbb{R}^d)} \leq \frac{\lambda}{\lambda^{\frac{d}{p}}} \|\nabla u\|_{L^p(\mathbb{R}^d)} \cdot c(d,p)$$

The two powers of λ should be the same (if not, we get a contradiction when $\lambda \to 0^+$ or $\lambda \to +\infty$)

Therefore $\quad \frac{d}{q} = \frac{d}{p} - 1 \quad \rightsquigarrow \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{d} \quad \rightsquigarrow \quad q = p_*$

In the same way we can show the optimality of the exponents in the pure estimate for the Hölder constant.

The enemy #1 of embeddings and extension



$\Omega = (-1,0) \cup (0,1)$

$u \in W^{m,p}$ for every $m \geq 1$ and p

But it is not Hölder continuous, and it is not possible to extend it to $\mathbb{R}$ in a Sobolev way.

— o — o —

Istituzioni di Analisi — LECTURE 31

Note Title 03/11/2021

$\boxed{\text{Sobolev embedding } p<d}$

$$\frac{1}{p} - \frac{1}{p_*} = \frac{1}{d}$$

$$u \in W^{1,p}(\mathbb{R}^d) \implies u \in L^{p_*}(\mathbb{R}^d) \quad \text{where} \quad p_* = \frac{pd}{d-p}$$

$\boxed{\text{Step 1}}$ Assume $u \in C_c^\infty(\mathbb{R}^d)$ and $p=1$ $\rightsquigarrow$ $p_* = \frac{d}{d-1}$

$$u(x_1, x_2, \dots, x_d) = \int_{\underset{\substack{\uparrow \\ \text{negative enough} \\ \text{number}}}{-\infty}}^{x_1} \frac{\partial u}{\partial x_1}(t, x_2, \dots, x_d)\,dt$$

$$|u(x_1, x_2, \dots, x_d)| \le \underbrace{\int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial x_1}(t, x_2, \dots, x_d) \right| dt}_{\varphi_1(\hat{x}_1)}$$

In the same way

$$|u(x_1, x_2, \dots, x_d)| \le \underbrace{\int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d) \right| dt}_{\varphi_i(\hat{x}_i)}$$

If we multiply these relations

$$|u(x)|^d \le \prod_{i=1}^d \varphi_i(\hat{x}_i)$$

and hence

$$|u(x)|^{\frac{d}{d-1}} \le \prod_{i=1}^d \varphi_i(\hat{x}_i)^{\frac{1}{d-1}}$$

We are in the assumptions of Gagliardo ineq.

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} dx \underset{\substack{\uparrow \\ \text{Gagliardo}}}{\leq} \prod_{i=1}^{d} \left\{ \int_{\mathbb{R}^{d-1}} \varphi_i(\hat{x}_i)\, d\hat{x}_i \right\}^{\frac{1}{d-1}}$$

$$\int_{\mathbb{R}^{d-1}} \varphi_i(\hat{x}_i)\, d\hat{x}_i = \int_{\mathbb{R}^{d-1}} d\hat{x}_i \int_{-\infty}^{+\infty} dt \left| \frac{\partial u}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d) \right|$$

$$= \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial x_i}(x) \right| dx$$

$$\leq \int_{\mathbb{R}^d} |\nabla u(x)|\, dx$$

If we plug this in the previous estimate we obtain

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} dx \leq \left\{ \int_{\mathbb{R}^d} |\nabla u(x)|\, dx \right\}^{\frac{d}{d-1}}$$

This implies that $\|u\|_{L^{1*}(\mathbb{R}^d)} \leq \|\nabla u\|_{L^1(\mathbb{R}^d)}$

$\boxed{\text{Step 2}}$ Assume that $u \in C_c^\infty(\mathbb{R}^d)$ and $p < d$.

Trick: consider $v(x) := |u(x)|^r u(x) = \varphi(u(x))$ $(r > 0)$

with $\varphi(\sigma) := |\sigma|^r \sigma$

Then $\varphi'(\sigma) = (r+1)|\sigma|^r$

$$\nabla v(x) = \varphi'(u(x)) \nabla u(x) = (r+1)|u(x)|^r \nabla u(x).$$

From Step 1 we know that

$$\|v\|_{L^{1*}(\mathbb{R}^d)} \leq \|\nabla v\|_{L^1(\mathbb{R}^d)}$$

$$\text{LHS} = \left\{ \int_{\mathbb{R}^d} |v(x)|^{\frac{d}{d-1}} \, dx \right\}^{\frac{d-1}{d}} = \left\{ \int_{\mathbb{R}^d} |u(x)|^{(r+1)\frac{d}{d-1}} \, dx \right\}^{\frac{d-1}{d}}$$

$$= \| u \|^{r+1}_{L^{(r+1)\frac{d}{d-1}}}$$

$$\text{RHS} = \int_{\mathbb{R}^d} |\nabla v(x)| \, dx = (r+1) \int_{\mathbb{R}^d} \underset{\frac{p-1}{p}}{|u(x)|^r} \cdot \underset{\frac{1}{p}}{|\nabla u(x)|} \, dx$$

$$\leq (r+1) \left\{ \int_{\mathbb{R}^d} |u(x)|^{\frac{rp}{p-1}} \, dx \right\}^{\frac{p-1}{p}} \cdot \| \nabla u \|_{L^p(\mathbb{R}^d)}$$

$$= (r+1) \, \| u \|^{r}_{L^{r\frac{p}{p-1}}} \cdot \| \nabla u \|_{L^p(\mathbb{R}^d)}$$

Putting things together we get

$$\boxed{\| u \|^{r+1}_{L^{(r+1)\frac{d}{d-1}}} \leq (r+1) \, \| u \|^{r}_{L^{r\frac{p}{p-1}}} \, \| \nabla u \|_{L^p(\mathbb{R}^d)}}$$

Idea: choose r in such a way that $(r+1)\frac{d}{d-1} = r\frac{p}{p-1}$

$$r\left(\frac{p}{p-1} - \frac{d}{d-1} \right) = \frac{d}{d-1} \rightsquigarrow r \, \frac{\cancel{pd} - p - \cancel{dp} + d}{(p-1)\cancel{(d-1)}} = \frac{d}{\cancel{d-1}}$$

$$\rightsquigarrow r = \frac{d(p-1)}{d-p} \rightsquigarrow (r+1)\frac{d}{d-1} = \left(\frac{d(p-1)}{d-p} + 1 \right) \frac{d}{d-1}$$

$$= \frac{dp - \cancel{d} + \cancel{d} - p}{d-p} \, \frac{d}{d-1} = \frac{pd}{d-p} = p_*$$

With this choice of r we obtain

$$\| u \|^{r+1}_{L^{p_*}} \leq \overset{C(p,d)}{\overset{\uparrow}{(r+1)}} \, \| u \|^{r}_{L^{p_*}} \cdot \| \nabla u \|_{L^p} \rightsquigarrow \text{conclusion}$$

$\boxed{\text{Step 3}}$ Let us consider now any $u \in W^{1,p}(\mathbb{R}^d)$, and let us approximate it in the deluxe way with $\{u_n\} \subseteq C^\infty_c(\mathbb{R}^d)$

$$u_n \to u \quad \text{in } L^p(\mathbb{R}^d) \qquad \nabla u_n \to \nabla u \quad \text{in } L^p(\mathbb{R}^d)$$

$\{\nabla u_n\}$ is a Cauchy sequence in $L^p(\mathbb{R}^d)$ (it is convergent)

As a consequence, $\{u_n\}$ is a Cauchy sequence in $L^{p_*}(\mathbb{R}^d)$ because

$$\| u_n - u_m \|_{L^{p_*}(\mathbb{R}^d)} \leq c(p,d) \, \| \underset{\nabla(u_n - u_m)}{\underset{\parallel}{\nabla u_n - \nabla u_m}} \|_{L^p(\mathbb{R}^d)}$$

Therefore $u_n \to \hat{u}$ in $L^{p_*}(\mathbb{R}^d)$
$u_n \to u$ in $L^p(\mathbb{R}^d)$

The two limits coincide (L^q convergence $\Rightarrow$ pointwise conv. of a subsequence)

At this point we can pass to the limit in

$$\| u_n \|_{L^{p_*}(\mathbb{R}^d)} \leq c(p,d) \, \| \nabla u_n \|_{L^p(\mathbb{R}^d)}$$

— o — o —

$\boxed{\text{Sobolev embedding in the case } p=d}$

$u \in W^{1,d}(\mathbb{R}^d) \Rightarrow u \in L^q(\mathbb{R}^d)$ $\forall q \in [p,+\infty)$ with improve estimate

$$\| u \|_{L^q(\mathbb{R}^d)} \leq c(d,q) \, \| u \|_{1,d,\mathbb{R}^d}$$

As in the previous case, we obtain that

$$\|u\|^{r+1}_{L^{(r+1)\frac{d}{d-1}}} \le (r+1)\, \|u\|^{r}_{L^{r\frac{d}{d-1}}}\, \|\nabla u\|_{L^d(\mathbb{R}^d)}$$

Key point: the summability in the LHS is better than in the RHS

Let us set $p_k := d + \frac{d}{d-1}k$ and let us prove by induction that $u \in L^{p_k}(\mathbb{R}^d)$.

This is enough because $p_k \to +\infty$.

$\boxed{\text{Case } k=0}$ Trivial, because $u \in L^d(\mathbb{R}^d)$ by assumption.
Here the estimate is already impure

$\boxed{\text{Case } k \Rightarrow k+1}$ Assume that $u \in L^{p_k}(\mathbb{R}^d)$.
We choose r in such a way that

$$r\frac{d}{d-1} = p_k \leadsto r\frac{\cancel{d}}{d-1} = \overset{1}{\cancel{d}} + \frac{\cancel{d}}{d-1}k$$

$$\leadsto r = d-1+k$$

Note: at each step the constant is growing

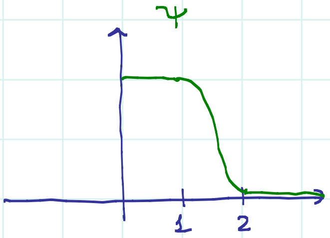
$$\|u\|_{L^{p_k}(\mathbb{R}^d)} \le \underset{\substack{\downarrow \\ +\infty \text{ as } k\to+\infty}}{c(d,k)} \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

— o — o —

How to prove deluxe approximation in $\mathbb{R}^d$

$\boxed{\text{Step 1: regularization}}$ Consider $u * \rho_\varepsilon = u_\varepsilon$ and observe that $u_\varepsilon \to u$ and $\nabla u_\varepsilon \to \nabla u$ in $L^p(\mathbb{R}^d)$

Step 2: we obtain the compact support



Take $v \in C^\infty(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$

Consider $v_n(x) := v(x) \cdot \psi\left(\frac{|x|}{n}\right)$

↑ coincides with v in $B(0,n)$
and $v_n(x) \equiv 0$ outside $B(0,2n)$

Easy to see (dominated convergence) that

$$v_n \to v \quad \text{and} \quad \nabla v_n \to \nabla v \quad \text{in } L^p(\mathbb{R}^d).$$

— o — o —

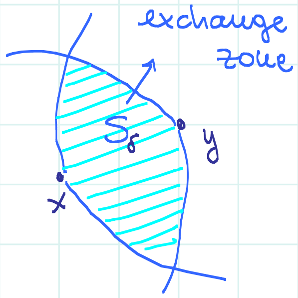
Istituzioni di Analisi — LECTURE 32

Note Title 03/11/2021

Sobolev embedding with $p>d$

$u \in W^{1,p}(\mathbb{R}^d) \implies u \in L^\infty(\mathbb{R}^d)$ with impure estimate and u is $1-\frac{d}{p}$ Hölder cont. with pure estimate of the Hölder constant

$\boxed{\text{Step 1}}$ Let us assume that $u \in C_c^\infty(\mathbb{R}^d)$.



MORREY TRICK (exchange zone)

$$|u(y)-u(x)| \le |u(y)-u(z)| + |u(z)-u(x)|$$

We set $\delta := |y-x|$ and we integrate in S_δ

$$\text{LHS} = \int_{S_\delta} |u(y)-u(x)|\,dz = |u(y)-u(x)| \cdot \text{meas}(S_\delta)$$

$$= |u(y)-u(x)| \cdot C(d)\,\delta^d$$

For the RHS, we have the following estimate

$$|u(z)-u(x)| \underset{\substack{\uparrow \\ \varphi(t)=u(x+t(z-x))}}{=} |\varphi(1)-\varphi(0)| = \left|\int_0^1 \varphi'(t)\,dt\right|$$

$$\le \int_0^1 |\varphi'(t)|\,dt$$

$$= \int_0^1 |\langle \nabla u(x+t(z-x)), z-x\rangle|\,dt$$

$$\le \underbrace{|z-x|}_{\le \delta} \cdot \int_0^1 |\nabla u(x+t(z-x))|\,dt \le \delta \int_0^1 |\nabla u(x+t(z-x))|\,dt$$

Now we integrate

$$\int_{S_\delta} |u(z)-u(x)|\,dz \leq \delta \int_{S_\delta} dz \int_0^1 |\nabla u(x+t(z-x))|\,dt$$

$$= \delta \int_0^1 dt \int_{S_\delta} |\nabla u(\underbrace{x+t(z-x)}_{w})|\,dz$$

$$dw = t^d\, dz$$

$$\leq \delta \int_0^1 t^{-d}\, dt \int_{B(x,\delta t)} \underset{\frac{1}{p}}{|\nabla u(w)|} \cdot \underset{\frac{p-1}{p}}{1}\, dw$$

where lies w? $|w-x| = t\,|z-x| \leq \delta t \Rightarrow w \in B(x,\delta t)$

$$\leq \delta \int_0^1 t^{-d}\, dt \underbrace{\left\{ \int_{B(x,\delta t)} |\nabla u(w)|^p dw \right\}^{\frac{1}{p}}}_{\leq \|\nabla u\|_{L^p(\mathbb{R}^d)}} \cdot \operatorname{meas}(B(x,t\delta))^{\frac{p-1}{p}}$$

$$\downarrow$$

$$C(d,p)\, t^{d\frac{p-1}{p}} \delta^{d\frac{p-1}{p}}$$

$$\leq C(d,p)\, \delta^{1+d-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)} \int_0^1 t^{\cancel{-d}+\cancel{d}-\frac{d}{p}}\, dt$$

$$= C_1(d,p)\, \delta^{1+d-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)} \qquad \overset{\shortparallel}{\boxed{\int_0^1 \frac{1}{t^{d/p}}\, dt}}$$

this is a number because $\frac{d}{p} < 1$

At the end of the day we find that

Hölder exponent that we expected

$$|u(x)-u(y)|\, C(d)\, \cancel{\delta}^{\cancel{d}} \leq 2C_1(d,p)\, \underset{\substack{\uparrow \\ |x-y|}}{\delta}^{\,1+\cancel{d}-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

$\boxed{\text{Step 2}}$ Again $u \in C_c^\infty(\mathbb{R}^d)$. We want the $L^\infty(\mathbb{R}^d)$ estimate

Let us consider any $x \in \mathbb{R}^d$. For sure

$$\underset{\substack{\| \\ |u(\hat{y})|^p \,\text{meas}\,(B(x,24))}}{\int_{B(x,24)} |u(y)|^p dy} \le \underset{\substack{\| \\ \|u\|^p_{L^p(\mathbb{R}^d)}}}{\int_{\mathbb{R}^d} |u(y)|^p dy}$$

$\uparrow$ misterious point in the Ball

$$\Rightarrow \quad |u(\hat{y})| \le c(d)\, \|u\|_{L^p(\mathbb{R}^d)}$$

$$\rightarrow \quad |u(x)| \le |u(\hat{y})| + |u(x) - u(\hat{y})|$$

$$\le c(d)\,\|u\|_{L^p(\mathbb{R}^d)} + c(p,d)\,\|\nabla u\|_{L^p(\mathbb{R}^d)} \cdot 24^{1-\frac{d}{p}}$$

$$\le c(d,p)\,\|u\|_{1,p,\mathbb{R}^d}$$

$\boxed{\text{Step 3}}$ Assume $u \in W^{1,p}(\mathbb{R}^d)$. Take deluxe approximation $\{u_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ and observe that in any ball of radius 24 (and actually any radius) they satisfy the assumptions of Ascoli-Arzelà.
The conclusion is standard.

— o — o —

$\boxed{\text{Rmk}}$ The previous proof does NOT work if $p = +\infty$ in the point where we apply Hölder inequality.
This case can be dealt with by convolution or simply by observing that Step 1 is almost trivial if $p = +\infty$ (in this case $|\nabla u|$ bounded $\Rightarrow u$ is Lip. with bound on the Lip. const.)

Corollary If $u \in W^{1,p}(\mathbb{R}^d)$ with $p>d$, then

→ $u \in C^0(\mathbb{R}^d)$ (and actually uniformly continuous)

→ $\lim\limits_{|x|\to +\infty} u(x) = 0$

↑ This DOES not follow from $\int_{\mathbb{R}^d} |u(x)|^p dx < +\infty$

$\boxed{\text{GENERAL FACT}}$ If $\int_{\mathbb{R}^d} |v(x)| dx < +\infty$ AND $v(x)$ is unif. cont.

then $v(x) \to 0$ as $|x| \to +\infty$

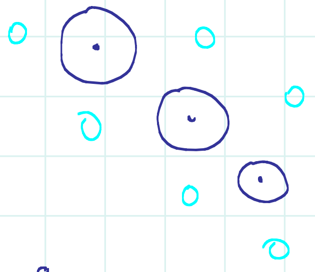
Exercise If $p<d$, then there exists $u \in C^\infty(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$ such that $u(x) \not\to 0$ as $|x| \to +\infty$

Let us use a scaling technique. Let us consider

$\varphi \in C^\infty_c(B(0,1))$ with $\varphi \geq 0$ and $\int_{B(0,1)} \varphi(x) dx = 1$

Let us consider a sequence of disjoint balls $B(x_n, r_n)$

Let us consider



$$u(x) := a_n \varphi\left(\frac{x-x_n}{r_n}\right) \quad \text{in } B(x_n, r_n)$$

$$\int_{B(x_n,r_n)} |u(x)|^q dx = a_n^q \int_{B(x_n,r_n)} \varphi\left(\frac{x-x_n}{r_n}\right)^q dx$$

$$= a_n^q r_n^d \int_{B(0,1)} \varphi(y)^q dy$$

$\nabla u(x) = \frac{a_n}{r_n} \nabla\varphi\left(\frac{x-x_n}{r_n}\right)$ and therefore

$$\int_{B(x_n,r_n)} |\nabla u(x)|^q = \frac{a_n^q}{r_n^q} r_n^d \int_{B(0,1)} |\nabla\varphi(y)|^q dy$$

If we want $u \in W^{1,p}(\mathbb{R}^d)$ we need

$$\sum a_n^p r_n^d < +\infty \quad \text{and} \quad \sum a_n^p r_n^{d-p} < +\infty$$

We can take $a_n \equiv 1$ so that $|u(x)| \not\to 0$ as $|x| \to +\infty$

(of course $|x_n| \to +\infty$)

$$\sum r_n^{\overset{>0}{\boxed{d}}} < +\infty \quad \text{and} \quad \sum r_n^{\overset{>0}{\boxed{d-p}}} < +\infty$$

enough to choose $r_n = \frac{1}{n!}$ (can also choose $a_n \to +\infty$!!)

— o — o —

Istituzioni di Analisi – LECTURE 33

Note Title 04/11/2021

EXTENDERS – MODEL CASE

Why do we need extenders? They are a suff. cond. for

→ deluxe approx. in $\Omega \subseteq \mathbb{R}^d$

→ Sobolev embeddings in $\Omega \subseteq \mathbb{R}^d$

Model case $A \subseteq \mathbb{R}^{d-1}$ open set

$$C_A := \{ (\underset{\overset{\uparrow}{d-1 \text{ var.}}}{x}, \underset{\overset{\nwarrow}{\in \mathbb{R}}}{y}) \in \mathbb{R}^d : x \in A, y \in (-1,1)\} \quad \text{cylinder}$$

$$C_A^+ := \{ (x,y) \in \mathbb{R}^d : x \in A, y \in (0,1)\}$$

We can consider also the "infinite cylinders"

$$C_A := A \times \mathbb{R} \qquad C_A^+ := A \times (0,+\infty)$$

Goal: given $u \in W^{1,p}(C_A^+)$, find $\hat{u} \in W^{1,p}(C_A)$ that extends u.

Def. For every $u \in W^{1,p}(C_A^+)$, let us consider

$$[E_{\text{even}}\, u](x,y) := \begin{cases} u(x,y) & \text{if } x \in A \text{ and } y \in (0,1) \\ u(x,-y) & \text{if } x \in A \text{ and } y \in (-1,0) \end{cases}$$

$$[E_{\text{odd}}\, u](x,y) := \begin{cases} u(x,y) & \text{as before} \\ -u(x,-y) & \text{as before} \end{cases}$$

$\boxed{\text{Theorem}}$ Let $u \in W^{1,p}(C_A^+)$. Then $\hat{u} := [E_{even}\, u](x) \in W^{1,p}(C_A)$, and

$$D_{x_i} \hat{u} = E_{even}\, D_{x_i} u \qquad \text{if } \ i \in \{1, \dots, d-1\}$$

$$D_y \hat{u} = E_{odd}\, D_y u$$

In addition $\quad \| \hat{u} \|_{1,p,C_A} = 2 \, \| u \|_{1,p,C_A}$

$\boxed{\text{Proof}}$ It is trivial that E_{even} is linear and extends u. Let us consider derivatives.

$\boxed{D_{x_i} \hat{u} \ \text{ with } \ i = 1, \dots, d-1}$ We need to prove that

$$\int_{C_A} \hat{u}(x,y)\, D_{x_i} \varphi(x,y)\, dx\, dy = - \int_{C_A} E_{even}\, D_{x_i} u(x,y)\, \varphi(x,y)\, dx\, dy$$

for every $\varphi \in C_c^\infty(C_A)$.

$$\text{LHS} = \int_A dx \int_0^1 dy \ u(x,y) D_{x_i} \varphi(x,y) + \int_A dx \int_{-1}^0 dy \ u(x, \overset{\substack{dz=-dy \\ z \\ \uparrow}}{-y}) D_{x_i} \varphi(x,y)$$

$$= \int_A dx \int_0^1 dy \ u(x,y) D_{x_i} \varphi(x,y) + \int_A dx \int_0^1 dy \ u(x,y) D_{x_i} \varphi(x,-y)$$

$$= \int_{C_A^+} u(x,y) \left\{ D_{x_i} \varphi(x,y) + D_{x_i} \varphi(x,-y) \right\} dx\, dy$$

$$= \int_{C_A^+} u(x,y)\, D_{x_i} \psi(x,y)\, dx\, dy \qquad \text{where } \ \psi(x,y) := \varphi(x,y) + \varphi(x,-y)$$

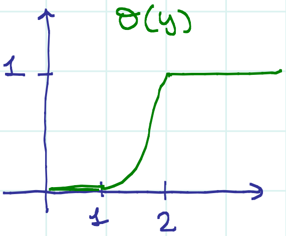
If we do the same with the RHS we find

$$\text{RHS} = -\int_{C_A^+} D_{x_i} u(x,y)\, \psi(x,y)\, dx\, dy \quad \text{with } \psi \text{ as above}$$

So we reduced ourselves to proving that

$$\int_{C_A^+} u(x,y)\, D_{x_i} \psi(x,y)\, dx\, dy = -\int_{C_A^+} D_{x_i} u(x,y)\, \psi(x,y)\, dx\, dy$$

This would be trivial if $\psi \in C_c^\infty(C_A^+)$, but in general this is not true.



Consider a cutoff function θ as in the figure.

Consider

$$\underset{\uparrow}{\psi_n}(x,y) := \psi(x,y) \cdot \theta(ny)$$

coincides with ψ if $y \geq \frac{2}{n}$
and vanishes if $y \leq \frac{1}{n}$

In particular, $\psi_n \in C_c^\infty(C_A^+)$ and therefore

$$\int_{C_A^+} u(x,y)\, D_{x_i} \psi_n(x,y)\, dx\, dy = -\int_{C_A^+} D_{x_i} u(x,y)\, \psi_n(x,y)\, dx\, dy$$

is true for every $n \geq 1$. Now we pass to the limit.

In the RHS we have pointwise convergence and domination

$$|D_{x_i} u(x,y)\, \psi_n(x,y)| \leq \text{constant}\ \underbrace{|D_{x_i} u(x,y)|}_{\in L^p}$$

The LHS is equal to

$$\int_{C_A^+} u(x,y)\,\theta(ny)\,D_{x_i}\psi(x,y)\,dx\,dy$$

and we conclude in the same way.

$\boxed{D_y\hat{u}}$ We need to prove that

$$\int_{C_A} \hat{u}(x,y)\,D_y\varphi(x,y)\,dx\,dy = -\int_{C_A} E_{odd}\,D_y u(x,y)\,\varphi(x,y)\,dx\,dy$$

for every $\varphi \in C_c^\infty(C_A)$.

With the same variable change we find that

$$\text{LHS} = \int_{C_A^+} u(x,y)\,D_y\psi(x,y)\,dx\,dy$$

$$\text{RHS} = -\int_{C_A^+} D_y u(x,y)\,\psi(x,y)\,dx\,dy$$

but now $\psi(x,y) := \varphi(x,y) \underset{\uparrow\uparrow}{-} \varphi(x,-y)$

We introduce $\psi_n(x,y) := \psi(x,y)\cdot\theta(ny)$ as before, and we observe that, for every $n \geq 1$,

$$\int_{C_A^+} u(x,y)\,D_y\psi_n(x,y)\,dx\,dy = -\int_{C_A^+} D_y u(x,y)\,\psi_n(x,y)\,dx\,dy$$

$$\downarrow$$

$$-\int_{C_A^+} D_y u(x,y)\,\psi(x,y)\,dx\,dy$$

$$\text{LHS} = \int_{C_A^+} u(x,y)\,\theta(ny)\,D_y\psi(x,y)\,dx\,dy + \int_{C_A^+} u(x,y)\,\psi(x,y)\,\underset{\uparrow\uparrow}{n}\,\theta'(ny)\,dx\,dy$$

The first term ⇝ no problem

As for the second term we observe that

- $m|\vartheta'(y)| \leq \text{const.} \cdot m$ and vanishes when $y \geq \frac{2}{m}$
- $\psi(x,0) = 0$ for every $x \in A$ and in addition ψ is Lip. cont. and therefore

$$|\psi(x,y)| = |\psi(x,y) - \psi(x,0)| \leq L y$$

As a consequence

$$\int_{C_A^+} |u(x,y)\psi(x,y)\, m\, \vartheta'(my)|\, dx\, dy$$

$$\leq \int_A dx \int_0^{\frac{2}{m}} dy\, |u(x,y)| \cdot L \underset{\overset{\uparrow}{\leq \frac{2}{m}}}{y} \cdot m \cdot \text{const}$$

$$\leq \int_A dx \int_0^{\frac{2}{m}} dy\, \text{const} \cdot L \cdot |u(x,y)| \underset{\overset{\uparrow}{\text{dominated convergence}}}{\longrightarrow} 0$$

— o — o —

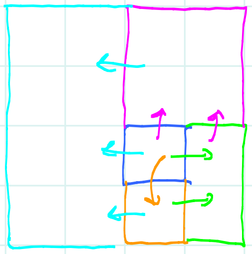
$\boxed{\text{Rmk}}$ In general it is NOT true that

$$\int_{C_A^+} u(x,y)\, D_{\substack{x_i \\ y}} \psi(x,y)\, dx\, dy = -\int_{C_A^+} D_{\substack{x_i \\ y}} u(x,y)\, \psi(x,y)\, dx\, dy$$

if ψ has not compact support in C_A^+.
One has to consider the boundary terms when $y = 0$

— o — o —

Extender for squares / rectangles



→ In four steps we extend by reflection to a bigger square

→ Then we multiply by a function of class C^∞ that is

- 1 in the smaller square
- 0 outside the big square

— o — o —

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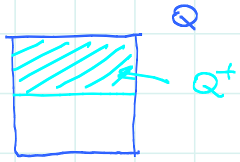
Note Title 04/11/2021

EXTENDER FOR SMOOTH OPEN SET

Let us consider the standard "cubes"



$$Q := \{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R} : \|x\| < 1,\ |y| < 1\}$$

$$Q^+ := \{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R} : \|x\| < 1,\ 0 < y < 1\}$$

Defn Let us assume that $\Omega \subseteq \mathbb{R}^d$ is an open set with the following properties

- $\partial\Omega$ is compact
- for every $x \in \partial\Omega$ there exists an open set $\mathcal{U} \ni x$ and a function

$$\Phi : \mathcal{U} \to Q$$

such that

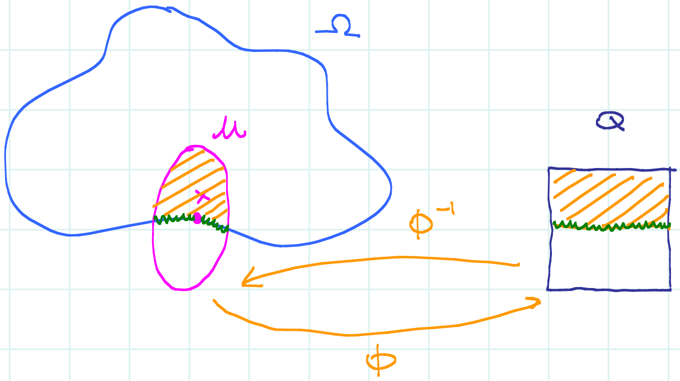
→ Φ is invertible

→ Φ and Φ^{-1} are of class C^1

→ J_Φ and $J_{\Phi^{-1}}$ are bounded matrices

→ $\Phi(\mathcal{U} \cap \Omega) = Q^+$

→ $\Phi(\mathcal{U} \cap \partial\Omega) = Q_0 = \{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R} : \|x\| < 1,\ y = 0\}$



Partition of the unity of type B

Let $\Omega \subseteq \mathbb{R}^d$ be an open set with $\partial\Omega$ compact.

Assume that $A_1, \dots, A_n$ is a FINITE open covering of $\partial\Omega$ by bounded open set.

Let us set $A_0 := \Omega$.

Then there exist functions $\overset{\downarrow\downarrow}{\psi_0}, \psi_1, \dots, \psi_n$ such that

(i) $\operatorname{supp}(\psi_i) \subseteq A_i$ for $i = 0, 1, \dots, n$

(this means compact support for $i \geq 1$ but not necessarily for $i = 0$ if Ω is not bounded)

(ii) $\psi_i(x) \geq 0$ for every $x \in \mathbb{R}^d$ and every $i = 0, 1, \dots, n$.

(iii) as expected

$$\sum_{i=0}^{n} \psi_i(x) = 1 \qquad \forall x \in \operatorname{Clos}(\Omega)$$

(and actually in an open set that contains $\operatorname{Clos}(\Omega)$).

Second ingredient (Composition with Sobolev outside)

Assume $\phi : \underset{\uparrow}{A} \longrightarrow \underset{\uparrow}{B}$ has the properties stated above

open sets (J_ϕ and $J_{\phi^{-1}}$ are bounded)

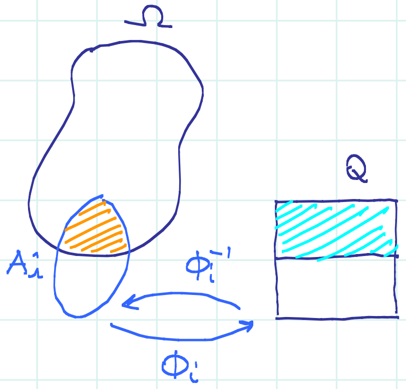
Then for every $v \in W^{1,p}(B)$ it turns out that $u = v \circ \Phi \in W^{1,p}(A)$ and we have a control on the full norms

$$m \, \|v\|_{1,p,B} \leq \|u\|_{1,p,A} \leq M \, \|v\|_{1,p,B}$$

The same using ϕ^{-1} instead of Φ.

$\boxed{\text{Theorem}}$ Let us assume that Ω is as in the previous definition.
Then Ω admits a $(1,p)$ - extender

$\boxed{\text{Proof}}$ Let $A_1, \dots, A_n$ be a finite covering of $\partial\Omega$ with open sets as in the definition.
For every $i = 1, \dots, n$, let
$$\Phi_i : A_i \longrightarrow Q$$
the corresponding map.



Let $\psi_0, \psi_1, \dots, \psi_n$ be the corresponding partition of the unity of type B. Let us set

$$u_i(x) = u(x)\,\psi_i(x) \qquad \forall x \in \Omega$$

so that
$$u(x) = \sum_{i=0}^{n} \underset{\uparrow \text{ has support } \subseteq A_i}{u_i(x)}$$

For every $i = 1, \dots, n$ we set

$$v_i(x) := u_i(\Phi_i^{-1}(x)) \qquad \forall x \in Q^+$$

$$\hat{v}_i(x) := E_{\text{even}}\, v_i(x) \qquad \forall x \in Q \leftarrow \text{FULL CUBE}$$

$$\hat{u}_i(x) := \hat{v}_i(\Phi_i(x)) \qquad \forall x \in A_i$$

and finally

$$\hat{u}(x) := u_0(x) + \sum_{i=1}^{n} \hat{u}_i(x) \qquad \forall x \in \overset{\substack{\text{open neighborhood} \\ \text{of } \operatorname{Clos}(\Omega) \\ \uparrow}}{\bigcup_{i=0}^{n} A_i}$$

The claim is that $u \longrightarrow \hat{u}$ is a (1,p) extender

- It is linear (almost trivial)
- It is an extension. To this end, it is enough to show that

$$\hat{u}_i(x) = u_i(x) \qquad \forall x \in A_i \cap \Omega$$

(if $x \in A_i \cap \Omega$ we did nothing)

- We need the norm control. It is enough to show that

$$\|\hat{u}_i\|_{1,p,A_i} \le M_i \|u_i\|_{1,p,A_i\cap\Omega}$$

and this is true because

$$\|\hat{u}_i\|_{1,p,A_i} \underset{\substack{\uparrow \\ \text{2nd ingr.}}}{\le} \text{const} \cdot \|\hat{v}_i\|_{1,p,Q}$$

$$\le 2 \cdot \text{const} \cdot \|v_i\|_{1,p,Q^+}$$

$$\le \text{const} \cdot 2 \cdot \text{const} \cdot \|u_i\|_{1,p,A_i\cap\Omega}$$

The conclusion follows because

$$\|\hat{u}\|_{1,p,\mathbb{R}^d} \le \|u_0\|_{1,p,\underset{\substack{\uparrow \\ \Omega}}{\mathbb{R}^d}} + \sum_{i=1}^{m} \|\hat{u}_i\|_{1,p,\underset{\substack{\uparrow \\ A_i}}{\mathbb{R}^d}}$$

$$\le \|u_0\|_{1,p,\Omega} + M \sum_{i=1}^{m} \|u_i\|_{1,p,A_i\cap\Omega}$$

$$\le \text{const} \, \|u\|_{1,p,\Omega}$$

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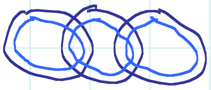
Proof of partition of the unity



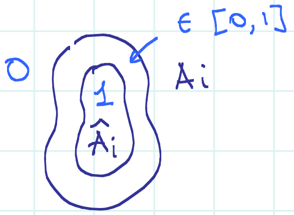
Step 1 We replace $A_0, A_1, \dots, A_n$ by smaller open sets defined as

$$A_{i,k} := \{ x \in A_i : \mathrm{dist}(x, \partial A_i) > \tfrac{1}{k} \}$$

We show that, if k is large enough, then $\{A_{i,k}\}_{i=0,1,\dots,n}$ are again an open covering of Clos(Ω).
We call these new sets

$$\hat{A}_0, \hat{A}_1, \dots, \hat{A}_n$$

Step 2 For every $i = 0, 1, \dots, n$ we find θ_i which is



Step 3 We set

$$\psi_0(x) := \theta_0(x)$$
$$\psi_1(x) := \theta_1(x)\,(1-\theta_0(x))$$
$$\psi_2(x) := \theta_2(x)\,(1-\theta_1(x))\,(1-\theta_0(x))$$
$$\vdots$$
$$\psi_k(x) := \theta_k(x) \prod_{i=0}^{k-1} (1-\theta_i(x))$$

This works! We only need to check that $\sum_{i=0}^{n} \psi_i(x) = 1$ for every $x \in \bigcup_{i=0}^{n} \hat{A}_i$. This follows from

$$1 - \sum_{i=0}^{n} \psi_i(x) = \prod_{i=0}^{n} (1-\theta_i(x)) \qquad \forall x \in \mathbb{R}^d$$

[This equality with k instead of n can be proved by induction]

Istituzioni di Analisi – LECTURE 35

Note Title 08/11/2021

Compactness results

Basic idea: assume that $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. We want to show that some subseq. converges in some $\boxed{\text{STRONG sense}}$ (weak conv. is trivial)

Compactness in metric spaces

Let X be a metric space. Then the following are equivalent

(i) X is covering compact

(ii) X is sequentially compact

(iii) X is complete and TOTALLY BOUNDED

Def. A metric space (X,d) is tot. bounded if for every $\varepsilon > 0$ there exists a FINITE subset $S \subseteq X$ such that

↑ ε-net

$$X \subseteq \bigcup_{s \in S} B(s,\varepsilon)$$

Prop. (relatively compact subsets)

Let $Y \subseteq X$ be a subset of a $\boxed{\text{COMPLETE}}$ metric space X.

Then the following are equivalent:

(i) $\operatorname{Clos}(Y)$ is compact (wrt the restriction of the metric)

(ii) for every $\{y_n\} \subseteq Y$ there exists $y_n \to y_\infty$ with y_∞ not necessarily in Y

(iii) Y is totally bounded (wrt the restriction of the metric).

Defn $Y \subseteq X$ is relatively compact if it satisfies any of (i), (ii), (iii)

↑ complete

Prop. Let X be a metric space, and let $Y \subseteq X$ be a subset. Then the following are equivalent

(i) Y is totally bounded, namely

$$\forall \varepsilon > 0 \ \exists \text{ FINITE } S \subseteq Y \text{ s.t. } Y \subseteq \bigcup_{y \in S} B(y, \varepsilon)$$

(ii) $\forall \varepsilon > 0 \ \exists K_\varepsilon \subseteq X$ s.t. K_ε is totally bounded (for example K_ε compact) and

$$Y \subseteq \bigcup_{y \in K_\varepsilon} B(y, \varepsilon)$$

"Existence of a compact ε-net $\Rightarrow$ existence of finite ε-net"

Proof of (ii) => (i) Given $\varepsilon > 0$ find K_ε s.t.

$$Y \subseteq \bigcup_{y \in K_\varepsilon} B(y, \tfrac{\varepsilon}{2})$$

then find $S \subseteq K_\varepsilon$ s.t. $K_\varepsilon \subseteq \bigcup_{y \in S} B(y, \frac{\varepsilon}{2})$. Conclude that

$$Y \subseteq \bigcup_{y \in S} B(y, \varepsilon)$$

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Conclusion of the summary Assume that X is a complete metric space. Assume that

$$\forall \varepsilon > 0 \ \exists K_\varepsilon \subseteq X \underset{\text{tot. bounded is enough}}{\text{compact}} \text{ s.t. } Y \subseteq \bigcup_{y \in K_\varepsilon} B(y, \varepsilon)$$

Then Y is relatively compact.

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$\boxed{\text{Theorem}}$ (Compactness in L^p) Let us assume that $\Omega \subseteq \mathbb{R}^d$ and $\mathcal{F} \subseteq L^p(\Omega)$.

Let us assume that

(i) Ω is BOUNDED

(ii) $\mathcal{F}$ is bounded in $L^p(\Omega)$, namely

$$\exists M \in \mathbb{R} \quad \forall f \in \mathcal{F} \quad \|f\|_{L^p(\Omega)} \leq M$$

(iii) $\mathcal{F}$ is "equicontinuous in $L^p(\Omega)$", namely

$$\forall \varepsilon > 0 \ \exists \delta > 0 \quad \forall f \in \mathcal{F} \quad \forall h \in \mathbb{R}^d$$

$$|h| \leq \delta \quad \Rightarrow \quad \int_{\mathbb{R}^d} |\hat{f}(x+h) - \hat{f}(x)|^p dx \leq \varepsilon^p$$

[After video: Ω is enough]

extension of u to 0 outside Ω

$$\|\tau_h \hat{f} - \hat{f}\|_{L^p(\mathbb{R}^d)} \leq \varepsilon$$

h-translation of u.

[Ω is enough]

[Some philosophy in the VIDEO]

Then $\mathcal{F}$ is relatively compact in $L^p(\Omega)$.

<u>Proof</u> We need to show that

$$\forall \varepsilon > 0 \ \exists \underset{\text{compact}}{K_\varepsilon} \text{ s.t. } \quad \mathcal{F} \subseteq \bigcup_{g \in K_\varepsilon} B(g, \varepsilon)$$

namely $\quad \forall f \in \mathcal{F} \ \exists g \in K_\varepsilon$ s.t. $\|f - g\|_{L^p(\Omega)} \leq \varepsilon$. $\qquad (*)$

Consider $\delta > 0$ corresponding to ε in (iii), and define

$$K_\varepsilon := \{ \underset{\text{convolution}}{f * \rho_\delta} : f \in \mathcal{F} \}$$

$\boxed{\text{Step 1}}$ Let us check $(*)$. Natural choice : $g = f * \rho_\delta$

$$f(x) - g(x) = f(x) - \int_{\mathbb{R}^d} f(x+\delta y)\,\rho(y)\,dy$$

$$= \int_{\mathbb{R}^d} f(x)\,\rho(y)\,dy - \int_{\mathbb{R}^d} f(x+\delta y)\,\rho(y)\,dy$$

$$= \int_{\mathbb{R}^d} \big(f(x) - f(x+\delta y)\big)\,\rho(y)\,dy$$

$$|f(x)-g(x)| \le \int_{\mathbb{R}^d} |f(x+\delta y) - f(x)|\,\rho(y)\,dy \qquad \frac{1}{p}+\frac{1}{p'}=1$$

$$= \int_{\mathbb{R}^d} \underbrace{|f(x+\delta y) - f(x)|\,\rho(y)^{\frac{1}{p}}}_{\frac{1}{p}}\ \underbrace{\rho(y)^{\frac{1}{p'}}}_{\frac{1}{p'}}\,dy$$

$$\le \left\{\int_{\mathbb{R}^d} |f(x+\delta y) - f(x)|^p\,\rho(y)\,dy\right\}^{\frac{1}{p}} \underbrace{\left\{\int_{\mathbb{R}^d} \rho(y)\,dy\right\}^{1/p'}}_{\overset{\shortparallel}{1}}$$

$$|f(x)-g(x)|^p \le \int_{\underset{\uparrow B(0,1)}{\mathbb{R}^d}} |f(x+\delta y) - f(x)|^p\,\rho(y)\,dy$$

Integrate over Ω

$$\int_\Omega |f(x)-g(x)|^p\,dx \le \underbrace{\int_{\underset{\uparrow B(0,1)}{\mathbb{R}^d}} dy\,\rho(y)}_{\overset{\shortparallel}{1}}\ \underbrace{\int_\Omega |f(x+\overset{\overset{h \text{ with } |h|\le\delta}{\uparrow}}{\delta y}) - f(x)|^p\,dx}_{\le\ \varepsilon^p}$$

$$\le \varepsilon^p$$

"Some f are actually $\hat{f}$".

$\boxed{\text{Step 2}}$ Elements of K_ε are bounded in the L^∞ sense.

$$|(f*\rho_\delta)(x)| \le \int_{\underset{\underset{B(0,1)}{\uparrow}}{\mathbb{R}^d}} \underset{\frac{1}{p}}{|\hat{f}(x+\varepsilon y)|}\cdot \underset{\frac{1}{p'}}{|\rho(y)|}\,dy$$

$$\le \left\{\int_{\mathbb{R}^d} |\hat{f}(\underset{\underset{\underset{\varepsilon^d dy = dz}{z}}{\shortparallel}}{x+\varepsilon y})|^p\,dy\right\}^{1/p} \left\{\int_{\mathbb{R}^d} \underset{\underset{\text{fixed number}}{\uparrow}}{\rho(y)^{p'}\,dy}\right\}^{1/p'}$$

$$\left\{\int_{\mathbb{R}^d} \underset{\underset{\text{bounded}}{\uparrow}}{|\hat{f}(z)|^p}\,\frac{1}{\varepsilon^d}\,dz\right\}^{1/p}$$

$\boxed{\text{Step 3}}$ Elements of K_ε are Lipschitz continuous with a common bound on the Lip. constant.
Since elements of K_ε are of class C^∞, it is enough to bound the gradient

$$\nabla(f*\rho_\delta)(x) = \nabla\left(\int_{\mathbb{R}^d} \hat{f}(\underset{\underset{z}{\shortparallel}}{x+\varepsilon y})\,\rho(y)\,dy\right)$$

$$= \nabla\left(\frac{1}{\varepsilon^d}\int_{\mathbb{R}^d} \hat{f}(z)\,\rho\left(\frac{z-x}{\varepsilon}\right)dz\right)$$

$$= -\frac{1}{\varepsilon^{d+1}}\int_{\mathbb{R}^d} \hat{f}(z)\,\nabla\rho\left(\frac{z-x}{\varepsilon}\right)dz$$

$$|\nabla(f*\rho_\delta)(x)| \le \frac{1}{\varepsilon^{d+1}}\int_{\mathbb{R}^d} \underset{\frac{1}{p}}{|\hat{f}(x)|}\,\underset{\frac{1}{p'}}{\left|\nabla\rho\left(\frac{z-x}{\varepsilon}\right)\right|}\,dz$$

$$\le \frac{1}{\varepsilon^{d+1}}\|f\|_{L^p(\Omega)}\underset{\text{number}}{\left\{\int_{\mathbb{R}^d}\left|\nabla\rho\left(\frac{z-x}{\varepsilon}\right)\right|^{p'}dz\right\}^{1/p'}}$$

Conclusion : the set K_ε satisfies the assumptions of the standard Ascoli - Arzelà in $\mathrm{Clos}(\Omega)$ which is bounded and compact.

Therefore K_ε is relatively compact wrt unif. convergence, and therefore also with respect to L^p convergence.

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Istituzioni di Analisi – LECTURE 36

Note Title 08/11/2021

COMPACT EMBEDDING RESULTS

$\boxed{\text{Theorem}}$ Let $\Omega \subseteq \mathbb{R}^d$, and let us assume that

(i) Ω is BOUNDED

(ii) Ω is "regular" (what is needed for Sobolev embeddings and deluxe approximation)

Then the following statements are true

$\boxed{p<d}$ The embedding $W^{1,p}(\Omega) \to L^q(\Omega)$ is compact for every $q \in [1, p_*)$ ↑ NOT INCLUDED

$\boxed{p=d}$ $W^{1,p}(\Omega) \to L^q(\Omega)$ is compact for every $q \in [1, +\infty)$

$\boxed{p>d}$ $W^{1,p}(\Omega) \to C^0(\text{Clos}(\bar{\Omega}))$ is compact

Defn Let X and Y be metric space. A map $f: X \to Y$ is compact if the image of every bounded subset of X is relatively compact in Y. In other words

$\forall \{x_n\} \subseteq X$ bounded, $\{f(x_n)\}$ admits a converging subsequence in Y.

$\boxed{\text{Proof for } p>d}$ Almost trivial. If $\{u_n\} \subseteq W^{1,p}(\Omega)$ is bounded then u_n is bounded in $L^\infty(\Omega)$ and we have a control on Hölder constants

Standard A.A. $\Rightarrow$ conclusion.

$\boxed{\text{Proof for } p=d}$ Very easy. Consider any $q<+\infty$. Then there exists $p_1<p=d$ such that

$$q<(p_1)_*$$

Observe that $\{u_n\}$ bounded in $W^{1,p}$ $\Rightarrow$ $\{u_n\}$ bounded in W^{1,p_1} because Ω is bounded.

The conclusion follows from the case $p_1<d$.

$\boxed{\text{Proof of the true case}: p<d}$ Let us start with the case $q=1$.

Let $\mathcal{F}\subseteq W^{1,p}(\Omega)$ be a bounded family. We want to prove that $\mathcal{F}$ satisfies the assumptions of the compactness thm. in $L^1(\Omega)$.

The first two assumptions are true for free. We need assumption (iii), namely

$$\int_{\mathbb{R}^d} |\hat{u}(x+h)-\hat{u}(x)|\,dx \le \varepsilon$$

for every $u\in\mathcal{F}$ if $|h|$ is small.

$\boxed{\text{Step 1}}$ Let us set $\Omega_k := \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>\frac{1}{k}\}$

$$\int_{\Omega\setminus\Omega_k} |\hat{u}(x+h)-\hat{u}(x)|\,dx \le \int_{\Omega\setminus\Omega_k} |\hat{u}(x+h)|\,dx+\int_{\Omega\setminus\Omega_k}|\hat{u}(x)|\,dx$$

$$\int_{\Omega\setminus\Omega_k} \underset{\frac{1}{p}}{|\hat{u}(x)|}\cdot \underset{\frac{1}{p'}}{1}\,dx \le \underset{\le \|u\|_{L^p(\Omega)}}{\left\{\int_{\Omega\setminus\Omega_k}|\hat{u}(x)|^p dx\right\}^{1/p}}\cdot\underbrace{\operatorname{meas}(\Omega\setminus\Omega_k)^{\frac{1}{p'}}}_{\downarrow\ 0 \text{ as } k\to+\infty}$$

The same for the other term. We obtain that

$$\int_{\Omega\setminus\Omega_k} |\hat{u}(x+h)-\hat{u}(x)|\,dx \le 2\|u\|_{L^p(\Omega)}\operatorname{meas}(\Omega\setminus\Omega_k)^{\frac{1}{p'}}$$

[After video: if $p=1$ one needs to use 1_* instead of 1]

$\boxed{\text{Step 2}}$ Assume that $|h| < \frac{1}{k}$. Assume also $u \in C_c^\infty(\mathbb{R}^d)$.
Assume that $x \in \Omega_k$. Then

$$|u(x+h) - u(x)| \underset{\underset{\varphi(t) := u(x+ht)}{\uparrow}}{=} |\varphi(1) - \varphi(0)|$$

$$\leq \int_0^1 |\varphi'(t)|\, dt$$

$$= \int_0^1 |\langle \nabla u(x+ht), h \rangle|\, dt$$

$$\leq |h| \cdot \int_0^1 |\nabla u(x+ht)|\, dt$$

Let us integrate wrt x:

$$\int_{\Omega_k} |u(x+h) - u(x)|\, dx \leq |h| \cdot \int_{\Omega_k} dx \int_0^1 |\nabla u(x+ht)|\, dt$$

$$= |h| \int_0^1 dt \int_{\Omega_k} |\nabla u(\underbrace{x+ht}_{y})|\, dx \quad dx = dy$$

$$= |h| \int_0^1 dt \int_{\substack{\Omega_k + th \\ \subseteq \Omega}} |\nabla u(y)|\, dy$$

$$\leq |h| \int_0^1 dt \int_{\Omega} |\nabla u(y)|\, dy$$

$$= |h| \int_{\Omega} |\nabla u(y)| \cdot 1\, dy$$

$$\leq |h| \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \operatorname{meas}(\Omega)^{\frac{1}{p'}}$$

Conclusion: if $u \in \mathcal{Y}$ and $|h| < \frac{1}{k}$, then

$$\int_{\Omega_k} |u(x+h) - u(x)|\, dx \leq |h| \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \operatorname{meas}(\Omega)^{\frac{1}{p'}}$$

provided that also $u \in C_0^\infty(\mathbb{R}^d)$.
Why is the same true for every $u \in W^{1,p}(\Omega)$?
This is true due to deluxe approximation!!

$\boxed{\text{Step 3}}$ The goal was the following

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall u \in \mathcal{Y} \quad \forall h \in \mathbb{R}^d$$

$$|h| \leq \delta \quad \Rightarrow \quad \int_\Omega |\hat{u}(x+h) - u(x)|\, dx \leq \varepsilon$$

We proved that

$$\int_\Omega |\hat{u}(x+h) - \hat{u}(x)|\, dx = \int_{\Omega \setminus \Omega_k} \dots + \int_{\Omega_k} \dots$$

$$\leq 2\, \|u\|_{L^p(\Omega)} \cdot \operatorname{meas}(\Omega \setminus \Omega_k)^{\frac{1}{p'}} + |h| \cdot \|\nabla u\|_{L^p} \operatorname{meas}(\Omega)^{\frac{1}{p'}}$$

Given $\varepsilon > 0$, we choose k s.t. first term is $\leq \frac{\varepsilon}{2}$.

Then, after choosing k, we choose δ such that $\delta < \frac{1}{k}$ and the second term is $\leq \frac{\varepsilon}{2}$.

$\boxed{\text{Step 4}}$ We need compactness up to p_*.

$\boxed{\text{GENERAL FACT}}$ small / converging in L^1 + bounded in L^{p_*}
$\Rightarrow$ small / converging in L^q for every $q \in [1, p_*)$

This is due to the interpolation inequality

$$\|u\|_{L^q(\Omega)} \leq \|u\|^{\alpha}_{L^1(\Omega)} \cdot \|u\|^{1-\alpha}_{L^{p_*}(\Omega)}$$

where $\frac{\alpha}{1} + \frac{1-\alpha}{p_*} = \frac{1}{q}$

More generally

$$\|u\|_{L^q(\Omega)} \leq \|u\|^{\alpha}_{L^{p_1}(\Omega)} \cdot \|u\|^{1-\alpha}_{L^{p_2}(\Omega)} \qquad \forall q \in [p_1, p_2]$$

where $\frac{\alpha}{p_1} + \frac{1-\alpha}{p_2} = \frac{1}{q}$ with $\alpha \in (0,1)$

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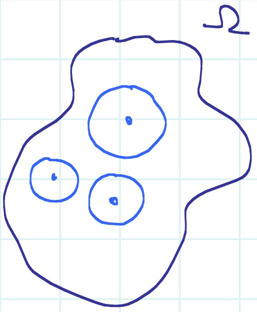
The embedding is NOT compact if $q = p_*$



$\boxed{\text{Step 1}}$ There exists a sequence of disjoint balls $B(x_n, r_n) \subseteq \Omega$

$\boxed{\text{Step 2}}$ Choose a nontrivial $\varphi \in C_c^\infty(B(0,1))$ and define

$$u_n(x) := \underset{\uparrow}{a_n}\, \varphi\left(\frac{x - x_n}{r_n}\right)$$

choose a_n in such a way that $\|\nabla u_n\| = 1$

$\uparrow$ in $L^p(\Omega)$ or $L^p(B(x_n, r_n))$

$\boxed{\text{Step 3}}$ Observe that

$$\|u_n\|_{L^{p_*}(\Omega)} \underset{\uparrow}{=} \text{constant} \quad (\text{does NOT depend on } n)$$

because of the defn. of p_*

$$\|u_n\|_{L^p(\Omega)} \underset{\uparrow}{\longrightarrow} 0 \quad \text{and therefore is bounded}$$

because $p < p_*$

In particular, the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$

$\boxed{\text{Step 4}}$ $\{u_n\}$ does NOT admit a converging subseq. in $L^{p_*}(\Omega)$ because

$$\int_\Omega |u_n(x) - u_m(x)|^{p_*} dx \underset{\substack{\uparrow \\ \text{balls are disjoint}}}{=} \int_\Omega |u_n(x)|^{p_*} dx + \int_\Omega |u_m(x)|^{p_*} dx$$

$$= 2 \cdot \text{constant}$$

$\Rightarrow$ $\{u_n\}$ has NO Cauchy subseq.

Rmk $u_n \to 0$ for every $p \in [1, p_*)$

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Istituzioni di Analisi — LECTURE 37

Note Title 10/11/2021

TRACES OF SOBOLEV FUNCTIONS

Problem: given $u \in W^{1,p}(\Omega)$, define $u_{|\partial\Omega}$ and more generally
$u_{|K}$ where $K \subseteq \mathrm{Clos}(\Omega)$ is a subset
→ either of codimension 1
→ or of " 2.

Rmk If $u \in L^p(\Omega)$ and nothing better, then there is no hope to define restrictions to sets of measure 0.

We look for a map

$$Tr: W^{1,p}(\Omega) \longrightarrow \mathrm{Space}(\partial\Omega)$$

with some reasonable properties
→ linear
→ it coincides with $u_{|\partial\Omega}$ if $u \in C^0(\mathrm{Clos}(\Omega))$
→ it is continuous, namely if $u_n \to u_\infty$ in some sense, then $Tr\, u_n \to Tr\, u_\infty$ in Space$(\partial\Omega)$
→ it satisfies estimates such as

$$\|Tr\, u\|_{\mathrm{Space}(\partial\Omega)} \leq \mathrm{const}\, \|u\|_{1,p,\Omega}$$

→ some int. by parts formula holds true.

Easy case 1 $\Omega = (a,b) \subseteq \mathbb{R}$. In this case functions in $W^{1,p}(\Omega)$ are continuous up to the boundary

Easy case 2 $\Omega \subseteq \mathbb{R}^d$ and $u \in W^{1,p}(\Omega)$ with $p > d$ and Ω regular.

As before u is continuous (also Hölder cont.) in $\mathrm{Clos}(\Omega)$, so that in this case we can take

$$\mathrm{Space}(\partial\Omega) = C^0(\partial\Omega)$$

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$\boxed{\text{FIRST NONTRIVIAL CASE}}$ Model case $\boxed{p < d}$ and Ω is a half-space

$$\Omega = \{ (\underset{d-1}{x}, \underset{1}{y}) \in \mathbb{R}^d : y > 0 \} = \mathbb{R}^d_+$$

$$\partial\Omega = \{ (x,y) \in \mathbb{R}^d : y = 0 \}$$

$\boxed{\text{Proposition}}$ (key estimates) Let us assume that $u \in C^\infty_c(\mathbb{R}^d)$.

(1) $$\boxed{\| u(x,0) \|_{L^p(\mathbb{R}^{d-1})} \leq c(p,d) \, \| u \|_{1,p,\mathbb{R}^d_+}}$$

IMPURE

u defined in $\mathbb{R}^d$
LHS is in $\mathbb{R}^{d-1}$
RHS with full norm in $\mathbb{R}^d_+$

(2) Let us assume $p < d$ and set $\hat{p}_* := \dfrac{p(d-1)}{d-p}$. Then

PURE

$$\boxed{\| u(x,0) \|_{L^{\hat{p}_*}(\mathbb{R}^{d-1})} \leq c(p,d) \, \| \nabla u \|_{L^p(\mathbb{R}^d)}}$$

u in $\mathbb{R}^d$
LHS in $\mathbb{R}^{d-1}$
RHS only ∇u in $\mathbb{R}^d$

(3) $$\boxed{\| u(x,0) \|_{L^q(\mathbb{R}^{d-1})} \leq c(p,d,q) \, \| u \|_{1,p,\mathbb{R}^d} \quad \forall q \in [p, \hat{p}_*]}$$

IMPURE RHS with full norm in $\mathbb{R}^d$

$\boxed{\text{Proof}}$ $\boxed{\text{Step 1}}$ Consider $p = 1$

$$|u(x,0)| = \left| -\int_0^{\overset{\nearrow \text{any large } R \text{ is enough}}{+\infty}} \frac{\partial u}{\partial y}(x,y)\,dy \right|$$

$$\leq \int_0^{+\infty} \left|\frac{\partial u}{\partial y}(x,y)\right| dy$$

We integrate wrt x:

$$\int_{\mathbb{R}^{d-1}} |u(x,0)|\,dx \leq \int_{\mathbb{R}^{d-1}} dx \int_0^{+\infty} \left|\frac{\partial u}{\partial y}(x,y)\right| dy$$

$$= \int_{\mathbb{R}^d_+} \left|\frac{\partial u}{\partial x}(x,y)\right| dx\,dy$$

$\boxed{\text{Step 2}}$ Any $p \geq 1$. We set $v(x,y) := |u(x,y)|^r u(x,y)$ and we apply Step 1 to $v(x,y)$.
We observe that

$$\frac{\partial v}{\partial y}(x,y) = (r+1)\,|u(x,y)|^r \frac{\partial u}{\partial y}(x,y)$$

$$\int_{\mathbb{R}^{d-1}} |u(x,0)|^{r+1}\,dx \leq \int_{\mathbb{R}^d_+} (r+1)\,\underset{\frac{1}{p'}}{|u(x,y)|^r}\,\underset{\frac{1}{p}}{\left|\frac{\partial u}{\partial y}(x,y)\right|}\,dx\,dy$$

$$\|u(x,0)\|^{r+1}_{L^{r+1}(\mathbb{R}^{d-1})} \leq (r+1) \left\|\frac{\partial u}{\partial y}\right\|_{L^p(\mathbb{R}^d_+)} \left\{\int_{\mathbb{R}^d_+} |u(x,y)|^{rp'}\,dx\,dy\right\}^{\frac{1}{p'}}$$

$$\left\{\int_{\mathbb{R}^d_+} |u(x,y)|^{rp'}\,dx\,dy\right\}^{\frac{1}{p'}} = \|u\|^r_{L^{rp'}(\mathbb{R}^d_+)}$$

We proved that

$$\|u(x,0)\|^{r+1}_{\underset{\uparrow\,\text{[After video]}}{L^{r+1}}(\mathbb{R}^{d-1})} \leq (r+1) \left\|\frac{\partial u}{\partial y}\right\|_{L^p(\mathbb{R}^d_+)} \|u\|^r_{L^{rp'}(\mathbb{R}^d_+)}$$

Choose r such that $rp' = p$ ⇝ $r = \frac{p}{p'} = \cancel{p}\,\frac{p-1}{\cancel{p}} = p-1$

In this case we obtain what we need

$$\|u(x,0)\|^{p}_{L^p(\mathbb{R}^{d-1})} \leq p \left\|\frac{\partial u}{\partial y}\right\|_{L^p(\mathbb{R}^d_+)} \|u\|^{p-1}_{L^p(\mathbb{R}^d_+)}$$

$$\searrow \qquad \swarrow$$

$$\|u\|^{p}_{1,p,\mathbb{R}^d_+}$$

$\boxed{\text{Step 3}}$ Choose r such that $rp' = p_*$

$$\frac{r\cancel{p}}{p-1} = \frac{\cancel{p}d}{d-p} \rightsquigarrow r = \frac{d(p-1)}{d-p} \rightsquigarrow r+1 = \frac{dp-d}{d-p}+1$$

$$= \frac{dp-\cancel{d}+\cancel{d}-p}{d-p} = \frac{p(d-1)}{d-p} = \hat{p}_*$$

We obtain that

$$\|u(x,0)\|^{\hat{p}_*}_{L^{\hat{p}_*}(\mathbb{R}^{d-1})} \leq c(p,d) \underset{\overset{\uparrow}{\leq \|\nabla u\|_{L^p(\mathbb{R}^d)}}}{\left\|\frac{\partial u}{\partial y}\right\|_{L^p(\mathbb{R}^d_+)}} \underbrace{\|u\|^{\hat{p}_*-1}_{L^{p_*}(\mathbb{R}^d_+)}}_{\leq \|u\|^{\hat{p}_*-1}_{L^{p_*}(\mathbb{R}^d)}}$$

$$\leq \hat{c}(p,d)\,\|\nabla u\|^{\hat{p}_*-1}_{L^p(\mathbb{R}^d)}$$

and the conclusion follows.

$\boxed{\text{Step 4}}$ The estimate for every $q \in [p, \hat{p}_*]$ follows by interpolation from the previous ones.

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<u>Rmk</u> The statements for the case $p = d$ follow in the same way. (exercise!)

$\boxed{\text{Theorem}}$ Let $p<d$, and let $q \in [p, \hat{p}_*]$.
Then there exists

$$Tr : W^{1,p}(\mathbb{R}^d_+) \longrightarrow L^q(\mathbb{R}^{d-1})$$

with the required properties

$\boxed{\text{Proof}}$ → Take $u \in W^{1,p}(\mathbb{R}^d_+)$

→ extend by reflection to $\hat{u} \in W^{1,p}(\mathbb{R}^d)$

→ approximate deluxe $\underset{\substack{\uparrow \\ \text{in } C^\infty_c(\mathbb{R}^d)}}{u_n} \longrightarrow \hat{u}$ in full norm

→ observe that $\{u_n\}$ is a Cauchy sequence wrt the full norm of $W^{1,p}(\mathbb{R}^d)$

→ by the Prop. we know that $\{u_n(x,0)\}$ are a Cauchy seq. in $L^q(\mathbb{R}^{d-1})$

→ define $Tr\, u$ as the limit in $L^q(\mathbb{R}^{d-1})$ of $u_n(x,0)$.

The defn. is well posed because if $v_n \to \hat{u}$ is a different deluxe approx, then $u_1, v_1, u_2, v_2, u_3, v_3, \ldots$ is again a Cauchy sequence...

If u is continuous, then $\hat{u}$ is continuous, then we can choose u_n defined by convolution and in this case

$$u_n \longrightarrow \hat{u} \quad \text{uniformly in } \mathbb{R}^d$$

and therefore $Tr\, u_n \longrightarrow \hat{u}(x,0)$

All the estimates and the int. by parts formula pass to the limit from u_n to u and $Tr\, u$.

Istituzioni di Analisi — LECTURE 38

Note Title 10/11/2021

CONTINUITY OF THE TRACE

Trivial statement: if $u_n \to u_\infty$ in $W^{1,p}(\mathbb{R}^d_+)$ (in full norm) then $\mathrm{Tr}\, u_n \to \mathrm{Tr}\, u_\infty$ in $L^q(\mathbb{R}^{d-1})$ for every $q \in [p, \hat{p}_*]$

Follows from linearity + estimates

Prop (Hölder continuity of the trace wrt to the level) $\boxed{p>1}$
For every $u \in W^{1,p}(\mathbb{R}^d_+)$ it turns out that

$$\|u(x,y_2)-u(x,y_1)\|_{L^p(\mathbb{R}^{d-1})} \le \left\|\frac{\partial u}{\partial y}\right\|_{L^p(\mathbb{R}^d_+)} |y_2-y_1|^{\frac{1}{p'}} \quad \forall\, 0<y_1<y_2$$

Proof $$|u(x,y_2)-u(x,y_1)| \le \int_{y_1}^{y_2} \underset{\overset{\uparrow}{\frac{1}{p}}}{\left|\frac{\partial u}{\partial y}(x,y)\right|} \cdot \underset{\overset{\uparrow}{\frac{1}{p'}}}{1}\, dy$$

$$\le |y_2-y_1|^{1/p'} \left\{\int_{y_1}^{y_2} \left|\frac{\partial u}{\partial y}(x,y)\right|^p dy\right\}^{1/p}$$

We raise both sides to power p and we integrate wrt x:

$$\int_{\mathbb{R}^{d-1}} |u(x,y_2)-u(x,y_1)|^p dx \le |y_2-y_1|^{\frac{1}{p'}} \underbrace{\int_{\mathbb{R}^{d-1}} dx \int_{y_1}^{y_2} \left|\frac{\partial u}{\partial y}(x,y)\right|^p dy}$$

$$\le \int_{\mathbb{R}^d_+} \left|\frac{\partial u}{\partial y}(x,y)\right|^p dx\,dy$$

This is good if $u \in C^\infty_c(\mathbb{R}^d)$. Then we conclude by approx.

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Theorem Assume $1 \underset{\substack{\uparrow \\ \text{IMPORTANT}}}{<} p < d$

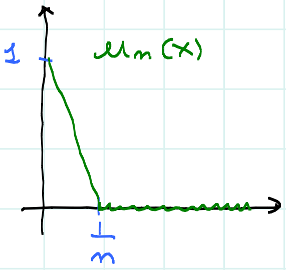
Let $\{u_n\} \subseteq W^{1,p}(\mathbb{R}^d_+)$. Assume that

(i) $u_n \longrightarrow u_\infty$ in $L^p(\mathbb{R}^d_+)$

(ii) there exists $M \in \mathbb{R}$ s.t. $\|\nabla u_n\|_{L^p(\mathbb{R}^d_+)} \leq M$

Then $\mathrm{Tr}\, u_n \longrightarrow \mathrm{Tr}\, u_\infty$ in $L^q(\mathbb{R}^{d-1})$ for every $q \in [p, \hat{p}_* \overset{\substack{\text{EXCLUDED} \\ \downarrow}}{)}$.

Example $p=1$ and $d=1$ and $\Omega = (0,1)$



We observe that $u_n \to 0$ in $L^p(\Omega)$ for every $p \in [1, +\infty)$ and

$\|\dot{u}_n\|_{\underset{\substack{\uparrow \\ \text{only in } L^1}}{L^1}((0,1))} = 1 \quad \forall n \geq 1$

In this case $\mathrm{Tr}\, u_n \not\to \mathrm{Tr}\, u_\infty$

General philosophy: the trace is well-defined for every $p \geq 1$ but it does NOT behave well if $p = 1$.

Proof **Step 1** It is enough to prove the case $q = p$. This is due to the usual interpolation inequality

conv. in L^p + $\underset{\substack{\uparrow \\ \text{follows from bound} \\ \text{on } \|u_n\|_{1,p,\mathbb{R}^d_+}}}{\text{bound in } L^{\hat{p}_*}}$ $\Rightarrow$ conv. in L^q for every $q \in [p, \hat{p}_*)$

Step 2 Observe that $u_\infty \in W^{1,p}(\mathbb{R}^d_+)$ and $\|\nabla u_\infty\|_{L^p(\mathbb{R}^d_+)} \leq M$.

Since $\|\nabla u_n\|_{L^p}$ are bounded and $p > 1$, up to subsequences we can assume that

$$\nabla u_n \rightharpoonup \text{something} \quad \text{weakly in } L^p(\mathbb{R}^d_+)$$

(Here we use weak compactness of balls in L^p when $p>1$)

In the usual way we see that something $= \nabla u_\infty$ and

$$\|\nabla u_\infty\|_{L^p(\mathbb{R}^d_+)} \leq M$$

(the norm of the gradient is convex, and therefore LSC wrt weak convergence).

Therefore the trace of u_∞ is well defined.

$\boxed{\text{Step 3}}$ We use Morrey's trick (exchange zone)

$$\begin{aligned}\|u_\infty(x,0)-u_n(x,0)\|_{L^p(\mathbb{R}^{d-1})} \leq\ & \|u_\infty(x,0)-u_\infty(x,y)\|_{L^p\ldots} \\ & + \|u_\infty(x,y)-u_n(x,y)\|_{L^p\ldots} \\ & + \|u_n(x,y)-u_n(x,0)\|_{L^p\ldots}\end{aligned}$$

for every $y>0$.

Let us raise to power p and use that

$$(A+B+C)^p \leq 3^{p-1}(A^p+B^p+C^p) \qquad \forall\,(A,B,C)\in[0,+\infty)^3$$

$$\begin{aligned}\|\mathrm{Tr}\,u_\infty - \mathrm{Tr}\,u_n\|^p_{L^p} \leq 3^{p-1}\Big\{ & \|u_\infty(x,0)-u_\infty(x,y)\|^p_{L^p} && ① \\ & + \|u_\infty(x,y)-u_n(x,y)\|^p_{L^p} && ② \\ & + \|u_n(x,y)-u_n(x,0)\|^p_{L^p}\Big\} && ③\end{aligned}$$

Now fix $\delta>0$ and integrate in $\underset{\substack{\uparrow \\ \text{exchange zone}}}{(0,\delta)}$ wrt y

$$\text{LHS} = \delta \, \| \operatorname{Tr} u_\infty - \operatorname{Tr} u_n \|^p_{L^p}$$

③ $$\int_0^\delta \underbrace{\| u_n(x,y) - u_n(x,0) \|^p_{L^p}}_{\leq \left\| \frac{\partial u_n}{\partial y} \right\|_{L^p(\mathbb{R}^d_+)} \cdot y^{\frac{p}{p'}}} dy \leq \underbrace{\left\| \frac{\partial u_n}{\partial y} \right\|_{L^p(\mathbb{R}^d_+)}}_{\leq M} \cdot \, \delta^{1+\frac{p}{p'}}$$

① $$\int_0^\delta \| u_\infty(x,y) - u_\infty(x,0) \|^p_{L^p} dy \leq M \delta^{1+\frac{p}{p'}}$$

② $$\int_0^\delta dy \int_{\mathbb{R}^{d-1}} dx \, | u_\infty(x,y) - u_n(x,y) |^p \leq \| u_\infty - u_n \|^p_{L^p(\mathbb{R}^d_+)}$$

In conclusion we obtain

$$\delta \, \| \operatorname{Tr} u_\infty - \operatorname{Tr} u_n \|^p_{L^p(\mathbb{R}^{d-1})} \leq 3^{p-1} \left\{ 2M \delta^{1+\frac{p}{p'}} + \underset{\substack{\downarrow \\ 0}}{\| u_\infty - u_n \|^p_{L^p(\mathbb{R}^d_+)}} \right\}$$

We take the limsup as $n \to +\infty$

$$\cancel{\delta} \limsup_{n \to +\infty} \| \operatorname{Tr} u_\infty - \operatorname{Tr} u_n \|^p_{L^p(\dots)} \leq 3^{p-1} \cdot 2M \, \delta^{\cancel{1+}\frac{p}{p'}}$$

Finally, let $\delta \to 0^+$ and observe that $\frac{p}{p'} > 0$ because $p > 1$.

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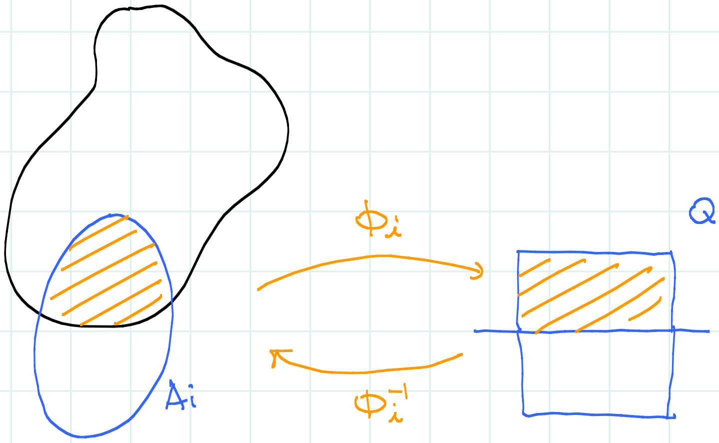
$\boxed{\text{How to define the trace for a general open set}}$

Assume Ω is regular in the sense that $\partial\Omega$ is compact and C^1

The key point is proving inequalities such that

$$\| u \|_{L^q(\partial\Omega)} \leq c(p,d,q,\Omega) \, \| u \|_{1,p,\Omega}$$

for every $u \in C_c^\infty(\mathbb{R}^d)$



→ Take any $u \in C_c^\infty(\mathbb{R}^d)$

→ Write u as a sum $u_0 + u_1 + \dots + u_n$ where u_i has support contained in A_i

→ Move everything to Q using Φ_i

→ In Q we are in the model case ($\mathbb{R}^d_+$)

→ Estimate the trace in Q

→ Go back to A_i using Φ_i^{-1}.

The key points are that if $u_i \in W^{1,p}(A_i)$ then $u_i \circ \Phi^{-1} \in W^{1,p}(\mathbb{R}^d)$ with an estimate such as

$$m \|u_i\|_{1,p,A_i} \leq \|u_i \circ \Phi^{-1}\|_{1,p,\mathbb{R}^d} \leq M \|u_i\|_{1,p,A_i}$$

and the same is true with $\|u_i\|_{L^q(\partial\Omega \cap A_i)}$ and $\|u_i(x,0)\|_{L^q(\mathbb{R}^{d-1})}$

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Note Title 11/11/2021

$\boxed{\text{The space } \underset{\uparrow\uparrow}{W_0^{m,p}}(\Omega)}$

<u>Def.</u> Let $\Omega \subseteq \mathbb{R}^d$ be an open set, let $m \geq 1$, let $p \in [1,+\infty]$.

(1) The space $W_0^{m,p}(\Omega)$ is the closure in $W^{m,p}(\Omega)$ of $\overset{\text{the same with } m}{\overset{\nearrow}{C_c^\infty}}(\Omega)$

(closure wrt the full norm of $W^{m,p}(\Omega)$)

(2) The space $W_0^{m,p}(\Omega)$ is the abstract completion of $\overset{\text{the same with } m}{\overset{\nearrow}{C_c^\infty}}(\Omega)$ with respect to the metric derived from the full norm of $W^{m,p}(\Omega)$.

<u>Easy Lemma</u> The two definitions are equivalent.

<u>Intuitive idea</u>: we expect elements in $W_0^{m,p}(\Omega)$ to be Sobolev functions that vanish on $\partial\Omega$ with all derivatives up to order $m-1$.

In general this is false, but true under suitable assumptions.

<u>Achtung!</u> Do NOT confuse spaces such as

$$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \qquad \text{and} \qquad W_0^{2,p}(\Omega)$$

For example, when $\Omega = (a,b) \subseteq \mathbb{R}$, then

$$W_0^{2,p}((a,b)) = \{u \in W^{2,p}((a,b)) : \overbrace{u(a) = u'(a) = u(b) = u'(b) = 0}^{u_n \to u \text{ and } u_n' \to u' \text{ unif.}}\}$$

$$W^{2,p}((a,b)) \cap W_0^{1,p}((a,b)) = \{u \in W^{2,p}((a,b)) : \underbrace{u(a) = u(b) = 0}_{\text{only } u_n \to u \text{ unif.}}\}$$

Rmk If $p > d$, and $u \in W_0^{1,p}(\Omega)$, then necessarily $u = 0$ at $\partial\Omega$

Proof Let $\{u_n\} \subseteq C_c^\infty(\Omega)$ be such that $u_n \to u$ and $\nabla u_n \to \nabla u$ in $L^p(\Omega)$.

Due to the bounds on $\|\nabla u_n\|_{L^p(\underset{\uparrow \mathbb{R}^d}{\Omega})}$ and $\|u_n\|_{L^p(\underset{\uparrow \mathbb{R}^d}{\Omega})}$

we can apply Ascoli – Arzelà in every ball $\subseteq \mathbb{R}^d$, and therefore $u_n \to \underset{\uparrow \text{extended to 0 outside } \Omega}{u}$ unif. on bounded sets of $\mathbb{R}^d$

Theorem All embedding theorems hold true in $W_0^{1,p}(\Omega)$ without assumptions on Ω and with the same pure inequalities as in the case of $\mathbb{R}^d$.

In particular if $u \in W_0^{1,p}(\Omega)$

(1) If $p < d$, then $\boxed{\|u\|_{L^{p*}(\Omega)} \leq c(p,d) \|\nabla u\|_{L^p(\Omega)}}$

(2) If $p = d$, then $\boxed{\|u\|_{L^q(\Omega)} \leq c(p,d,q) \|u\|_{1,p,\Omega}}$

(3) If $p > d$, then $\boxed{\|u\|_{L^\infty(\Omega)} \leq c(p,d) \|u\|_{1,p,\Omega}}$

Moreover, u is $1-\frac{d}{p}$ Hölder cont., with a pure estimate on the Hölder constant.

Proof The estimates are true for every $u \in C_c^\infty(\underset{\uparrow \mathbb{R}^d}{\Omega})$

and then we can pass to the limit.

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Rmk Same is true for compact embeddings.

<u>Prop.</u> (Always true, even if Ω is "bad")

If $u \in W_0^{1,p}(\Omega)$, then $\underset{\substack{\uparrow \\ \text{extension to 0} \\ \text{outside } \Omega}}{\hat{u}} \in W^{1,p}(\mathbb{R}^d)$.

<u>Proof</u> Let us consider $\{u_n\} \subseteq C_c^\infty(\Omega)$ such that

$u_n \to u$ and $\nabla u_n \to \nabla u$ in $L^p(\Omega)$

Observe that actually $u_n \in C_c^\infty(\mathbb{R}^d)$ and

$\hat{u}_n = u_n$ and $\widehat{\nabla u_n} = \nabla u_n$

and in addition

$$\left.\begin{aligned} \hat{u}_n &\to \hat{u} \quad \text{in } L^p(\mathbb{R}^d) \\ \nabla \hat{u}_n &\to \nabla \hat{u} \quad \text{in } L^p(\mathbb{R}^d) \end{aligned}\right\} \text{ outside } \Omega \text{ everything is } 0.$$

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<u>Theorem</u> Assume now that $\partial\Omega$ is compact and of class C^1.

Then the following are equivalent (for a function in $W^{1,p}(\Omega)$).

↑ [Added after video]

(1) $u \in W_0^{1,p}(\Omega)$

(2) $\hat{u} \in W^{1,p}(\mathbb{R}^d)$ extension to 0

(3) $\operatorname{Tr} u = 0$ in $\partial\Omega$ (the trace is well-defined)

$\boxed{\text{Proof}}$ (1) ⇒ (2) is always true.

We need to show that (3) ⇒ (2) ⇒ (1) ⇒ (3)

$\boxed{(1) \Rightarrow (3)}$ <u>Easy way</u>: consider the definition of $\operatorname{Tr} u$.

Extend u to $\underset{\substack{\uparrow \\ \text{extension to 0}}}{\hat{u}}$. Approximate $\hat{u}$ in a deluxe way. In this

case we know that we can do it with $u_n \in C_c^\infty(\Omega)$.

$\operatorname{Tr} u$ is the limit in $L^p(\partial\Omega)$ of the restrictions of u_n to the boundary, which are 0.

Alternative way Consider again the sequence $\{u_n\} \subseteq C^\infty_c(\Omega)$ that converges to u in full norm. Then exploit the continuity of the trace.

$\boxed{(3) \Rightarrow (2)}$ We need to prove that

$$\int_{\mathbb{R}^d} \hat{u}(x)\, D_{x_i}\varphi(x)\,dx = -\int_{\mathbb{R}^d} \underset{\substack{\uparrow \\ \text{should be } \widehat{D_{x_i}u}(x)}}{D_{x_i}\hat{u}(x)}\, \varphi(x)\,dx \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^d)$$

$$\text{LHS} = \int_\Omega u(x)\, D_{x_i}\varphi(x)\,dx$$

$$\text{RHS} = -\int_\Omega D_{x_i}u(x)\,\varphi(x)\,dx$$

They are equal if $\varphi \in C^\infty_c\underset{\uparrow\uparrow}{(\Omega)}$ but this is NOT enough

On the other hand, the integration by parts is always true

$$\int_\Omega u(x) \underset{\substack{\uparrow \\ \operatorname{div} E(x)}}{D_{x_i}\varphi(x)}\,dx = -\int_\Omega \underset{\substack{\uparrow \\ \langle \nabla u(x), \vec{E}(x) \rangle}}{D_{x_i}u(x)\,\varphi(x)}\,dx + \int_{\partial\Omega} \underbrace{\operatorname{Tr} u}_{\substack{\shortparallel \\ 0}} \langle \vec{E}, \vec{n} \rangle\, d\sigma$$

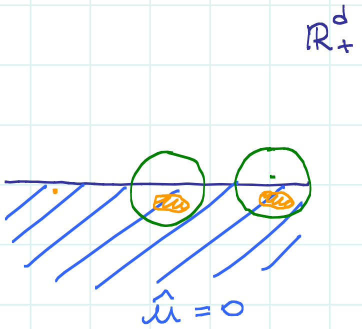
$$E(x) = (0, \dots, 0, \underset{\substack{\uparrow \\ i}}{\varphi}, 0 \dots 0)$$

$\boxed{(2) \Rightarrow (1)}$ In the usual way we reduce to the model case where $\Omega = \mathbb{R}^n_+$

We have $\hat{u} \in W^{1,p}(\mathbb{R}^d)$.

We want to approximate

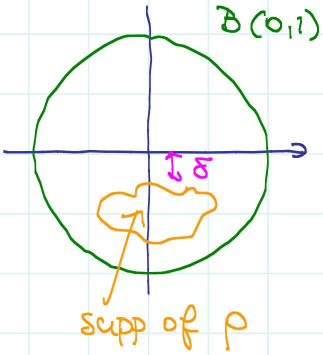
$$\underset{\substack{\uparrow \\ \in C^\infty_c(\mathbb{R}^d_+)}}{u_m} \longrightarrow \hat{u} \quad \text{in } W^{1,p}(\mathbb{R}^d)$$



Consider as usual $u_\varepsilon := u * \rho_\varepsilon$

The support of u_ε in general is not contained in $\mathbb{R}^d_+$

Trick to fix: consider a convolution kernel ρ



When I compute the convolution, it is equal to zero provided that the distance from the hyperplane is $\geq \varepsilon\delta$.

In this case $u * \rho_\varepsilon$ has compact support in $\mathbb{R}^d_+$.

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Note Title 11/11/2021

Question : are there cases where $W^{1,p}(\Omega) = W_0^{1,p}(\Omega)$?

First case: TRUE if $\Omega = \mathbb{R}^d$ (Deluxe approximation)

Second example: $d=1$ and $\Omega = \mathbb{R} \setminus \{0\}$. In this case it is FALSE because

$$W_0^{1,p}(\Omega) = \{ u \in W^{1,p}(\Omega) : u(0) = 0 \}$$

This is because Sobolev functions are continuous up to the boundary and if $u_n \to u$ in the full norm of Ω, then $u_n \to u$ uniformly on compact subsets of $\mathrm{Clos}(\Omega)$

(this is true and easy if $p>1$ because bound on full norm $\Rightarrow$ Ascoli – Arzelà; if $p=1$ it is true but less easy)

Third example : $d \geq 2$ and $\Omega = \mathbb{R}^d \setminus \{\text{origin}\}$

↑ TRUE in general if what we remove is "small"

It is TRUE that

$$W_0^{1,p}(\Omega) = W^{1,p}(\Omega)$$

provided that $p \leq d$.

What we need to prove: for every $u \in W^{1,p}(\Omega)$ there exists $\{u_n\} \subseteq C_c^\infty(\Omega)$ s.t. $u_n \to u$ in full norm.

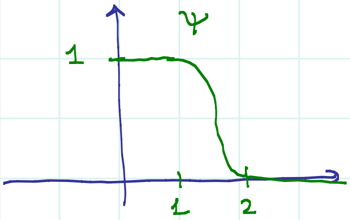
Equivalently: for every $\varepsilon > 0$ there exists $u_\varepsilon \in C_c^\infty(\Omega)$ s.t.

$$\| u - u_\varepsilon \|_{1,p,\Omega} \leq \varepsilon .$$

Let us prove the equivalent statement

$\boxed{\text{Step 1}}$ For every $\varepsilon > 0$ there exists $v_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$\| u - v_\varepsilon \|_{1,p,\Omega} \le \frac{1}{2}\varepsilon$$



Consider $u_n(x) := u(x) \cdot \psi\left(\frac{|x|}{n}\right)$

It is easy to see that $u_n \to u$ in full norm of $W^{1,p}(\Omega)$ (pointwise convergence + domination)

If n is large enough then

$$\| u - u_n \|_{\underset{\uparrow \atop \text{or } \mathbb{R}^d}{1,p,\Omega}} \le \frac{1}{4}\varepsilon$$

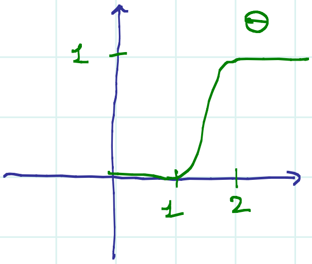
Now consider a convolution $u_n * \rho_\delta$ and if δ is small enough then

$$\| u_n - u_n * \rho_\delta \|_{1,p,\Omega} \le \frac{1}{4}\varepsilon$$

and of course $\underbrace{u_n * \rho_\delta}_{{}'' v_\varepsilon}$ is in $C_c^\infty(\mathbb{R}^d)$

$\boxed{\text{Step 2}}$ We modify v_ε in order to obtain $u_\varepsilon \in C_c^\infty(\Omega)$ and

$$\| v_\varepsilon - u_\varepsilon \|_{1,p,\Omega} \le \frac{1}{2}\varepsilon.$$



Define $v_n(x) := v_\varepsilon(x) \cdot \theta\left(n|x|\right)$

$\uparrow$ coincides with v_ε if $|x| \ge \frac{2}{n}$

Claim: $v_n \longrightarrow v_\varepsilon$ in full norm

$v_n \longrightarrow v_\varepsilon$ in $L^p(\Omega)$ (pointwise convergence + domination)

$$\nabla v_n(x) = \nabla v_\varepsilon(x)\cdot\theta(n|x|) + v_\varepsilon(x)\cdot n\,\theta'(n|x|)\,\frac{x}{|x|}$$

$\downarrow$

$\nabla v_\varepsilon(x)$ in L^p

We need that the second term $\to 0$ in $L^p(\Omega)$

$$\int_\Omega \overbrace{|v_\varepsilon(x)|^p}^{\le C}\cdot n^p\cdot\overbrace{|\theta'(n|x|)|^p}^{\le C}\,dx \le \text{const}\cdot n^p\cdot \text{meas}\left(B\left(0,\tfrac{2}{n}\right)\right)$$

$\uparrow B(0,\frac{2}{n})$ because otherwise $\theta'=0$

$$\sim\frac{\text{const}}{n^d}$$

$$\sim \text{const}\;\frac{n^p}{n^d}$$

This goes to 0 if $p<d$.

The statement is true if $p<d$.

— o — o —

$\boxed{\text{Exercise}}$ Given $E \subset\subset \Omega \subset\subset \mathbb{R}^d$

$\uparrow$ any meas. set $\quad\uparrow$ open set

Define

$$\text{Cap}_p(E,\Omega) := \inf\left\{\int_\Omega |\nabla u(x)|^p\,dx : u\in W_0^{1,p}(\Omega),\ \underset{\downarrow}{u(x)\ge 1 \text{ in } E}\right\}$$

$\uparrow$ Capacity $\quad u\in C_c^\infty(\Omega) \to$ makes sense

<u>Statement</u> $\quad \text{Cap}_p(\text{point},\text{ball}) = \begin{cases}\nearrow 0 & \text{if } p\le d\\ \searrow >0 & \text{if } p>d\end{cases}$

$\boxed{\text{Cap}_p(\text{point, ball})>0 \text{ if } p>d}$ Claim: there exists

$$\min\left\{\int_{\underset{\uparrow\text{ball}}{\Omega}}|\nabla u|^p\,dx : u\in W_0^{1,p}(\Omega),\ u(0)\ge 1\right\}$$

$\uparrow$ well defined because $p>d$

<u>Direct method</u> Let u_n be a minimizing sequence.

Then $\|\nabla u_n\|_{L^p(\Omega)} \leq M$

In addition $\|u_n\|_{L^\infty(\Omega)} \leq M'$ (because $\|\nabla u_n\|$ controls the Hölder constant of u_n, which is 0 at the boundary)

A.A. $\Rightarrow$ up to subsequences $u_n \to u_\infty$ unif. in $\mathrm{Clos}(\Omega)$ and therefore also $u_\infty(0) \geq 1$.

As usual, up to further subsequences

$$\nabla u_n \rightharpoonup \nabla u_\infty \quad \text{weakly in } L^p(\Omega)$$

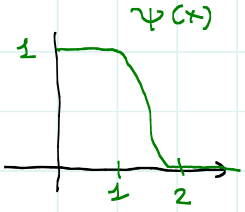
and finally

$$\liminf_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^p\, dx \underset{\substack{\uparrow \\ \text{convexity of the norm}}}{\geq} \int_\Omega |\nabla u_\infty(x)|^p\, dx$$

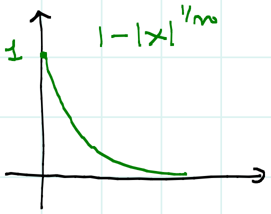
Weierstrass theorem $\Rightarrow$ the min exists.

Claim: it is >0. If not, then $\nabla u(x) = 0$ and this implies (true, but not trivial) that $u \equiv$ const, but $u=0$ at the boundary and $u = 1$ in the center.

How to prove that $\mathrm{Cap}_p(\text{center}, \text{ball}) = 0$ if $p \leq d$



- If $p<d$, consider
$$u_n(x) = \psi(n|x|)$$



- If $p=d$, consider
$$u_n(x) = 1-|x|^{1/n}$$

(we can modify it near 0 if we want $u_n \in C^\infty$)

— o — o —

Istituzioni di Analisi – LECTURE 41

Note Title 15/11/2021

POINCARÉ INEQUALITY

Setting : $\Omega \subseteq \mathbb{R}^d$ is a BOUNDED set and $p \in [1,+\infty]$.

$$\boxed{\|u\|_{L^p(\Omega)} \leq c(p,d,\Omega)\|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W_0^{1,p}(\Omega)}$$

NATURAL INEQUALITY : 0 at $\partial\Omega$ and $|\nabla u|$ small $\Rightarrow u$ small

$\uparrow\uparrow$ NECESSARY (think of a constant function)

POINCARÉ - SOBOLEV INEQ

$\Omega \subseteq \mathbb{R}^d$ is bounded

$$\boxed{\|u\|_{L^q(\Omega)} \leq c(p,q,d,\Omega)\|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W_0^{1,p}(\Omega)}$$

This inequality holds true for $q \in [1,p_*]$ if $p<d$.
True for every $q \in [1,+\infty)$ if $p=d$
True for every $q \in [1,+\infty]$ if $p>d$.

Proof Almost trivial if $p>d$. The case $p=d$ follows from $p<d$. Let us consider $p<d$.

[After video : c depends on q, p_*, meas(Ω)]

$$\|u\|_{L^q(\Omega)} \leq c(q,\Omega)\|u\|_{L^{p_*}(\Omega)}$$

$\uparrow$ Ω is bounded

$$\leq c(p,d,q,\Omega)\|\nabla u\|_{L^p(\Omega)}$$

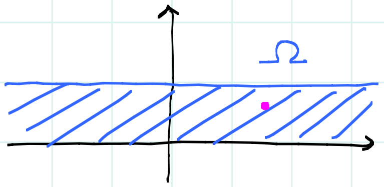
$\uparrow$ Sobolev embedding with PURE inequality → TRUE because $u \in W_0^{1,p}(\Omega)$

<u>Rmk</u> In the previous proof we do NOT need any regularity on $\partial\Omega$.

$\boxed{\text{THE EXAMPLE}}$ The boundedness of Ω is not really needed.

Let us consider a strip in $\mathbb{R}^2$

$$\Omega := \{ (x,y) \in \mathbb{R}^2 : \quad 0 < y < 1 \}$$



I claim that $\qquad \|u\|_{L^p(\Omega)} \leq \|u_y\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}$

$\boxed{\text{Proof}}$ Assume $u \in C_c^\infty(\Omega)$. Then

$$|u(x,y)| = \left| \int_0^y u_y(x,t)\,dt \right| \leq \int_0^1 \underset{\frac{1}{p}}{|u_y(x,t)|} \cdot \underset{\frac{1}{p'}}{1}\,dt$$

$$\leq \left\{ \int_0^1 |u_y(x,t)|^p dt \right\}^{1/p} \cdot 1$$

We raise to power p and integrate in Ω

$$|u(x,y)|^p \leq \int_0^1 |u_y(x,t)|^p dt$$

$$\int_{\mathbb{R}} dx \int_0^1 dy\, |u(x,y)|^p \leq \int_{\mathbb{R}} dx \underbrace{\int_0^1 dy}_{\overset{\shortparallel}{1}} \int_0^1 |u_y(x,t)|^p dt = \|u_y\|^p_{L^p(\Omega)}$$

If $u \in W_0^{1,p}(\Omega)$, then exploit the usual approx. argument

— o — o —

<u>Conclusion</u> The true assumption for Poincaré – Sobolev ineq. is that each point $x \in \Omega$ is "close enough" to some point in $\partial\Omega$, where $u = 0$.

<u>Question</u> In the previous inequality, can we consider different exponents in LHS and RHS? [Usual homothety ...]

When u is NOT 0 at $\partial\Omega$

POINCARÉ – SOBOLEV – WIRTINGER

$$\boxed{\|u-u_\Omega\|_{L^q(\Omega)} \leq c(d,p,q,\Omega)\,\|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W^{1,p}(\Omega)}$$

where $\boxed{u_\Omega := \frac{1}{\text{meas}(\Omega)}\int_\Omega u(x)\,dx}$ ← Average of u in Ω

True under the following assumptions

- → $\Omega \subseteq \mathbb{R}^d$ is BOUNDED
- → Ω is regular enough for Sobolev embeddings
- → $q \in [1, p^*]$ (at least if $p<d$, otherwise it is simpler)
- → Ω is CONNECTED (if not, consider $u \equiv 1$ in a ball and $u \equiv -1$ in an equal disjoint ball)

Proof Some initial reductions

- → we can assume that $u_\Omega = 0$ (if not, subtract u_Ω)
- → we can assume that $\|u\|_{L^q(\Omega)} = 1$ (because the ineq. is homogeneous of degree 1)
- → It is enough to prove the case $q=p$.

$$\|u\|_{L^q(\Omega)} \underset{\substack{\uparrow\\ \Omega \text{ is bounded}}}{\leq} c_1(d,p,q,\Omega)\,\|u\|_{L^{p^*}(\Omega)}$$

$$\underset{\substack{\uparrow\\ \text{Sobolev embedd.}}}{\leq} c_2(d,p,q,\Omega)\left\{\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\right\}$$

$$\underset{\substack{\uparrow\\ \text{Ineq. for } q=p}}{\leq} c_3(d,p,q,\Omega)\,\|\nabla u\|_{L^p(\Omega)}$$

$\boxed{\text{The case } p=q}$ Assume the inequality

$$\boxed{\begin{array}{c} \|u\|_{L^p(\Omega)} \leq c(d,p,\Omega)\, \|\nabla u\|_{L^p(\Omega)} \\ \text{for every } u \in W^{1,p}(\Omega) \text{ with } \int_\Omega u(x)\,dx = 0 \text{ and } \|u\|_{L^p(\Omega)} = 1 \end{array}}$$

is FALSE.

Then there exists a sequence $\{u_n\} \subseteq W^{1,p}(\Omega)$ s.t.

$$\|u_n\|_{L^p(\Omega)} = 1 \qquad \int_\Omega u_n(x)\,dx = 0 \qquad \|\nabla u_n\|_{L^p(\Omega)} \to 0$$

In particular $\|u_n\|_{1,p,\Omega}$ is bounded. Due to the compact embedding theorem, there exists

$$u_{n_k} \longrightarrow u_\infty \quad \text{in } L^p(\Omega),$$

and by assumption

$$\nabla u_{n_k} \longrightarrow 0 \quad \text{in } L^p(\Omega)$$

In the usual way we see that $\nabla u_\infty = 0$, and therefore u_∞ is locally constant in Ω. But Ω is connected, and hence u_∞ is constant.

On the other hand

$$\left.\begin{array}{l} 1 = \|u_{n_k}\|_{L^p(\Omega)} \longrightarrow \|u_\infty\|_{L^p(\Omega)} \\ 0 = \int_\Omega u_{n_k}(x)\,dx \underset{\substack{\uparrow \\ \text{pointwise conv. +} \\ \text{domination}}}{\longrightarrow} \int_\Omega u_\infty(x)\,dx \end{array}\right\} \text{impossible if } u_\infty \text{ is constant}$$

— o — o —

$\boxed{\text{Exercise}}$ P-S-W inequality is equivalent to proving that

$$\inf\left\{ \int_\Omega |\nabla u(x)|^p dx \;:\; u \in W^{1,p}(\Omega),\ \|u\|_{L^q(\Omega)} = 1,\ \int_\Omega u(x)dx = 0 \right\}$$

is strictly positive. (in this case $C = \frac{1}{\inf}$)

One could try to prove, using the DIRECT METHOD, that inf = min (when it is min, it is strictly positive)

This proof works almost always, but there are problems in two special case

→ $q = p_*$ (the embedding is not compact)

→ $p = 1$ (balls in L^1 are not weakly compact)

— o — o —

Istituzioni di Analisi — LECTURE 42

Note Title 15/11/2021

Two variational problems in higher dimension

Let $\Omega \subseteq \mathbb{R}^d$ be a ball (any regular bounded open set is the same)
We want to solve

$$\begin{cases} \Delta u = \sin u & \text{in } \Omega \\ \text{DBC} & \text{on } \partial\Omega \end{cases} \qquad\qquad \begin{cases} \Delta u = u^{2021} & \text{in } \Omega \\ \text{DBC} & \text{on } \partial\Omega \end{cases}$$

We expect a variational formulation of the form

$$\int_\Omega \left\{ \frac{1}{2} |\nabla u(x)|^2 - \cos u(x) \right\} dx \qquad\qquad \int_\Omega \left\{ \frac{1}{2} |\nabla u(x)|^2 + \frac{1}{2022} u^{2022}(x) \right\} dx$$

$$+ \text{DBC} \qquad\qquad + \text{DBC}$$

Discussion on DBC Does it make sense to minimize them with pointwise constraints such as
u (origin) = something

Answer: depends on p and d. It makes sense if we have

$$\int_\Omega |\nabla u(x)|^p dx \quad \text{with } p > d.$$

For the future: prescribing the values on a subset $E \subseteq \Omega$ makes sense if the p-capacity of E is $\neq 0$.

If we want to prescribe the values of u in $\partial\Omega$ its enough to say that $\underset{\substack{\uparrow \\ \text{trace}}}{\mathrm{Tr}}\, u$ = given function

but the given function has to be in the image of the trace, but the image of the trace is strange, for example in dim = 2 it is not true that any continuous function on S^1 is the trace of a function $u \in H^1 = W^{1,2}$ (ball).

Strange answer: $\mathrm{Tr} : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$ is surjective

fractional Sobolev space, with "fractional derivatives"

What we do in general is to prescribe DBC in the form

$$\boxed{\mathrm{Tr}\, u = \mathrm{Tr}\, u_0} \qquad\qquad \boxed{u - u_0 \in W_0^{1,p}(\Omega)}$$

given function in $W^{1,p}(\Omega)$

Let us start with the direct method.

$$\int_\Omega \frac{1}{2}|\nabla u|^2 - \cos u \qquad\qquad \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{u^{2022}}{2022}$$

Weak formulation in $H^1(\Omega) = W^{1,2}(\Omega)$. So $F : H^1(\Omega) \to$??

In the first case real values, in the second case it depends on d: If $2022 < \underset{\substack{\shortparallel \\ \frac{2d}{d-2}}}{2^*}$, then real values, otherwise $\mathbb{R} \cup \{+\infty\}$ (the only case is $d = 2$)

$\boxed{\text{Compactness}}$ Let us assume that $F(u_n) \leq M$.

In both cases we obtain

$$\|\nabla u_n\|_{L^2(\Omega)} \leq M' \qquad \text{(usual argument)}$$

We need a bound on u_n is some space.

In the second case we have

$$\|u_n\|_{L^{2022}(\Omega)} \leq M'' \quad \text{and therefore} \quad \|u_n\|_{L^2(\Omega)}$$

In the first case we use Poincaré inequality if $u \in W_0^{1,p}(\Omega)$:

$$\|u_n\|_{L^2(\Omega)} \leq \text{const} \cdot \|\nabla u\|_{L^2(\Omega)}$$

If $u - u_0 \in W_0^{1,p}(\Omega)$, then

$$\|u_n\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} + \|u_n - u_0\|_{L^2(\Omega)}$$

$$\leq \|u_0\|_{L^2(\Omega)} + \text{const}\ \|\nabla(u_n - u_0)\|_{L^2(\Omega)}$$

$$\leq \underbrace{\|u_0\|_{L^2(\Omega)} + \text{const}\ \|\nabla u_0\|_{L^2(\Omega)}}_{\text{given}} + \text{const}\ \|\nabla u_n\|_{L^2(\Omega)}$$

In any case from $F(u_n) \leq M$ we get $\underbrace{\|u_n\|_{1,2,\Omega}}_{\text{FULL NORM IN } H^1(\Omega)} \leq M'$

Compact embedding $\Rightarrow$ $u_{n_k} \longrightarrow u_\infty$ in $\underset{\substack{\uparrow \\ \text{more generally in} \\ L^q(\Omega) \text{ for } q < 2^*}}{L^2(\Omega)}$

Up to a further subsequence

$$\nabla u_{n_k} \longrightarrow \nabla u_\infty \quad \text{weakly in } L^2(\Omega)$$

$\boxed{\text{LSC}}$ In both cases we have that

$$\liminf_{k\to+\infty} \int_\Omega |\nabla u_{n_k}(x)|^2 dx \underset{\substack{\uparrow \\ \text{LSC of the norm} \\ \text{wrt weak conv.}}}{\geq} \int_\Omega |\nabla u_\infty(x)|^2 dx$$

In the first case $\displaystyle\lim_{k\to+\infty} \int_\Omega \cos(u_{n_k}(x))dx \underset{\substack{\uparrow \\ \text{pointwise conv.} \\ \text{+ trivial domination}}}{=} \int_\Omega \cos(u_\infty(x))dx$

[Here we DO NOT need a further subsequence, because any subseq. has a further subsubseq. for which convergence is true]
↑ (SUB-SUB lemma)

In the second case we have that

$$\liminf_{k\to+\infty} \int_\Omega u_{n_k}^{2022} dx \underset{\substack{\uparrow \\ \text{FATOU lemma} \\ \text{+ equi-bound from below}}}{\geq} \int_\Omega u_\infty^{2022} dx$$

Back to compactness: we need to be sure that $\text{Tr}\, u_\infty = \text{Tr}\, u_0$

This is true because of the continuity of the trace

$$\left.\begin{array}{l} u_n \to u_\infty \text{ in } L^2 \\ \|\nabla u_n\|_{L^2} \text{ bounded} \end{array}\right\} \Rightarrow \text{Tr}\, u_n \to \text{Tr}\, u_\infty \quad \text{in } L^2(\partial\Omega)$$

In this moment a min. exists in $H^1(\Omega)$.

$\boxed{\text{ELE}}$ In both cases take a min. point u, and consider
$\varphi(t) := F(u+tv)$ with $v \in C_c^\infty(\Omega)$

In the usual way we discover that

$$0 = \varphi'(0) = \int_\Omega \langle \nabla u(x), \nabla v(x) \rangle + \sin u(x) \cdot v(x)\, dx$$

If we $\boxed{\text{could}}$ integrate by parts ... then

$$0 = \int_\Omega \left(-\Delta u(x) + \sin u(x)\right) v(x)\, dx$$

FLCV $\Rightarrow$ $\Delta u = \sin u$

$\boxed{\text{UNFORTUNATELY WE CANNOT}}$

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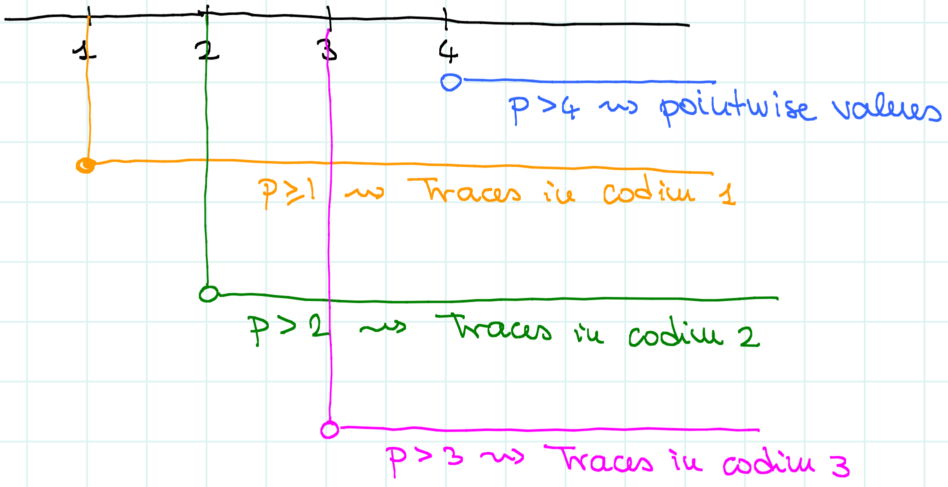
<u>Traces again</u> Let us assume $d = 4$, and $u \in W^{1,p}(\Omega)$



Istituzioni di Analisi – LECTURE 43

Note Title 17/11/2021

ELLIPTIC REGULARITY

$$\Delta u = \sin u \qquad\qquad \Delta u = u^{2021}$$

DBC in $\partial\Omega$ DBC in $\partial\Omega$

We proved that there exists $u \in H^1(\Omega)$ s.t.

1-st int. form of ELE

$$-\int_\Omega \langle \nabla u, \nabla \varphi \rangle dx = \int_\Omega \sin u \, \varphi \qquad -\int_\Omega \langle \nabla u, \nabla\varphi \rangle dx = \int_\Omega u^{2021} \varphi \, dx$$

$\forall \varphi \in C_c^\infty(\Omega)$ ⇝ and therefore also $\forall \varphi \in H_0^1(\Omega)$

ELLIPTIC EQUATION

1.1 Basic linear equation $\Delta u = f(x)$

- weak formulation $$\boxed{-\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega f \varphi}$$

- Functional $$F(u) = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + f(x) u \right\} dx$$

$$L(x,s,p) := \frac{1}{2} |p|^2 + f(x) s$$

1.2 Linear equation with coefficient $\operatorname{div}(\underset{\uparrow}{A(x)} \nabla u) = f$

symmetric square matrix with entries depending on x

- weak formulation $$\boxed{-\int_\Omega \underset{\uparrow}{\langle} A(x) \nabla u(x), \nabla \varphi(x) \rangle = \int_\Omega f \varphi}$$

scalar product

- Functional

$$F(u) = \int_\Omega \{ \underbrace{<A(x)\nabla u(x), \nabla u(x)>}_{\text{quadratic form}} + fu \} \, dx$$

$$L(x,s,p) = < A(x)p, p > + f(x)s$$

Def. The matrix $A(x)$ is $\boxed{\text{uniformly elliptic}}$ in Ω if there exists $\nu > 0$ s.t.

$$\boxed{< A(x)p, p > \geq \nu \|p\|^2 \qquad \forall p \in \mathbb{R}^d \quad \forall x \in \Omega}$$

The KEY POINT is that

$$\nu \int_\Omega |\nabla u|^2 dx \leq \int_\Omega < A(x)\nabla u(x), \nabla u(x) > dx$$

With sums

$$< A(x)\nabla u(x), \nabla \varphi(x) > = \sum_{i=1}^{d} \sum_{j=1}^{d} A_{i,j}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial \varphi}{\partial x_j}(x)$$

$$\operatorname{div}(A(x)\nabla u(x)) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{d} A_{i,j}(x) \frac{\partial u}{\partial x_j}(x) \right)$$

$\boxed{1.3}$ General linear equation

... $L(x,s,p)$ = general polynomial of degree 2 in the variables s and p with coeff. depending on x.

$\boxed{2}$ SEMILINEAR EQUATION

$$\boxed{\operatorname{div}(A(x)\nabla u(x)) = g(\underset{\uparrow\uparrow}{x,u})}$$

- Weak formulation

$$\boxed{-\int_\Omega \langle A(x)\nabla u(x), \nabla\varphi(x)\rangle dx = \int_\Omega g(x,u(x))\,\varphi(x)\,dx}$$

- Functional

$$F(u) := \int_\Omega \{\langle A(x)\nabla u, \nabla u\rangle + \underset{\uparrow}{G}(x,u)\}\,dx$$

$$\frac{\partial G}{\partial s}(x,s) = g(x,s)$$

$$L(x,s,p) = \dots$$

$\boxed{3}$ QUASILINEAR EQUATION

$$\operatorname{div}(A(\underset{\uparrow\uparrow}{x,u},\nabla u)\nabla u) = g(x,u,\nabla u)$$

The matrix depends on $x, u, \nabla u$

- Weak formulation : what we expect

- Functional $\displaystyle\int_\Omega L(x,u,\nabla u)\,dx$

$\boxed{4}$ FULLY NONLINEAR EQUATIONS

$$F(x,u,\nabla u, \underset{\uparrow}{D^2 u}) = 0$$

Hessian matrix

— o — o —

Elliptic regularity → INTERNAL
→ BOUNDARY REGULARITY

$$\Delta u(x) = f(x) \quad \text{in } \Omega$$

Internal regularity : $f \in \text{Space}_{loc}(\Omega) \Rightarrow D^2 u \in \text{Space}_{loc}(\Omega)$

Boundary regularity : $f \in \text{Space}(\Omega) \Rightarrow D^2 u \in \text{Space}(\Omega)$

Surprise : we assume the sum of pure 2-nd order deriv. in some space, and we obtain ALL 2-nd order deriv. in same space

" If we control $x^2+y^2+z^2$ we control $x^2, y^2, z^2, xy, yz, zx$ "

$f \in H^k_{loc}(\Omega) \Rightarrow D^2 u \in H^k_{loc}(\Omega)$ L^2- theory

$f \in W^{k,p}_{loc}(\Omega) \Rightarrow D^2 u \in W^{k,p}_{loc}(\Omega)$ L^p- theory $p \in (1, +\infty)$

$f \in C^{0,\alpha}_{loc}(\Omega) \Rightarrow D^2 u \in C^{0,\alpha}_{loc}(\Omega)$ SCHAUDER THEORY $\alpha \in (0,1)$

The same without "loc"

$f \in L^\infty \not\Rightarrow D^2 u \in L^\infty$
$f \in L^1 \not\Rightarrow D^2 u \in L^1$
$f \in C^0 \not\Rightarrow D^2 u \in C^0$
$f \in Lip \not\Rightarrow D^2 u \in Lip$

L^2 theory and L^p theory include the case $k=0$.

$\Delta u = \underbrace{\sin u}_{f(x)\in L^2}$ and Ω = ball

$f(x)\in L^2(\Omega) = D^2 u \in L^2(\Omega) \rightsquigarrow \Delta u \in L^2$ and therefore the equation is satisfied in diff. form

$$\Downarrow$$

$$u\in H^2(\Omega)$$

$\Rightarrow \sin u = f(x) \in H^2(\Omega) \Rightarrow u\in H^4(\Omega) \Rightarrow \dots\ u\in H^k$ for every k and therefore $u\in C^\infty(\Omega)$

IT DOES NOT WORK

It is false that $u\in H^2 \Rightarrow \sin u\in H^2$

$u\in H^1 \Rightarrow \sin u\in H^1$ TRUE

$\boxed{d=1}$ $v = \sin u(x)$

$v'(x) = \cos u(x)\cdot u'(x)$

$$v''(x) = -\underset{\substack{\uparrow\\ \text{bounded}}}{\sin u(x)}\cdot \underset{\substack{\uparrow\\ \text{a priori not}\\ \text{in } L^2,\ \text{unless}\\ u'\in L^4}}{[u'(x)]^2} + \underset{\substack{\uparrow\\ \text{bounded}}}{\cos u(x)}\cdot \underset{\substack{\uparrow\\ L^2}}{u''(x)}$$

If we want the implication $u\in H^2 \Rightarrow \sin u\in H^2$ we need

$H^1 \longrightarrow L^4$, namely $4 \le 2^* = \dfrac{2d}{d-2}$

$\rightsquigarrow 4d-8 \le 2d \rightsquigarrow d\le 4$

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Istituzioni di Analisi — LECTURE 44

Note Title 17/11/2021

Theorem (Internal regularity)

Let $d \geq 1$, let $\Omega \subseteq \mathbb{R}^d$ be any open set. Let us assume that

$$\operatorname{div}(A(x)\nabla u(x)) = f(x) \quad \text{in } \Omega$$

in the sense that

$$-\int_\Omega \langle A(x)\nabla u(x), \nabla\varphi(x)\rangle dx = \int_\Omega f(x)\varphi(x)dx \qquad \forall \varphi \in C_c^\infty(\Omega)$$

Let us assume that

(i) $f \in L^2_{loc}(\Omega)$

(ii) $A(x) \in C^1(\Omega)$

(iii) $u \in H^1_{loc}(\Omega)$

under these assumptions the weak formulation makes sense

(iv) $A(x)$ unif. elliptic in Ω

Then $u \in H^2_{loc}(\Omega)$ and for every $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$

$$\boxed{\|D^2 u\|_{L^2(\Omega_1)} \leq c(\Omega_1, \Omega_2)\{\|f\|_{L^2(\Omega_2)} + \|u\|_{1,2,\Omega_2}\}}$$

UN-NATURAL ESTIMATE

Rmk The theorem is true also in iterated form, namely

(i) $f \in H^k_{loc}(\Omega)$

(ii) $A \in C^{k+1}$

(iii) $u \in H^{k+1}_{loc}(\Omega)$

$\rightsquigarrow \quad u \in H^{k+2}_{loc}(\Omega)$

A PRIORI ESTIMATES

Assume everything is C^∞ and prove the estimates

Baby case 1 $d = 2$, $\Omega = \mathbb{R}^2$, $u \in C_c^\infty(\mathbb{R}^2)$, $f \in C_c^\infty(\mathbb{R}^2)$

$$-\int_{\mathbb{R}^2} u_x \varphi_x + u_y \varphi_y = \int_{\mathbb{R}^2} f \varphi$$

Idea: use $\varphi = u_{xx}$ (I can because $u \in C_c^\infty$)

$$-\int_{\mathbb{R}^2} u_x u_{xxx} + u_y u_{xxy} = \int_{\mathbb{R}^2} f u_{xx}$$

$$\text{RHS} \leq \|f\|_{L^2} \cdot \|u_{xx}\|_{L^2} \leq \|f\|_{L^2} \left(\|u_{xx}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2\right)^{1/2}$$

$$\text{LHS} = \int_{\mathbb{R}^2} (u_{xx}^2 + u_{xy}^2) = \|u_{xx}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2$$

When I compare I get

$$\|u_{xx}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2 \leq \|f\|_{L^2} \cdot (\dots)^{1/2}$$

and therefore $\left(\|u_{xx}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2\right)^{1/2} \leq \|f\|_{L^2}$

If we use $\varphi = u_{yy}$ we obtain

$$\left(\|u_{yy}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2\right)^{1/2} \leq \|f\|_{L^2}$$

and finally

$$\boxed{\|u_{xx}\|_{L^2}^2 + \|u_{yy}\|_{L^2}^2 + 2\|u_{xy}\|_{L^2}^2 \leq 2\|f\|_{L^2}^2}$$

Baby case 2 any d, $\Omega = \mathbb{R}^d$, u and f in $C_c^\infty(\mathbb{R}^d)$

We use $\varphi = u_{x_i x_i}$ and in the same way we estimate all second derivatives with one x_i.

At the end of the day

$$\boxed{\sum_{i=1}^{d} \|u_{x_i x_i}\|_{L^2}^2 + \sum_{i \neq j} \|u_{x_i x_j}\|_{L^2}^2 \leq d \, \|f\|_{L^2}^2}$$

Baby case 3 any d, $\Omega = \mathbb{R}^d$, $u, f, A \in C_c^\infty(\mathbb{R}^d)$, equation

$$-\int_\Omega \langle A(x) \nabla u, \nabla \varphi \rangle = \int_\Omega f \varphi$$

Use again $\varphi = u_{x_i x_i}$

$$\text{RHS} = \int_\Omega f \, u_{x_i x_i} \leq \|f\|_{L^2} \, \|u_{x_i x_i}\|_{L^2}$$

$$\leq \|f\|_{L^2} \left\{ \sum_{j=1}^{d} \|u_{x_i x_j}\|_{L^2}^2 \right\}^{1/2}$$

$$\text{LHS} = -\int_\Omega \langle A(x) \nabla u, \nabla u_{x_i x_i} \rangle \, dx$$

$$= \int_\Omega \langle A_{x_i}(x) \nabla u, \nabla u_{x_i} \rangle \, dx + \underbrace{\int_\Omega \langle A(x) \nabla u_{x_i}, \nabla u_{x_i} \rangle}_{\substack{\geq \ (\text{ellipticity}) \\ \nu \int_\Omega \|\nabla u_{x_i}\|^2}}$$

$$\nu \int_\Omega \|\nabla u_{x_i}\|^2 \leq \|f\|_{L^2} \, \|\nabla u_{x_i}\|_{L^2}^2 - \int_\Omega \langle \underbrace{A_{x_i}(x)}_{\text{bounded}} \nabla u, \nabla u_{x_i} \rangle$$

In conclusion

$$\nu \|\nabla u_{x_i}\|^{\cancel{2}}_{L^2} \le \|f\|_{L^2} \|\nabla \cancel{u_{x_i}\|_{L^2}} + \|A\|_{C^1} \|\nabla u\|_{L^2} \|\nabla \cancel{u_{x_i}}\|$$

and we obtain

$$\sum_{j=1}^{d} \|u_{x_i x_j}\|^2_{L^2} \le \|f\|_{L^2} + \|A\|_{C^1} \cdot \|\nabla u\|_{L^2}$$

$\boxed{\text{Case 4}}$ $\Omega \subseteq \mathbb{R}^d$ any open set, $\Delta u = f$ in weak sense in Ω, u and f in $C^\infty(\Omega)$ (no cpt. support)

Consider $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$

Consider $\psi \in C^\infty_c(\mathbb{R}^d)$ with

$\psi \equiv 1$ in Ω_1

$\psi \equiv 0$ outside Ω_2

$0 \le \psi \le 1$ in between

Define $v = u\psi$. Then

$$\Delta v = \Delta(u\psi) = \underset{\substack{\shortparallel \\ f}}{\Delta u}\, \psi + u\, \Delta\psi + 2\langle \nabla u, \nabla \psi\rangle =: g$$

Then apply baby case 2 to v and g

$$\|D^2 u\|^2_{L^2(\Omega_1)} \underset{\substack{\uparrow \\ u = v \\ \text{in } \Omega_1}}{\le} \|D^2 v\|_{L^2(\mathbb{R}^d)} \le d\, \|g\|^2_{L^2(\mathbb{R}^d)}$$

$$\underset{\substack{\nearrow \\ \psi \equiv 0 \text{ outside } \Omega_2 \\ \text{and } \psi \le 1 \text{ in } \Omega_2}}{\le} \overset{\substack{\text{depends on } \psi \\ \uparrow}}{\text{const}} \left\{ \|f\|^2_{L^2(\Omega_2)} + \|u\|^2_{L^2(\Omega_2)} + \|\nabla u\|^2_{L^2(\Omega_2)} \right\}$$

Case 4 revisited Assume everything as in Case 4, but only $f \in L^2_{loc}$ and $u \in H^1_{loc}$

Can we say that v solves the same equ $\Delta v = g$?

Hyp : $$-\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega f \varphi$$

Th : $$-\int_\Omega \langle \nabla v, \nabla \varphi \rangle = \int_\Omega g \varphi$$

$$-\int_\Omega \langle \nabla v, \nabla \varphi \rangle = -\int_\Omega \langle \nabla (u\psi), \nabla \varphi \rangle$$

$$= -\int_\Omega \langle \nabla u \cdot \psi + u \nabla \psi, \nabla \varphi \rangle$$

$$= \underbrace{-\int_\Omega \langle \nabla u, \psi \nabla \varphi \rangle}_{①} \underbrace{-\int_\Omega \langle u \nabla \psi, \nabla \varphi \rangle}_{②}$$

$$① = -\int_\Omega \langle \nabla u, \nabla (\psi\varphi) - \varphi \nabla \psi \rangle$$

$$= \underbrace{-\int_\Omega \langle \nabla u, \nabla (\psi\varphi) \rangle}_{\boxed{\int_\Omega f \psi \varphi}\ ☺} + \underbrace{\boxed{\int_\Omega \langle \nabla u, \nabla \psi \rangle \varphi}}_{☺}$$

$$② = -\int_\Omega \langle \underbrace{u \nabla \psi}_{\vec{E}}, \nabla \varphi \rangle = \int_\Omega \varphi \underbrace{\operatorname{div} (u \nabla \psi)}_{\operatorname{div} \vec{E}}$$

$$= \underbrace{\boxed{\int_\Omega \varphi \langle \nabla u, \nabla \psi \rangle + \varphi u \Delta \psi}}_{☺}$$

The sum of ☺ is $\int_\Omega g \varphi$ —o—o—

Istituzioni di Analisi — LECTURE 45

Note Title 18/11/2021

DIFFERENCE QUOTIENTS | DISCRETE DERIVATIVES

Let $u: \mathbb{R}^d \to \mathbb{R}$, let $\alpha \in \mathbb{R}^d \setminus \{0\}$, let $\varepsilon > 0$. We set

$$\boxed{D_\alpha^\varepsilon u(x) := \frac{u(x+\varepsilon\alpha) - u(x)}{\varepsilon \|\alpha\|}}$$

Diff. quotient in direction α

If $\alpha = (0, \dots, 0, \underset{\substack{\uparrow \\ \text{pos. } i}}{1}, 0, \dots, 0) = e_i$ then we write $D_{x_i}^\varepsilon u(x)$

If u is defined in $\Omega \subseteq \mathbb{R}^d$ open set, then $D_\alpha^\varepsilon u$ is defined in every $\Omega' \subset\subset \Omega$ provide that ε is small enough.

SIMPLE PROPERTIES

(i) $u \to D_\alpha^\varepsilon u$ is linear

(ii) D_α^ε preserves regularity: if $u \in W^{m,p}(\mathbb{R}^d)$, then $D_\alpha^\varepsilon u \in W^{m,p}(\mathbb{R}^d)$

(iii) D_α^ε commutes with weak derivatives $D_\alpha^\varepsilon (D^\beta u) = D^\beta (D_\alpha^\varepsilon u)$ for every multiindex β.

(iv) There is a formula for the product

$$D_\alpha^\varepsilon (u(x) v(x)) = D_\alpha^\varepsilon u(x) \cdot v(x) + u(x+\varepsilon\alpha) \cdot D_\alpha^\varepsilon v(x)$$

(v) There is a formula for the composition. Assume $g \in C^1$

$$D_\alpha^\varepsilon (g(u(x))) = g'(\;\underset{\uparrow}{}\;)\, D_\alpha^\varepsilon u(x)$$

some point between $u(x)$ and $u(x+\varepsilon\alpha)$

$\boxed{\text{Proposition}}$

(1) If $u \in W^{1,p}(\mathbb{R}^d)$, then $\| D^{\varepsilon}_{x_i} u(x) \|_{L^p(\mathbb{R}^d)} \leq \| D_{x_i} u(x) \|_{L^p(\mathbb{R}^d)}$

(2) Discrete integration by parts

$$\int_{\mathbb{R}^d} D^{\varepsilon}_{\alpha} u(x) \cdot \varphi(x)\, dx = -\int_{\mathbb{R}^d} u(x) \cdot D^{-\varepsilon}_{\alpha} \varphi(x)\, dx$$

for every choice of u and φ for which the integrals make sense for example

→ $u \in L^1_{loc}(\mathbb{R}^d)$ and $\varphi \in C^{\overset{\circ}{\infty}}_{c}(\mathbb{R}^d)$

→ $u \in L^p(\mathbb{R}^d)$ and $\varphi \in L^{p'}(\mathbb{R}^d)$ if $\frac{1}{p} + \frac{1}{p'} = 1$.

$\boxed{\text{Proof}}$ (1) $|u(x+\varepsilon e_i) - u(x)| \underset{\substack{\uparrow \\ \varphi(t) := u(x+\varepsilon e_i t)}}{=} |\varphi(1) - \varphi(0)|$

$$\leq \int_0^1 |\varphi'(t)|\, dt = \int_0^1 \underset{\frac{1}{p}}{|D_{x_i} u(x+\varepsilon e_i t)|} \cdot \underset{\frac{1}{p'}}{|\varepsilon|}\, dt$$

$$\leq |\varepsilon| \left\{ \int_0^1 |D_{x_i} u(x+\varepsilon e_i t)|^p\, dt \right\}^{1/p}$$

Therefore

$$\int_{\mathbb{R}^d} |u(x+\varepsilon e_i) - u(x)|^p\, dx \leq |\varepsilon|^p \int_{\mathbb{R}^d} dx \int_0^1 |D_{x_i} u(x+\varepsilon e_i t)|^p\, dt$$

$$= |\varepsilon|^p \int_0^1 dt \int_{\mathbb{R}^d} |D_{x_i} (\underbrace{x+\varepsilon e_i t}_{y})|^p\, dx \qquad dx = dy$$

$$= |\varepsilon|^p \int_{\mathbb{R}^d} |D_{x_i} (y)|^p\, dy$$

We divide by $|\varepsilon|^p$ and we get the conclusion

(2)

$$\int_{\mathbb{R}^d} \frac{u(x+\varepsilon\alpha)-u(x)}{\varepsilon\|\alpha\|}\,\varphi(x)\,dx$$

$$= \frac{1}{\varepsilon\|\alpha\|}\left\{\int_{\mathbb{R}^d} u(\underbrace{x+\varepsilon\alpha}_{y})\,\underset{\substack{\uparrow\\ y-\varepsilon\alpha}}{\varphi(x)}\,dx - \int_{\mathbb{R}^d} u(x)\varphi(x)\,dx\right\}$$

$$= \frac{-1}{-\varepsilon\|\alpha\|}\left\{\int_{\mathbb{R}^d} u(y)\,\varphi(y-\varepsilon\alpha)\,dy - \int_{\mathbb{R}^d} u(y)\,\varphi(y)\,dy\right\}$$

This proves (2).

— o — o —

$\boxed{\text{Theorem}}$ (Sobolev spaces and discrete derivatives)

Let us assume that $u\in L^p(\mathbb{R}^d)$ with $p\in(\underset{\uparrow}{1},\underset{\uparrow}{+\infty})$

excluded, but something is true

Then the following are equivalent

(1) $u\in W^{1,p}(\mathbb{R}^d)$

(2) $D^{\varepsilon}_{x_i}u \longrightarrow$ something in $L^p(\mathbb{R}^d)$ as $\varepsilon\to0^+$ for every $i=1,\dots,d$

↑ a posteriori is $D_{x_i}u$

(3) $D^{\varepsilon}_{x_i}u \rightharpoonup$ something weakly in $L^p(\mathbb{R}^d)$

(4) $\|D^{\varepsilon}_{x_i}u\|_{L^p(\mathbb{R}^d)}$ are bounded indep. of ε

$\boxed{\text{Proof}}$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are almost trivial by general facts

(1) $\Rightarrow$ (2) Less easy, but again an exercise

[Hint: very similar to the proof that $u*\rho_\varepsilon\to u$]

(4) $\Rightarrow$ (1) Due to the bound on the norms, there exists $\varepsilon_k \to 0^+$ such that (we need also $p>1$)

$$D^{\varepsilon_k}_{x_i} u(x) \rightharpoonup v_i(x) \quad \text{weakly in } L^p(\mathbb{R}^d)$$

Claim: $v_i(x) = D_{x_i} u(x)$, namely

$$\int_{\mathbb{R}^d} u(x) D_{x_i} \varphi(x)\, dx = -\int_{\mathbb{R}^d} v_i(x) \varphi(x)\, dx$$

We start with the discrete integration by parts

$$\int_{\mathbb{R}^d} D^{\varepsilon_k}_{x_i} u(x) \varphi(x)\, dx = -\int_{\mathbb{R}^d} u(x) D^{-\varepsilon_k}_{x_i} \varphi(x)\, dx$$

$\downarrow$ $\qquad\qquad$ $\downarrow$ also uniformly if $\varphi \in C^\infty_c(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} v_i(x) \varphi(x)\, dx = -\int_{\mathbb{R}^d} u(x) D_{x_i} \varphi(x)\, dx$$

— o — o —

$\boxed{\text{Theorem}}$ (Internal regularity)

Assume that $f \in L^2(\mathbb{R}^d)$ and $u \in H^1(\mathbb{R}^d)$ is a solution to

$$\Delta u = f$$

in the weak formulation

$$-\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle\, dx = \int_{\Omega} f\varphi \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^d)$$

$\forall \varphi \in H^1(\mathbb{R}^d)$

Then $u \in H^2(\mathbb{R}^d)$ and

$$\boxed{\|D^2 u\|^2_{L^2(\mathbb{R}^d)} \le d\, \|f\|^2_{L^2(\mathbb{R}^d)}}$$

$\boxed{\text{Proof}}$ Last lecture we used $\varphi = u_{x_i x_i}$.

Now we use

$$\varphi = D^{-\varepsilon}_{x_i} D^{\varepsilon}_{x_i} u(x) \in H^1(\mathbb{R}^d)$$

$$-\int_{\mathbb{R}^d} \langle \nabla u(x), \nabla D^{-\varepsilon}_{x_i} D^{\varepsilon}_{x_i} u(x) \rangle dx = \int_{\mathbb{R}^d} f D^{-\varepsilon}_{x_i} D^{\varepsilon}_{x_i} u(x) dx$$

$$\text{RHS} \le \|f\| \cdot \|D^{-\varepsilon}_{x_i} D^{\varepsilon}_{x_i} u\| \underset{\substack{\uparrow \\ \text{Prop}}}{\le} \|f\| \cdot \|D_{x_i} D^{\varepsilon}_{x_i} u\|$$

$$\le \|f\| \cdot \|\overset{\curvearrowleft}{\nabla D^{\varepsilon}_{x_i}} u\|$$

$$= \|f\| \cdot \|D^{\varepsilon}_{x_i} \nabla u\|$$

$$\text{LHS} = -\int_{\mathbb{R}^d} \langle \nabla u(x), \underset{\curvearrowright}{\nabla D^{-\varepsilon}_{x_i}} D^{\varepsilon}_{x_i} u(x) \rangle dx$$

$$= -\int_{\mathbb{R}^d} \langle \nabla u(x), D^{-\varepsilon}_{x_i} \nabla D^{\varepsilon}_{x_i} u(x) \rangle dx$$

$$\underset{\nearrow}{=} \int_{\mathbb{R}^d} \langle D^{\varepsilon}_{x_i} \nabla u(x), \underset{\curvearrowright}{\nabla D^{\varepsilon}_{x_i}} u(x) \rangle dx$$

integr. by parts to each term of the sc. prod

$$= \int_{\mathbb{R}} \langle D^{\varepsilon}_{x_i} \nabla u(x), D^{\varepsilon}_{x_i} \nabla u(x) \rangle dx$$

$$= \|D^{\varepsilon}_{x_i} \nabla u(x)\|^2$$

In conclusion $\|D^{\varepsilon}_{x_i} \nabla u(x)\|^{\cancel{2}} \le \|f\| \cdot \cancel{\|D^{\varepsilon}_{x_i} \nabla u(x)\|}$

This implies that $\|D_{x_i} \nabla u(x)\|^2 \le \|f\|^2$

$$\uparrow \quad \sum_{j=1}^{d} \|D_{x_i x_j} u(x)\|^2 .$$

Sum over i and get conclusion.

— o — o —

Istituzioni di Analisi — LECTURE 46

Note Title 18/11/2021

Boundary regularity for Dirichlet problem

Theorem Let $\Omega \subseteq \mathbb{R}^d$ be an open set, let $f \in L^2(\Omega)$ and let $u \in H^1_{\underset{\text{DBC}}{\underset{\uparrow\uparrow}{0}}}(\Omega)$ be a solution to $\Delta u = f$ in the usual weak sense.

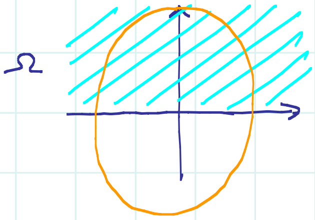
Assume that $\partial\Omega$ is compact and of class C^2.

Then $u \in H^2(\Omega)$ and

$$\boxed{\|D^2 u\|_{L^2(\Omega)} \le c(\Omega)\left\{\|f\|_{L^2(\Omega)} + \|u\|_{1,2,\Omega}\right\}}$$

The same is true for $\operatorname{div}(A(x)\nabla u) = f$ provided that $A \in C^1(\operatorname{Clos}(\Omega))$ and unif. elliptic.

Model case $\Omega = \mathbb{R}^d_+$



Proof Let us show that a priori estimates

$$(\underset{\mathbb{R}^{d-1}}{\underset{\uparrow}{x}}, \underset{y\in\mathbb{R}_+}{\underset{\uparrow}{y}})$$

Let us assume that f and u are in $C^\infty(\mathbb{R}^d_+)$ and they vanish outside a large enough ball of $\mathbb{R}^d$.

For every $i = 1, \dots, d-1$ we use $\varphi = u_{x_i x_i}$

$$-\int_{\mathbb{R}^d_+} \underset{\uparrow}{\langle \nabla u, \nabla u_{x_i x_i} \rangle}\, dx = \int_{\mathbb{R}^d_+} f\, u_{x_i x_i}\, dx$$

we can integrate by parts wrt x_i because we do not cross the boundary

As in the case of $\mathbb{R}^d$ we obtain that

$$\sum_{j=1}^{d} \|u_{x_i x_j}\|^2_{L^2(\mathbb{R}^d_+)} \leq \|f\|^2_{L^2(\mathbb{R}^d_+)} \qquad i = 1, \dots, d-1$$

In this way we can estimate all second order derivatives of u with the exception of u_{yy}. But

$$u_{yy} = \underset{L^2}{\underset{\cap}{f}} - \sum_{i=1}^{d-1} \underset{L^2}{\underset{\cap}{u_{x_i x_i}}} \Rightarrow u_{yy} \in L^2 \text{ with estimate}$$

In the case where $f \in L^2$ and $u \in H^1$ we use discrete derivatives in the same way as for the internal regularity.

$$\varphi = D^{-\varepsilon}_{x_i} D^{\varepsilon}_{x_i} u \in H^1_0(\mathbb{R}^d_+) \text{ because } i = 1, \dots, d-1$$

Again we obtain that all second der. of u, but u_{yy}, are in $L^2(\mathbb{R}^d_+)$.

$$-\int_\Omega f\varphi = +\int_\Omega \langle \nabla u, \nabla \varphi \rangle$$

$$= \sum_{i=1}^{d-1} \int_\Omega D_{x_i} u \, D_{x_i} \varphi + \int_\Omega D_y u \, D_y \varphi$$

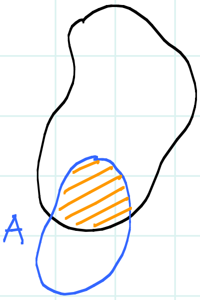
$\downarrow$ here we can integrate by parts

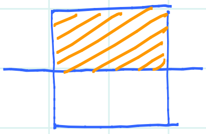
$$= -\sum_{i=1}^{d-1} \int_\Omega D_{x_i x_i} u \cdot \varphi + \int_\Omega D_y u \, D_y \varphi$$

$$-\int_\Omega D_y u \, D_y \varphi = \int_\Omega \underbrace{\{ f - D_{x_i x_i} u \}}_{g \in L^2} \varphi$$

This is equivalent to saying that $D_y u$ has a weak deriv wrt y and $D_y D_y u = g$

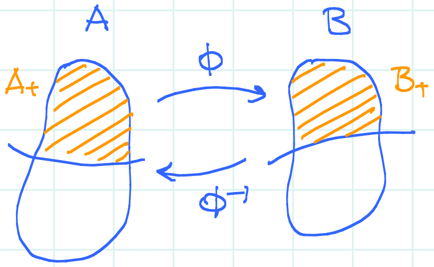
General case





As always we write u as a sum of functions u_i with support is suitable open sets that are diffeomorphic to the standard cube.

Problem Consider two diffeom. open sets A and B



$\phi : A \to B$

$\phi : A^+ \to B^+$

$\phi : \partial A^+ \to \partial B^+$

If u solves

$$\mathrm{div}(\hat{A}(x)\nabla u) = f \quad \text{in } A_+$$

Let us define $v(y)$, $g(y)$, $\hat{B}(y)$ for $y \in B$ or B^+ such that

$$v(\phi(x)) = u(x)$$

$$g(\phi(x))\,|\mathrm{Det}\, J\phi(x)| = f(x)$$

$$\hat{B}(\phi(x))\,|\mathrm{Det}\, J\phi(x)| = J\phi(x)\,\hat{A}(x)\,J\phi(x)^t$$

We claim that $\mathrm{div}(\hat{B}(y)\, v(y)) = g(y)$ in B_+ in the usual weak sense

$\underline{\text{Hyp}}$: $-\int_{A_+} \langle \hat{A}(x)\nabla u(x), \nabla\varphi(x)\rangle = \int_{A^+} f\varphi \qquad \forall \varphi \in C_c^\infty(A^+)$

$\underline{\text{Th}}$: $-\int_{B_+} \langle \hat{B}(y)\nabla v(y), \nabla\psi(y)\rangle = \int_{B_+} g\psi \qquad \forall \psi \in C_c^\infty(B^+)$

$$\mathrm{RHS} = \int_{B_+} g(y)\,\psi(y)\,dy \underset{\overset{\uparrow}{y=\Phi(x)}}{=} \int_{A_+} \underline{g(\Phi(x))}\,\underbrace{\psi(\Phi(x))}_{\varphi(x)}\,\underline{|\mathrm{Det}\, J\Phi(x)|}\,dx$$

$$= \int_{A_+} f(x)\,\varphi(x)\,dx$$

Let us observe that $\underset{\boxed{}}{\nabla\varphi(x)} = \underset{\boxed{}}{\nabla\psi(\Phi(x))}\,\underset{\boxed{}}{J\Phi(x)}$

$$\nabla u(x) = \nabla v(\Phi(x)) \cdot J\Phi(x)$$

$$-\mathrm{LHS} = \int_{B_+} \langle \hat{B}(y)\,\nabla v(y), \nabla\psi(y) \rangle\,dy$$

$$= \int_{B_+} \nabla v(y)\,\hat{B}(y)\,\nabla\psi(y)^t\,dy$$

$$= \int_{A_+} \nabla v(\Phi(x))\,\hat{B}(\Phi(x))\,\nabla\psi(\Phi(x))^t\,|\mathrm{Det}\, J\Phi(x)|\,dx$$

$$= \int_{A_+} \underbrace{\nabla v(\Phi(x))\,J\Phi(x)}_{\nabla u(x)}\,\hat{A}(x)\,\underbrace{J\Phi(x)^t\,\nabla\psi(\Phi(x))^t}_{\nabla\varphi(x)^t}\,dx$$

$= -$ LHS in the assumption

what we need is that

$$\hat{B}(\Phi(x))\,|\mathrm{Det}\, J\Phi(x)| = J\Phi(x)\,\hat{A}(x)\,J\Phi(x)^t$$

—o—o—

$\boxed{\text{Rmk}}$ Where we used that $\partial\Omega$ is of class C^2.

In at least two points

1 - We need $\hat{B}$ to be of class C^1

2 - We need that $v \in H^2 \Rightarrow u \in H^2$ and this requires Φ to be of class C^2.

$\boxed{\text{Rmk}}$ We need $\hat{B}$ to be unif. elliptic.

<u>Lemma</u> (lin. alg.) If A is coercive and M is invertible, then MAM^t is coercive

$\boxed{\text{Rmk}}$ What about the case where $u = u_0$ on $\partial\Omega$ in the sense that $u - u_0 \in H_0^1$

$$\Delta(u-u_0) = \Delta u - \Delta u_0 = f - \Delta u_0$$

If $u_0 \in H^2 \rightsquigarrow f - \Delta u_0 \in L^2 \rightsquigarrow u - u_0 \in H^2$
$\rightsquigarrow u \in H^2$.

— o — o —