

A.A. 2021/2022
Istituzioni di Analisi Matematica
Printout of lectures
(Volume 1)

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Istituzioni di Analisi – LECTURE 01

Note Title

27/09/2021

Introduction to Calculus of VariationsGeneral problem Set X and a function $F: X \rightarrow \mathbb{R}$

↑
sometimes
 $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

Goal: determine whether there exists

$$\min \{ F(x) : x \in X \} \leftarrow \text{number (unique if it exists)}$$

→ If min exists, then find

$$\operatorname{argmin} \{ F(x) : x \in X \} \leftarrow \text{set of min. points (a subset of } X, \text{ with one or more elements)}$$

$$= \{ x \in X : F(x) = \min \{ F(x) : x \in X \} \}$$

→ If min does NOT exist, then find inf & characterize minimizing sequences, namely $\{x_n\} \subseteq X$ s.t.

$$F(x_n) \rightarrow \inf \quad \text{"inf-izing sequence"}$$

Two main approaches

- INDIRECT METHOD
- DIRECT METHOD

INDIRECT METHOD Road map

- ① Trying to figure out which are the min. points
- ② Proving they are min. points.

Example $\mathbb{X}: \mathbb{R}^2 \quad f(x, y) := x^2 + y^2 + 2x - 6y$

- ① $\nabla f = (2x+2, 2y-6) = (0, 0) \Leftrightarrow x = -1, y = 3$
I conjecture that $(x_0, y_0) = (-1, 3)$ is the unique min. point
- ② I prove that (x_0, y_0) is the min. point.
Every other competitor can be written as

$$(x, y) = (-1+u, 3+v) = (x_0+u, y_0+v)$$

$$\begin{aligned} f(x, y) &= (x_0+u)^2 + (y_0+v)^2 + 2(x_0+u) - 6(y_0+v) \\ &= \underbrace{x_0^2}_{\text{orange}} + \underbrace{2x_0u}_{\text{blue}} + \underbrace{u^2}_{\text{green}} + \underbrace{y_0^2}_{\text{orange}} + \underbrace{2y_0v}_{\text{blue}} + \underbrace{v^2}_{\text{green}} + \underbrace{2x_0}_{\text{orange}} + \underbrace{2u}_{\text{blue}} - \underbrace{6y_0}_{\text{orange}} - \underbrace{6v}_{\text{blue}} \\ &= f(x_0, y_0) + \underbrace{(2x_0+2)u + (2y_0-6)v}_{\substack{\text{by definition of } (x_0, y_0) \\ 0}} + (u^2+v^2) \\ &\geq f(x_0, y_0) \quad \text{with } = \text{ iff } u=v=0. \end{aligned}$$

DIRECT METHOD Road map

- ① Proving that min. exists
- ② Trying to find min point

In the same example we can observe that

$$\lim_{x^2+y^2 \rightarrow +\infty} f(x, y) = +\infty$$

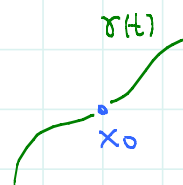
Thanks to generalized Weierstrass theorem we know that min exists (we can restrict to $\{f(x, y) \leq 2021\}$)

Now we proceed as in step 1 before.

INDIRECT METHOD IN GENERAL

Let X be a set, and let $F: X \rightarrow \mathbb{R}$ be a function

Let $x_0 \in X$ be a point, let $\delta > 0$, let $\gamma: (-\delta, \delta) \rightarrow X$ be a curve with $\gamma(0) = x_0$



I can consider $\varphi: (-\delta, \delta) \rightarrow \mathbb{R}$ defined by

$$\varphi(t) := F(\gamma(t))$$

I can compute $\varphi'(0)$. If it exists, I call it the first variation of F in x_0 wrt the curve γ

$$\delta F(x_0, \gamma) := \varphi'(0)$$

Theorem (Necessary condition) If x_0 is a min point for F , then

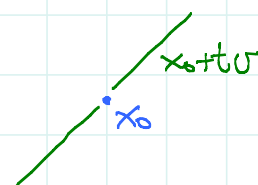
$$\delta F(x_0, \gamma) = 0$$

whenever it exists.

Equivalent statement: for every curve with $\gamma(0) = x_0$, we have that $\varphi'(0)$

→ either it does not exist

→ or is equal to 0.



Special case X is an AFFINE space with corresponding vector space V .

For every $x_0 \in X$ and $v \in V$ we can consider

$$\varphi(t) := F(x_0 + tv)$$

In this case

$$\varphi'(0) = \delta F(x_0, v)$$

GATEAUX derivative of F
in x_0 in the direction v

↑ usual directional derivative

DIRECT METHOD IN GENERAL

Def. Let X be a set.

Let $\text{Seq}(X)$ be the set of sequences with values in X

A NOTION OF CONVERGENCE in X is any subset of

$$\text{Seq}(X) \times X$$

In brutal words, it is the list of converging sequences in X with their limits.

WITHOUT ANY RULE AT ALL.

Def. We say that X is compact if every sequence admits a converging subsequence (a subseq. in the list)

We say that a subset $K \subseteq X$ is compact if for every sequence $\{x_n\} \subseteq K$ there exist $n_k \rightarrow +\infty$ and $x_\infty \in K$ such that $x_{n_k} \rightarrow x_\infty$

Def. A function $f: X \rightarrow \mathbb{R}$ is lower semicontinuous (LSC) if for every converging sequence $x_n \rightarrow x_\infty$ it turns out that

$$\liminf_{n \rightarrow \infty} f(x_n) \geq f(x_\infty)$$

Theorem (Weierstrass) Let $F: X \rightarrow \mathbb{R}$. Assume that

(i) X is compact and $\neq \emptyset$

(ii) F is LSC

Then $\min \{F(x) : x \in X\}$ exists.

Proof. Let $I := \inf \{F(x) : x \in X\} \in \mathbb{R} \cup \{-\infty\}$

Let $\{x_n\} \subseteq X$ be a inf-izing sequence.

Since X is compact, there exists $x_{n_k} \rightarrow x_\infty$. Now

$$\begin{array}{ccccccc}
 I \leq F(x_\infty) & \leq & \liminf_{k \rightarrow \infty} F(x_{n_k}) & = & \lim_{n \rightarrow \infty} F(x_n) & = & I. \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{defn. of} & & F \text{ is LSC} & & \text{subsequence} & & \text{defn. of } x_n \\
 \text{inf} & & & & \text{of conv. sequence} & & \\
 & & & & \text{of numbers.} & &
 \end{array}$$

This proves that $F(x_\infty) = I \Rightarrow x_\infty$ is a min. point and $I \in \mathbb{R}$.

Def. A function $F: X \rightarrow \mathbb{R}$ is called COERCIVE if there exists a compact set $K \subseteq X$ s.t.

$$\inf \{F(x) : x \in X\} = \inf \{F(x) : x \in K\}$$

Theorem (Generalized W) Assume that

(i) F is coercive

(ii) F is LSC.

Then again min exists.

Proof: exercise

Remark F is coercive provided that there exists $M \in \mathbb{R}$ such that

$\{x \in X : F(x) \leq M\}$ is $\neq \emptyset$ and compact (or contained in a compact set).

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LECTURE 02

Note Title

27/09/2021

Integral functionals

SYNONIMS but in different contexts

- Function
- Functional
- Map (mapping)
- Application
- Operator

Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let
 $L: (a, b) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function
 ↑ LAGRANGIAN (with some regularity)

Then we can consider

$$F: \mathbb{X} \rightarrow \mathbb{R}$$

 where $\mathbb{X} \subseteq C^1([a, b]) = \{\text{functions } u: [a, b] \rightarrow \mathbb{R} \text{ of class } C^1\}$

defined by

$$F(u) := \int_a^b L(x, u(x), u'(x)) dx$$

We can extend the notion in several ways

- by allowing $u: [a, b] \rightarrow \mathbb{R}^m$
- " " $u: \Omega \rightarrow \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^d$
- " " higher order derivatives
- " " $(a, +\infty)$ instead of (a, b) .

$$L(x, s, p)$$

↑ position

derivative
velocity

Classical examples

$$F(u) = \int_a^b \dot{u}(x)^2 dx \quad G(u) = \int_a^b |\dot{u}(x)| dx \quad H(u) = \int_a^b |\dot{u}(x)|^{1/2} dx$$

Compute $\min \{ F(u) : \underbrace{u \in C^1([0,1])}_{\text{set } \mathbb{X}}, \underbrace{u(0)=1, u(1)=9}_{\text{Boundary conditions}} \}$

INDIRECT METHOD Observe that $\mathbb{X}$ is an affine space with corresponding vector space

$$V = \{ v \in C^1([0,1]) : v(0) = v(1) = 0 \}$$

\begin { brutal }

Assume $u_0(x)$ is a min. point. in $\mathbb{X}$. For every $v \in V$ we can define

$$\begin{aligned} F(u_0 + tv) &= \int_0^1 (\dot{u}_0(x) + t \dot{v}(x))^2 dx \\ &= \int_0^1 \dot{u}_0(x)^2 dx + 2t \int_0^1 \dot{u}_0(x) \dot{v}(x) dx + t^2 \int_0^1 \dot{v}(x)^2 dx \\ &=: \varphi(t) \quad (t \in \mathbb{R}) \end{aligned}$$

$$\delta F(u_0, v) = \varphi'(0) = 2 \int_0^1 \dot{u}_0(x) \dot{v}(x) dx$$

If $u_0(x) \in \mathbb{X}$ is a min. point. then

$$\int_0^1 \dot{u}_0(x) \dot{v}(x) dx = 0 \quad \forall v \in V$$

First integral
form of
ELE

Euler-Lagrange equ.

Now I integrate by parts:

$$\int_0^1 \ddot{u}_0(x) \dot{v}(x) dx = \left[\ddot{u}_0(x) v(x) \right]_{x=0}^{x=1} - \int_0^1 \ddot{u}_0(x) v(x) dx$$

$\parallel$
 $\ddot{u}_0(1) \underset{0}{v(1)} - \ddot{u}_0(0) \underset{0}{v(0)}$

Therefore

$$\boxed{\int_0^1 \ddot{u}_0(x) v(x) dx = 0 \quad \forall v \in V} \quad \text{Second integral form of ELE}$$

Achtung! I can not integrate by parts because I do not know that $\ddot{u}_0$ exists, but I DO NOT CARE.

It is reasonable to deduce that $\ddot{u}_0(x) = 0$ for every $x \in [0, 1]$

$$\begin{cases} \ddot{u}_0(x) = 0 & \leftarrow \text{ELE in differential form} \\ u_0(0) = 1 \\ u_0(1) = 9 \end{cases} \quad \text{BCs (boundary conditions)}$$

The unique solution is $u_0(x) = 1 + 8x$

I conjecture this is the unique min point.

end {brutal}

Let us prove that $u_0(x)$ is the unique min. point.

Any other competitor $u(x) \in \mathbb{X}$ can be written as

$$u(x) = u_0(x) + v(x) \quad \begin{matrix} \downarrow \in V \\ \nearrow \text{integrate by parts} \end{matrix} \quad = 0 \text{ because now I can}$$

$$F(u) = \int_0^1 (\ddot{u}_0 + \ddot{v})^2 dx = \int_0^1 \ddot{u}_0^2 dx + 2 \int_0^1 \ddot{u}_0 \ddot{v} dx + \int_0^1 \ddot{v}^2 dx = 0$$

$$\geq F(u_0)$$

with equality if and only if $\int_0^1 \dot{v}^2 dx = 0$, namely

iff $\dot{v}(x) \equiv 0 \leadsto v(x)$ is constant $\leadsto v(x) \equiv 0$ because $v(0) = 0$.

— 0 — 0 —
 $\min \left\{ \int_0^1 |\dot{u}(x)| dx : u \in C^1([0,1]), u(0)=1, u(1)=9 \right\}$

$$\int_0^1 |\dot{u}(x)| dx \geq \int_0^1 \dot{u}(x) dx = u(1) - u(0) = 8$$

↑
 equality iff $\dot{u}(x) \geq 0$ for every $x \in [0,1]$

In this case the min exists, BUT there are infinitely many minimizers (any $u_0(x)$ with $\dot{u}_0(x) \geq 0$ and $u_0(0)=1$ and $u_0(1)=9$ is a minimizer).

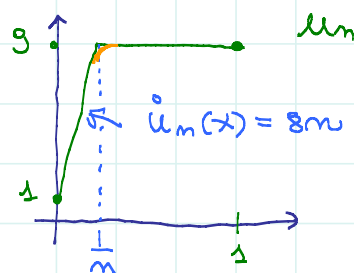
— 0 — 0 —
 $\min \left\{ \int_0^1 |\dot{u}(x)|^{1/2} dx : u \in C^1(\dots), u(0)=1, u(1)=9 \right\}$

Surprise! The min does NOT exist, and $\inf = 0$.

Easy: if $\inf = 0$, the min does NOT exist (it should be $\dot{u}(x) \equiv 0 \leadsto u(x)$ constant $\leadsto$ NO BCs)

Enough to FIND a sequence $\{u_m\} \subseteq \mathbb{X}$ s.t. $F(u_m) \rightarrow 0$

$$F(u_m) = \int_0^{\frac{1}{m}} \sqrt{8m} dx = \frac{1}{m} \cdot \sqrt{8m} \rightarrow 0$$



We can modify u_m around $\frac{1}{m}$
 in such a way that it is of class C^1

The difference between the three examples is the convexity of $L(x, s, p)$ with respect to p

$L(x, s, p) = p^2 \rightsquigarrow$ existence and uniqueness

$L(x, s, p) = |p| \rightsquigarrow$ existence without uniqueness

$L(x, s, p) = \sqrt{|p|} \rightsquigarrow$ big troubles

The ambient space is important.

For standard Lagrangians the infimum in C^1 coincides with the infimum in piecewise- C^1 .

It is enough to make a smooth connection near the intermediate points.

— o — o —

Istituzioni di Analisi

LECTURE 03

Note Title

29/09/2021

FUNDAMENTAL LEMMA IN THE CALC. OF VARIATIONS

→ FLCV

→ FLCV with 0 average

→ DBR Lemma (Du Bois Reymond)

Given $u: \Omega \rightarrow \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^d$ for some $d \geq 1$.

Assume

$$(\star) \quad \int_{\Omega} u(x) \varphi(x) dx = 0 \quad \forall \varphi \in C_c^{\infty}(\Omega)$$

$\uparrow$
compact support

FLCV (classical setting) Assume that $u \in C^0(\Omega)$ and $(\star)$.
Then $u(x) \equiv 0$ for every $x \in \Omega$.

FLCV (Lebesgue setting) Assume that $u \in L^1_{loc}(\Omega)$ and $(\star)$.
Then $u(x) = 0$ for almost every $x \in \Omega$.

$$[u \in L^1_{loc}(\Omega) \Leftrightarrow u \in L^1(\Omega') \text{ for every } \Omega' \subset\subset \Omega]$$

$\uparrow$
 $\text{clos}(\Omega') \subseteq \Omega$ and compact

FLCV with 0 average (classical setting)

Assume $u \in C^0(\Omega)$ and $(\star)$ is true for every $\varphi \in C_c^{\infty}(\Omega)$ such that

$$\int_{\Omega} \varphi(x) dx = 0$$

Then there exists $c \in \mathbb{R}$ s.t. $u(x) = c$ for every $x \in \Omega$.

The same is true in the Lebesgue setting with $u \in L^1_{\text{loc}}(\Omega)$ and the conclusion is $u(x) = c$ for almost every $x \in \Omega$.

DBR Lemma (Lebesgue setting)

Assume that $(a, b) \subseteq \mathbb{R}$ is an interval, assume that $u \in L^1_{\text{loc}}(a, b)$ satisfies

$$\int_a^b u(x) \varphi'(x) dx = 0 \quad \forall \varphi \in C_c^\infty((a, b))$$

Then again $\exists c \in \mathbb{R}$ s.t. $u(x) = c$ for almost every $x \in (a, b)$.

Proof (FLCV - classical) Assume that $u(x) \not\equiv 0$, wlog $\exists x_0 \in \Omega$ s.t. $u(x_0) = \alpha > 0$

Since u is continuous, then

$$\exists r > 0 \text{ s.t. } u(x) \geq \frac{\alpha}{2} \text{ in } B(x_0, r) \subset \Omega$$



There exists $\varphi \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$(i) \quad 0 \leq \varphi(x) \leq 1 \quad \forall x \in \mathbb{R}^d$$

$$(ii) \quad \varphi(x) = 0 \quad \forall x \text{ in } \mathbb{R}^d \setminus B(x_0, r) \rightsquigarrow \varphi \in C_c^\infty(\Omega)$$

$$(iii) \quad \varphi(x) = 1 \quad \forall x \in B(x_0, \frac{r}{2})$$

$$\begin{aligned} \int_{\Omega} u(x) \varphi(x) dx &= \int_{B(x_0, r)} u(x) \varphi(x) dx \\ &\geq \int_{B(x_0, \frac{r}{2})} \underbrace{u(x)}_{\geq \frac{\alpha}{2}} \underbrace{\varphi(x)}_1 dx \\ &= \frac{\alpha}{2} |B(x_0, \frac{r}{2})| > 0. \\ &\quad \text{--- } 0 \text{ --- } 0 \text{ ---} \end{aligned}$$

Proof FLCV (Lebesgue)

Idea: "it would be nice to choose
 $\varphi(x) = \text{sign } u(x)$ "

Take any $\Omega' \subset \subset \Omega$ (It is enough to consider $\Omega' = \text{any ball}$)

Define $v(x) := \begin{cases} \text{sign}(u(x)) & \text{if } x \in \Omega' \\ 0 & \text{otherwise} \end{cases}$

Then $\exists \{\varphi_m\} \subseteq C_c^\infty(\Omega)$ such that

(i) $\varphi_m(x) \rightarrow v(x)$ for almost every $x \in \Omega$

(ii) $|\varphi_m(x)| \leq 1$ for every $x \in \Omega$ and every $m \in \mathbb{N}$

(iii) support of $\varphi_m \subseteq \Omega'' \subset \subset \Omega$

For every $m \in \mathbb{N}$

$$\int_{\Omega} u(x) \varphi_m(x) dx = 0$$

$\downarrow \leftarrow$ Dominated convergence

$$\int_{\Omega} u(x) v(x) dx = \int_{\Omega'} |u(x)| dx \leadsto u(x) \equiv 0 \text{ for almost every } x \in \Omega'$$

$$|u(x) \varphi_m(x)| \leq |u(x) \cdot \mathbb{1}_{\Omega''}(x)|$$

$\uparrow$
characteristic function of Ω''

FLCV with 0 average.

The key idea is the following

$$\int_{\Omega} u(x) \varphi(x) dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega) \text{ and } \int_{\Omega} \varphi(x) dx = 0$$

then for every $c \in \mathbb{R}$ it turns out that

$$\int_{\Omega} (u(x) - c) \varphi(x) dx = 0 \quad \forall \varphi \text{ as above}$$

Idea: choose c s.t. $u(x) - c$ has 0 average, and then use $\varphi = u - c$ or $\varphi = \text{sign}(u - c)$

Let us take any $\Omega' \subset \subset \Omega$ (balls are enough). Let

$$c := \frac{1}{|\Omega'|} \int_{\Omega'} u(x) dx$$

Let us consider $\text{sign}(u(x) - c)$ and let us choose $d \in \mathbb{R}$ s.t.

$$\int_{\Omega'} (\text{sign}(u - c) - d) dx = 0$$

Approximation result: $\exists \{\varphi_m\} \subseteq C_c^\infty(\Omega)$ s.t.

$$(i) \quad \varphi_m(x) \rightarrow \begin{cases} \text{sign}(u - c) - d & \text{in } \Omega' \\ 0 & \text{otherwise} \end{cases}$$

$$(ii) \quad \exists M \in \mathbb{R} \text{ s.t. } |\varphi_m(x)| \leq M \quad \forall x \in \Omega \quad \forall m \in \mathbb{N}$$

$$(iii) \quad \text{support of } \varphi_m \subseteq \Omega'' \subset \subset \Omega.$$

$$(iv) \quad \int_{\Omega} \varphi_m(x) dx = 0$$

For every $m \in \mathbb{N}$

$$\int_{\Omega} (u(x) - c) \varphi_m(x) dx = 0$$

↓

$$\int_{\Omega'} (u(x) - c) [\text{sign}(u(x) - c) - d] dx$$

$$= \int_{\Omega'} \{|u(x) - c| - d(u(x) - c)\} dx = \int_{\Omega'} |u(x) - c| dx$$

This proves that $u(x) \equiv c$ for almost every $x \in \Omega$!

In order to conclude, we need to show that c does not depend on Ω !

Proof of DBR It is enough to observe that

$$\{\varphi'(x) : \varphi \in C_c^\infty((a,b))\} = \{\psi(x) : \psi \in C_c^\infty \text{ and } \int_a^b \psi dx = 0\}$$

$$\int_a^b \varphi'(x) dx = \varphi(b) - \varphi(a) = 0$$

Remark In DBR we can replace (a,b) by $(a, +\infty)$ or $\mathbb{R}$, but connectedness is essential.

Ex Give a proof of FLCV with 0 average in the classical setting.

— o — o —

How to approx with 0 average.

Assume $v(x)$ has 0 average and $v_m \in C_c^\infty$ is s.t.

$v_m \rightarrow v(x)$ (pointwise + domination)

$$\text{In any case } \int_{\Omega} \underbrace{v_m(x)}_{C_m} dx \rightarrow \int_{\Omega} v(x) dx = 0$$

Consider $\hat{v}_m(x) := v_m(x) - C_m \psi(x)$

where ψ is any function of class C_c^∞ with $\int_{\Omega} \psi(x) dx = 1$

Istituzioni di Analisi – LECTURE 04

Note Title

29/09/2021

GENESIS OF BOUNDARY CONDITIONS

$$F(u) = \int_0^2 \{ \dot{u}(x)^2 + x^2 u(x) \} dx$$

Example minimize $F(u)$ in

$$\mathbb{X} := \{ u \in C^1([0,2]) : u(0) = 1, u(2) = 7 \}$$

Observe that $\mathbb{X}$ is an affine space with con. vect. space

$$\mathbb{V} := \{ v \in C^1([0,2]) : v(0) = v(2) = 0 \}$$

Assume $u_0 \in \mathbb{X}$ is a min. point, and consider

$$\begin{aligned} \varphi(t) := F(u_0 + tv) &= \int_0^2 \dot{u}_0(x)^2 dx + 2t \int_0^2 \dot{u}_0(x) \dot{v}(x) dx + t^2 \int_0^2 \dot{v}(x)^2 dx \\ &\quad + \int_0^2 x^2 u_0(x) dx + t \int_0^2 x^2 v(x) dx \end{aligned}$$

Then

$$\delta F(u_0, v) = \varphi'(0) = 2 \int_0^2 \dot{u}_0(x) \dot{v}(x) dx + \int_0^2 x^2 v(x) dx$$

Then we integrate by parts...

$$\begin{aligned} &2 \left[\dot{u}_0(x) v(x) \right]_{x=0}^{x=2} - 2 \int_0^2 \ddot{u}_0(x) v(x) dx + \int_0^2 x^2 v(x) dx \\ &\quad \stackrel{\text{"0 because } v \in \mathbb{V}}{\sim} \int_0^2 \{ -2 \ddot{u}_0(x) + x^2 \} v(x) dx = 0 \end{aligned}$$

$$\begin{cases} 2\ddot{u}_0(x) = x^2 \rightarrow \text{ELE in diff. form} \\ u_0(0) = 1 \\ u_0(2) = 7 \end{cases} \quad \text{DIRICHLET BC (prescribe the values of } u_0 \text{ at the boundary)}$$

$\leadsto$ there exists a unique solution, which is the unique min. point.

— o — o —

Example 2 $\min \{ F(u) : u \in C^4 \dots, \underbrace{u(0) = 3}_{\text{only one BC}} \}$

Now $V = \{ v \in C^1([0,2]) : v(0) = 0 \}$

As before we obtain that

$$\int_0^2 \{ 2\ddot{u}_0(x) \dot{v}(x) + x^2 v(x) \} dx = 0 \quad \forall v \in V$$

When we integrate by parts we obtain that

$$2\ddot{u}_0(2)v(2) + \int_0^2 \{ -2\ddot{u}_0(x) + x^2 \} v(x) dx = 0 \quad \forall v \in V \quad (*)$$

Step 1 If we limit ourselves to $v \in C^1([0,2])$ with $v(0) = v(2) = 0$, or we limit ourselves to $v \in C_c^\infty((0,2))$

we get

$$\int_0^2 \{ -2\ddot{u}_0(x) + x^2 \} v(x) dx = 0 \quad \forall v \in C_c^\infty((0,2))$$

$\leadsto 2\ddot{u}_0(x) = x^2 \rightarrow \text{AGAIN ELE as before}$

STEP 2 Going back to (*), now we know that

$$\forall v \in V$$

It is enough to consider $v \in V$ with $v(2) \neq 0$ to conclude that

$$u_0(z) = 0.$$

We have "proved"

$$\begin{cases} 2\ddot{u}_0(x) = x^2 & \rightarrow \text{ELE} \\ u_0(0) = 3 & \rightarrow \text{Dirichlet BC} \quad \text{DBC} \\ \dot{u}_0(2) = 0 & \rightarrow \text{NEUMANN Boundary Condition} \quad \text{NBC} \end{cases}$$

Example 3 If we minimize without BCs we obtain

$$\left\{ \begin{array}{l} 2\ddot{u}_0(x) = x^2 \\ \dot{u}_0(0) = 0 \\ \dot{u}_0(2) = 0 \end{array} \right\} \text{ NBC originated "on the road"}$$

Now...
$$u_0(2)v(2) - u_0(0)v(0) + \int_0^2 \{-2\ddot{u}_0(x) + x^2\} v(x) dx = 0$$

Step 1 Consider $v \in C_c^\infty((0, 2))$ and deduce $2\ddot{u}_0 = x^2$

Step 2 Consider $v \in V = \mathbb{X}$ with $v(2) = 1$ and $v(0) = 0$
 " $v(2) = 0$ and $v(0) = 1$
 and we deduce the NBC.

Achtung! These problems are NOT Cauchy problems, and there are no existence and uniqueness results

Example 4 minimize with BC $u(0) = u(2) + 7$

The vector space is

$$V = \{v \in C^1([0, 2]) : v(0) = v(2)\}$$

$$\dots \quad \dot{u}_0(2)v(2) - \dot{u}_0(0)v(0) = 0 \quad \forall v \in V$$

We take $v \in V$ s.t. $v(2) = v(0) = 1$ and we obtain that
 $\dot{u}_0(2) = \dot{u}_0(0)$

$$\begin{cases} 2\ddot{u}_0(x) = x^2 \\ u_0(0) = u_0(2) + 7 \\ \dot{u}_0(2) = \dot{u}_0(0) \end{cases} \quad \begin{array}{l} \text{PBC} \\ \uparrow \text{periodic.} \end{array}$$

Example 5 Minimize with $\int_0^2 u(x) dx = 9$

$$\text{Now } V = \{v \in C^1([0, 2]) : \int_0^2 v(x) dx = 0\}$$

In the usual way ...

$$\dot{u}_0(2)v(2) - \dot{u}_0(0)v(0) + \int_0^2 \{-2\ddot{u}_0(x) + x^2\}v(x) dx = 0 \quad \forall v \in V$$

Now use $v \in C_c^\infty((0, 2))$ with 0 average and deduce

$$2\ddot{u}_0(x) = x^2 + \lambda \quad \begin{array}{l} \uparrow \text{constant} \end{array} \quad \rightsquigarrow \text{integral} = 0 \rightsquigarrow \text{(step 2)} \quad \text{NBC}$$

We have "proved" that

$$\begin{cases}
 2\ddot{u}_0(x) = x^2 + \lambda & \leftarrow \text{ELE} \\
 \dot{u}_0(0) = 0 \\
 \dot{u}_0(2) = 0
 \end{cases}
 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{NBC on the road}$$

$\int_0^2 u_0(x) dx = 9$ given integral constraint
 — o — o —

Istituzioni di Analisi

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LECTURE 05

Note Title

30/09/2021

ELE in general Setting:

$$\int_a^b L(x, u(x), \dot{u}(x)) dx = F(u)$$

where $L: [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ $L(x, s, p)$

$x \quad s \quad p$

Goal: compute the first variation. We consider

$$\varphi(t) := F(u + tv) = \int_a^b L(x, u(x) + tv(x), \dot{u}(x) + t\dot{v}(x)) dx$$

$$\varphi'(t) = \int_a^b \{L_s(x, u + tv, \dot{u} + t\dot{v})v + L_p(\dots)\dot{v}\} dx$$

$$\delta F(u, v) := \varphi'(0) =$$

FIRST INTEGRAL FORM

$$\int_a^b \{L_s(x, u(x), \dot{u}(x))v(x) + L_p(x, u(x), \dot{u}(x))\dot{v}(x)\} dx$$

Sufficient conditions: $\rightarrow u$ and v in $C^1([a, b])$ $\rightarrow L$ of class C^1 (we just need L_s and L_p)

If we integrate by parts, we obtain

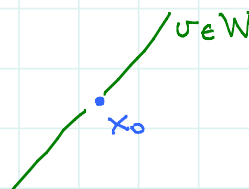
SECOND INTEGRAL FORM

$$\delta F(u, v) = \int_a^b \left\{ L_s(x, u(x), \dot{u}(x)) - \frac{d}{dx} L_p(x, u(x), \dot{u}(x)) \right\} v(x) dx + [L_p(x, u(x), \dot{u}(x))v(x)]_{x=a}^{x=b}$$

Sufficient conditions: $\rightarrow u \in C^2([a,b])$
 $\rightarrow v \in C^1([a,b])$
 $\rightarrow L$ of class C^2 (L of class C^1)

Of course, if u is a minimizer, then $\delta F(u, v) = 0$ for every "admissible" v

Def. Let X be an affine space with vect. space V . Let $F: X \rightarrow \mathbb{R}$ be a functional, let $W \subseteq V$ be a vect. subspace, let $x_0 \in X$. We say that x_0 is a directional local minimum wrt W if, for every $v \in W$, the function $t \rightarrow F(x_0 + tv)$ has a loc. min. in $t=0$.



Remark x_0 min. point in $X \Rightarrow x_0$ is a DLM for every $W \subseteq V$
 The converse is not true even if $W = V = X = \mathbb{R}^2$

Coming back to integral functionals, let us assume that u_0 is a DLM with $W = C_c^\infty((a,b))$.

Then we deduce that

$$\frac{d}{dx} L_p(x, u_0(x), \dot{u}_0(x)) = L_s(x, u_0(x), \dot{u}_0(x))$$

DIFF. FORM
OF ELE

Suff. conditions: the same as in 2nd int. form.

The derivative in the LHS is a total derivative. If we expand

$$L_{px}(\dots) + L_{ps}(\dots) \dot{u}_0 + L_{pp}(\dots) \ddot{u}_0 = L_s(\dots)$$

If $L_{pp}(x, s, p) > 0$ for every admissible values of the parameters, then ELE can be written as

$$\ddot{u}_0(x) = \Phi(x, u_0(x), \dot{u}_0(x))$$

For a suitable Φ . ($L_{pp} > 0$ is stronger than strict convexity)

If $u_0(x)$ is DLM with respect to a class W that contains $C_c^\infty(a, b)$ but also functions with $v(a) = 0$ and $v(b) = 1$, then a NBC appears of the form

$$L_p(b, u_0(b), \dot{u}_0(b)) = 0 \quad \text{NBC}$$

The same if $v(a) = 1$ and $v(b) = 0$.

ELE in DBR form

$$\int_a^b \{ L_s(x, u(x), \dot{u}(x)) v(x) + L_p(x, u(x), \dot{u}(x)) \dot{v}(x) \} dx$$

Let us set $\hat{L}(x) := \int_a^x L_s(t, u(t), \dot{u}(t)) dt$

If we integrate by parts the first term we obtain

$$\int_a^b \{ -\hat{L}(x) + L_p(x, u(x), \dot{u}(x)) \} \dot{v}(x) dx + [\hat{L}(x) v(x)]_{x=a}^{x=b}$$

Again, if $u(x)$ is DLM with $W = C_c^\infty(a, b)$ then we apply DBR lemma and we obtain

$$L_p(x, u(x), \dot{u}(x)) = c + \int_a^x L_s(t, u(t), \dot{u}(t)) dt$$

↑
free parameter

Remark The same can be obtained by integrating the diff. form, BUT the diff. form requires 2nd order deriv. of u and L , while here we used just 1st order derivative.

Remark DBR Lemma is trivial if we could int. by parts

$$\int_a^b u(x) \ddot{\varphi}(x) dx = 0 \quad \rightsquigarrow \quad \int_a^b \dot{u}(x) \dot{\varphi}(x) dx = 0$$

$$\rightsquigarrow \dot{u}(x) \equiv 0 \quad \rightsquigarrow \quad u(x) \equiv \text{constant}$$

— 0 — 0 —

ELE in the autonomous case Assume that $L(s, p)$ does NOT depend on x .

Start with ELE

$$[L_p(u, \dot{u})]' = L_s(u, \dot{u})$$

Multiply by $\ddot{u}$: $[L_p(u, \dot{u})]' \ddot{u} = L_s(u, \dot{u}) \ddot{u}$

$$[L_p(u, \dot{u}) \ddot{u}]' - L_p(u, \dot{u}) \ddot{u}$$

and therefore

$$[L_p(u, \dot{u}) \ddot{u}]' = L_p(u, \dot{u}) \ddot{u} + L_s(u, \dot{u}) \ddot{u} = [L(u, \dot{u})]'$$

and this implies that

$$L_p(u(x), \dot{u}(x)) \ddot{u}(x) = L(u(x), \dot{u}(x)) + c$$

First order diff. equ. in u with a free parameter

More general situation

$$F(u) = \int_a^b L(x, u, \dot{u}, \ddot{u}) dx \quad L(x, s, p_1, p_2)$$

$$\delta F(u, v) = \int_a^b \{ L_s(\dots) v + L_{p_1}(\dots) \dot{v} + L_{p_2}(\dots) \ddot{v} \} dx$$

Assume $v \in C_c^\infty(a, b)$. Int. by parts we obtain

$$\delta F(u, v) = \int_a^b \{ \underbrace{L_s(\dots) - (L_{p_1}(\dots))' + (L_{p_2}(\dots))''}_{=0} \} v(x) dx$$

$= 0 \leadsto$ ELE in diff. form

(equation of order 4)

— 0 — 0 —

Istituzioni di Analisi

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LECTURE 06

Note Title

30/09/2021

FIRST VARIATION FOR MULTIPLE INTEGRALS

$$F(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) dx$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded open set with "smooth" boundary and

$$L: \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

$$L(x, s, \underset{\substack{\uparrow \\ \text{vector}}}{p})$$

Gauss-Green formula (integration by parts for multiple int.)

$$\int_{\Omega} f \cdot \operatorname{div} \vec{E} dx = \int_{\partial\Omega} f \langle \vec{E}, \vec{n} \rangle d\sigma - \int_{\Omega} \langle \nabla f, \vec{E} \rangle dx$$

$$f: \Omega \rightarrow \mathbb{R}$$

$$\vec{E}: \Omega \rightarrow \mathbb{R}^d$$

↑
outward unit
normal to $\partial\Omega$

↑
integral in
(d-1) dimensions

In the special case where $\vec{E} = \nabla g$ for some $g: \Omega \rightarrow \mathbb{R}$ the formula becomes

$$\int_{\Omega} f \Delta g dx = \int_{\partial\Omega} f \langle \nabla g, \vec{n} \rangle d\sigma - \int_{\Omega} \langle \nabla f, \nabla g \rangle dx$$

$$\int_{\partial\Omega} f \frac{\partial g}{\partial \vec{n}} d\sigma$$

↑ derivative of g
in the direction $\vec{n}$

The formula requires some "smoothness" of the boundary $\partial\Omega$. For example $\partial\Omega$ piecewise C^1 is enough.

Come back to $F(u)$ defined above. let us start with the special case

$$F(u) = \int_{\Omega} |\nabla u(x)|^2 dx$$

↑ norm of the gradient

(DIRICHLET
INTEGRAL)

$$\begin{aligned} \varphi(t) &:= F(u+tv) = \int_{\Omega} |\nabla u(x) + t \nabla v(x)|^2 dx \\ &= \int_{\Omega} |\nabla u(x)|^2 dx + 2t \int_{\Omega} \langle \nabla u(x), \nabla v(x) \rangle dx \\ &\quad + t^2 \int_{\Omega} |\nabla v(x)|^2 dx \end{aligned}$$

$$\delta F(u, v) = \varphi'(0) = 2 \int_{\Omega} \langle \nabla u(x), \nabla v(x) \rangle dx$$

FIRST INT.
FORM

If we have more regularity, we int. by parts and we obtain

$$\delta F(u, v) = -2 \int_{\Omega} \Delta u(x) v(x) dx + 2 \int_{\partial \Omega} \frac{\partial u(x)}{\partial \vec{n}} \cdot v(x) d\sigma$$

SECOND INTEGRAL FORM

If u is a DLM wrt every $v \in C_c^\infty(\Omega)$, then FLCV implies that

$$\Delta u(x) = 0 \quad \forall x \in \Omega$$

ELE in diff. form

If we can use functions v which are not 0 at the boundary we obtain "on the road" NBC of the form

$$\frac{\partial u}{\partial \vec{n}} = 0 \quad \text{at } \partial \Omega$$

Consider now a more general

$$F(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) dx$$

$$\varphi(t) = \int_{\Omega} L(x, u(x) + t v(x), \nabla u(x) + t \nabla v(x)) dx$$

$$\delta F(u, v) = \varphi'(0) = \int_{\Omega} \{ L_s(x, u(x), \nabla u(x)) v(x) + \\ + \langle \nabla_p L(x, u(x), \nabla u(x)), \nabla v(x) \rangle \} dx$$

After integration by parts we obtain

$$\delta F(u, v) = \int_{\Omega} \{ L_s(x, u(x), \nabla u(x)) - \operatorname{div}(\nabla_p L(x, u(x), \nabla u(x))) \} v(x) dx \\ + \int_{\partial\Omega} v(x) \langle \nabla_p L(x, u(x), \nabla u(x)), \vec{n} \rangle d\sigma$$

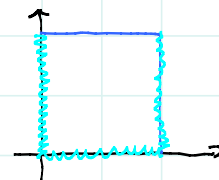
If u is DLM wrt every $v \in C_c^\infty(\Omega)$ (and both u and L are regular enough)

then u satisfies

$$\operatorname{div}(\nabla_p L(x, u(x), \nabla u(x))) = L_s(x, u(x), \nabla u(x))$$

Example $\Omega = (0,1)^2$ $F(u) = \int_{\Omega} \{ |\nabla u(x)|^2 - u(x) \} dx$

Problem: minimize $F(u)$ over all u 's of class C^1 which are 0 on three sides of the boundary



Assume that $u_0(x, y)$ is a min. of class C^2 . Then nec.

$$2 \Delta u = -1$$

$\uparrow$ L_S

$$L(x, y, p_1, p_2) = -S + p_1^2 + p_2^2$$

$$(L_{p_1}(\dots), L_{p_2}(\dots)) = (2p_1, 2p_2)$$

$$\nabla_p L(u, \nabla u) = 2(u_x, u_y)$$

$u \equiv 0$ on the 3 sides DBC from the very beginning

$u_y = 0$ on the 4th side NBC born on the road

$$\mathbb{X} = \{u \in C^1(\bar{\Omega}) : u \equiv 0 \text{ on the 3 sides}\}$$

$$\mathbb{V} = \{ \dots \}$$

When we compute the 2nd int. form of $\delta F(u, v)$, the only boundary term that survives is

$$\int_S v(x, y) u_y(x, y) d\sigma = \int_0^1 v(x, 1) \underbrace{u_y(x, 1)}_{\substack{\uparrow \\ \text{param. of 4th side}}} dx$$

$\uparrow$ 4th side

FLCV on the 4th side $\Rightarrow u_y(x, 1) = 0 \quad \forall x \in [0, 1]$.

— o — o —

Example

$$F(u) = \int_{\Omega} (y^2 \underbrace{u_x^2}_{x^2} + u_y^2) dx dy$$

$\uparrow$ square

Compute ELE

$$L(x, y, s, p_1, p_2) = y^2 \underbrace{p_1^2}_{x^2} + p_2^2$$

$$L_S = 0 \quad (L_{p_1}, L_{p_2}) = (2y^2 p_1, 2p_2) = (2y^2 u_x, 2u_y)$$

x^2

$$2 \operatorname{div} (\underbrace{y^2 u_x}_{x^2}, u_y) = 0 \quad \text{an ELE}$$

$$y^2 u_{xx} + u_{yy} = 0$$

$2x u_x + x^2 u_{xx} + u_{yy} = 0$

— o — o —

Istituzioni di Analisi – LECTURE 07

Note Title

04/10/2021

Setting: S is a set, $F: S \rightarrow \mathbb{R}$ is a function, $s_0 \in S$
 How to prove that s_0 is a min point?

Two main strategies: $\rightarrow$ via CONVEXITY
 $\rightarrow$ via AUXILIARY FUNCTIONAL

MINIMALITY VIA CONVEXITY

Main example $\min \left\{ \int_a^b \varphi(u'(x)) dx : u \in C^1([a,b]), \right.$
 $\left. u(a) = A, u(b) = B \right\}$

If $u_0(x)$ is a minimizer, then $[\varphi'(u'_0(x))] = 0$, namely

$$\varphi''(u'_0(x)) u''_0(x) = 0 \quad \text{ELE}$$

Let $u_0(x)$ denote the straight line that satisfies DBC.

Claim: $u_0(x)$ is the unique minimizer.

Prop. The claim is true if φ is CONVEX.

If φ is strictly convex, then the minimizer is unique.

Proof. Let us consider the inequality

$$(*) \quad \varphi(a+b) \geq \varphi(a) + \varphi'(a)b \quad \forall a \in \mathbb{R} \quad \forall b \in \mathbb{R}$$

Taylor-Lagrange: $\varphi(a+b) = \varphi(a) + \varphi'(a)b + \underbrace{\frac{1}{2} \varphi''(\xi) b^2}_{\geq 0}$

$\uparrow$
somewhere

Let $u(x)$ be any other competitor. Then $u(x) = u_0(x) + v(x)$
 $\uparrow$
 $0 \in u[a, b]$

$$\begin{aligned}
 F(u) &= \int_a^b \varphi(\underbrace{u'_0(x)}_a + \underbrace{v'(x)}_b) dx \geq \int_a^b \varphi(\underbrace{u'_0(x)}_{(**)}) dx + \int_a^b \underbrace{\varphi'(u'_0(x))}_{\text{constant } m} v'(x) dx \\
 &= F(u_0) + \varphi'(m) \int_a^b v'(x) dx \geq F(u_0) \\
 &\quad \quad \quad \uparrow \\
 &\quad \quad \quad v(b) - v(a) = 0
 \end{aligned}$$

If φ is strictly convex, then $(*)$ is strict whenever $b \neq 0$, namely $(**)$ is strict whenever $v'(x) \not\equiv 0$.

Remark If φ is a convex function, then for every $a \in \mathbb{R}$ there exists $\alpha \in \mathbb{R}$ such that

$$\varphi(a+b) \geq \varphi(a) + \alpha b \quad \forall b \in \mathbb{R}$$



It is enough to take any $\alpha \in [\varphi'_-(a), \varphi'_+(a)]$
 $\uparrow$ $\uparrow$
 left and right derivative

— 0 — 0 —

Theorem Let us consider a Lagrangian $L(x, s, p)$.

Assume that, for every $x \in [a, b]$, the function

$$(s, p) \mapsto L(x, s, p)$$

is convex in the pair (s, p) (convexity in $\mathbb{R}^2$).

Then any solution to ELE is a minimizer, unique if L is strictly convex.

Proof Start from the inequality

$$L(x, s_0 + s_1, p_0 + p_1) \geq L(x, s_0, p_0) \\ + L_s(x, s_0, p_0) s_1 \\ + L_p(x, s_0, p_0) p_1$$

for every $x \in [a, b]$ and $(s_0, s_1, p_0, p_1) \in \mathbb{R}^4$.

Let $u_0(x)$ be a solution to ELE (just in the first int. form)

Then $u(x) = u_0(x) + v(x)$

$$F(u) = \int_a^b L(x, \underset{\uparrow s_0}{u_0(x)} + \underset{\uparrow s_1}{v(x)}, \underset{\uparrow p_0}{u'_0(x)} + \underset{\uparrow p_1}{v'(x)}) dx \\ \geq F(u_0) + \int_a^b \underbrace{\{L_s(x, u_0(x), u'_0(x)) v(x) + L_p(\dots) v'(x)\}}_{=0 \text{ because of ELE}} dx$$

If we have strict convexity, we argue as before.
— o — o —

MINIMALITY VIA AUXILIARY FUNCTIONAL

↑ "OSCULATING"

Lemma S is a set, $F: S \rightarrow \mathbb{R}$, $G: S \rightarrow \mathbb{R}$, $s_0 \in S$.

Assume that

(i) $F(x) \geq G(x) \quad \forall x \in S$

(ii) $G(x) \geq G(s_0) \quad \forall x \in S$

(iii) $G(s_0) = F(s_0)$

Then s_0 is a min. point of F .

Proof $F(x) \geq \underset{\uparrow (i)}{G(x)} \geq \underset{\uparrow (ii)}{G(s_0)} = \underset{\uparrow (iii)}{F(s_0)} \quad \forall x \in S \quad \text{☺}$

Easy but not trivial example

$$F(u) = \int_0^1 (u'(x)^2 - 1)^2 dx$$

$$\min \{ F(u) : u \in C^1(\dots), \\ u(0) = 0, u(1) = 4 \}$$

Claim: $u_0(x)$ is the unique minimizer

Consider $G(u) := \int_0^1 \hat{\varphi}(u'(x)) dx$

It is true that (i) $F(u) \geq G(u)$ for every admissible u
 $\uparrow$
 $\varphi(p) \geq \hat{\varphi}(p)$

(ii) $G(u) \geq G(u_0)$ (because $\hat{\varphi}$ is convex)

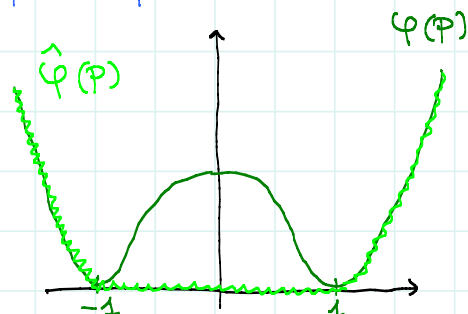
(iii) $G(u_0) = F(u_0)$ (because $\hat{\varphi}(4) = \varphi(4)$)

From the lemma we conclude that $u_0(x)$ minimizes F .

Actually it is the unique min. because $G(u) \geq G(u_0)$ is strict whenever $u(x) \neq u_0(x)$ (because the graph of $\hat{\varphi}$ is strictly above the tangent line in $p = 4$).

The same argument applies if we replace 4 with any B with $|B| \geq 1$.

$$\varphi(p) = (p^2 - 1)^2$$



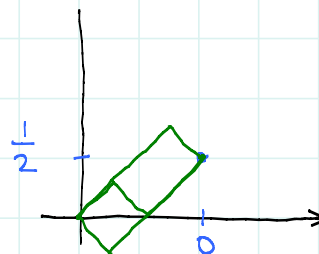
DOUBLE WELL POTENTIAL

$\uparrow$
POZZO
(BRÜNNEN)

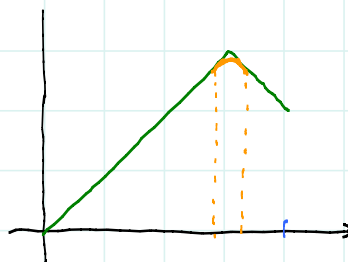
$$\min \{ F(u) : u \in C^1(\dots), u(0) = 0, u(1) = \frac{1}{2} \}$$

The inf = 0 and min DOES NOT EXIST.

A inf-izing sequence need to have derivative ± 1 "almost everywhere"

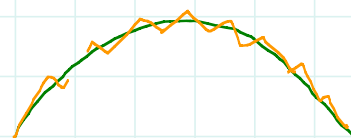


The cost of a smooth connection is at most the length of the interval where it takes place.



$$\min \left\{ \int_0^1 \{ (\dot{u}(x)^2 - 1)^2 + (u - \sin x)^2 \} dx : u \in C^1 \right\}$$

Again inf = 0 because I can approx $\sin x$ with u which has derivative ± 1 .



— 0 — 0 —

Istituzioni di Analisi

LEZIONE 08

Note Title

04/10/2021

Horizontal variation (inner variation)Vertical variation : $u_0(x) + tv(x)$ Horizontal variation : $u_0(x + tv(x))$ Assume $u_0 \in C^1([a, b])$. Assume that $v \in C_c^\infty(a, b)$.Then there exists $\delta > 0$ s.t.

$$a \leq x + tv(x) \leq b \quad \forall x \in [a, b] \quad \forall t \in [-\delta, \delta]$$

and therefore $u_0(x + tv(x))$ is a good competitor.

Two cases where it is fundamental

- case where u_0 has values in S^1 or some manifold
- geometric problems, where the unknown is a set identified with its characteristic function.

Let us compute the first variation of

$$F(u) = \int_a^b L(x, u(x), u'(x)) dx$$

wrt to a hor. var.

$$\varphi(t) := \int_a^b L(x, u(x + tv(x)), \underbrace{u'(x + tv(x)) (1 + t \dot{v}(x))}_{\text{deriv. wrt } x \text{ of } u(x + tv(x))}) dx$$

$$\begin{aligned} \varphi'(t) = & \int_a^b L_s(\dots) u(x + tv(x)) v(x) + \\ & + L_p(\dots) [u(x + tv(x)) v(x) (1 + t \dot{v}(x)) + \dot{u}(x + tv(x)) \dot{v}(x)] \end{aligned}$$

and therefore

$$\begin{aligned} \varphi'(0) &= \int_a^b \left\{ L_s(x, u(x), \dot{u}(x)) \dot{u}(x) v(x) + L_p(x, u(x), \dot{u}(x)) \right. \\ &\quad \left. [\ddot{u}(x) v(x) + \dot{u}(x) \dot{v}(x)] \right\} dx \\ &\quad \quad \quad [\dot{u}(x) v(x)]' \\ &= \int_a^b \left\{ L_s(x, u(x), \dot{u}(x)) - [L_p(x, u(x), \dot{u}(x))]' \right\} \dot{u}(x) v(x) dx \\ &\quad + \text{boundary terms} \end{aligned}$$

If we ignore boundary terms, FLCV implies that

$$\left\{ L_s(\dots) - [L_p(\dots)]' \right\} \dot{u}(x) = 0$$

$\uparrow$
previous ELE
 $\uparrow$
we should have expected this $\circ$

If $u_0(x)$ satisfies this new ELE and $\dot{u}_0(x)$ vanishes only in a finite number of points, then $u_0(x)$ satisfies the classical ELE.

— $\circ$ — $\circ$ —

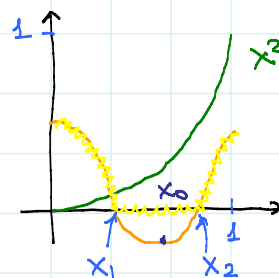
TRUNCATION ARGUMENTS VS MAXIMUM PRINCIPLES

Example $\min \left\{ \int_0^1 [\dot{u}^2 + (u - x^2)^2] dx : u \in C^1([0, 1]) \right\}$

Claim: a unique minimiser $u_0(x)$ exists and $0 \leq u_0(x) \leq 1$.

There is NO REASON for $u_0(x)$ to be > 1 or < 0 .

Assume that $u(x_0) < 0$. Then we obtain a better competitor by TRUNCATING $u(x)$



Let $u(x_0) < 0$. Let

$$x_1 := \sup \{ x \in [0, x_0] : u(x) = 0 \}$$

$$x_2 := \inf \{ x \in [x_0, 1] : u(x) = 0 \}$$

Then define

$$\hat{u}(x) := \begin{cases} u(x) & \text{if } x \in [0, x_1] \\ 0 & \text{if } x \in [x_1, x_2] \\ u(x) & \text{if } x \in [x_2, 1] \end{cases}$$

and observe that

$$F(\hat{u}) < F(u)$$

↑
we save on both terms in $F(u)$

Complain #1: $\hat{u}$ is not C^4 , BUT it is piecewise C^4 , and I can keep the strict inequality even with a smooth connection in x_1 and x_2

Complain #2: x_1 and x_2 , as defined, could not exist. But in that case it is even easier (use 0 instead of x_1 and 1 instead of x_2).

Remark If $u \in C^4$, then $\max \{ u(x), 0 \}$ is NOT necessarily piecewise C^4 .

In this case ELE is

$$(2\ddot{u})' = 2(u - x^2) \quad \leadsto \quad \ddot{u} = u - x^2$$

Assume that $\min \{ u_0(x) : x \in [0, 1] \} < 0$. Let $x_0 \in (0, 1)$ be a min. point of $u_0(x)$.

Then necessarily $\ddot{u}_0(x_0) \geq 0$. On the other hand

$$\ddot{u}_0(x_0) = u_0(x_0) - x_0^2 \leq u_0(x_0) < 0$$

Complain: it could be $x_0 = 0$ or $x_0 = 1$.

But even in this case the same argument applies because again $i''_0(x_0) \geq 0$ (please consider that this is not trivial: for example $\log x$ has second derivative which is negative and min. at the boundary in every interval!)

BUT this is true if we know that $i'(x_0) = 0$.

In this case we know that $i'(x_0)$ because of XBC.

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LECTURE 09

Note Title

06/10/2021

Vector spaces + structure

 $\mathbb{R}^d$

Hilbert spaces (everything works, with some notable exceptions)

Banach spaces (nothing works, ")

Normed spaces

Locally convex top. vect. spaces (as Banach spaces, but with more work)

Topological vector spaces

HILBERT SPACE **Vector space** + positive definite **scalar product**scal. prod. $\leadsto$ norm $\leadsto$ distance $\leadsto$ notion of convergence**Complete** metric space wrt this distance**HILBERT BASIS** Let H be a Hilbert space.A subset $\{e_i\}_{i \in I}$ (I is an index set) is **orthonormal** if

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

A Hilbert basis is an orthonormal subset $\{e_i\}_{i \in I}$ such thatSpan $\{e_i\}$ is dense in H
 $\uparrow$ set of FINITE linear combinations of $\{e_i\}$
COMPONENTS OF A VECTORLet H be a Hilbert space, let $v \in H$ let $\{e_i\}_{i \in I}$ be orthonormal setThen $v_i := \langle v, e_i \rangle$ is called the component of v wrt $\{e_i\}$
 $\uparrow$
real number

Theorem 1 Every Hilbert space H admits a Hilbert basis $\{e_i\}_{i \in I}$

If H is **SEPARABLE**, namely there exists a COUNTABLE dense subset, then I is either FINITE or COUNTABLE.

Theorem 2 Let H be a Hilbert space.

Let us assume that there exists a countable Hilbert basis $\{e_n\}_{n \geq 1}$

Let $v \in H$, and let $v_n = \langle v, e_n \rangle$ be its components. Then the following facts are true

$$v = \sum_{n=1}^{\infty} v_n e_n = \sum_{n=1}^{\infty} \langle v, e_n \rangle e_n$$

$$\|v\|^2 = \sum_{n=1}^{\infty} v_n^2 = \sum_{n=1}^{\infty} \langle v, e_n \rangle^2 \quad (\text{PITAGORA'S THM})$$

$$\langle v, w \rangle = \sum_{n=1}^{\infty} \langle v, e_n \rangle \cdot \langle w, e_n \rangle$$

In particular, all the series are convergent.

Remark Due to thm 2, we can identify any vector $v \in H$ with the sequence $\{v_n\}$ of its components.

In a brutal sense, a separable Hilbert space is " $\mathbb{R}^\infty$ ".

Lemma 1 (Continuity of norm and scalar product)

Let us assume that $v_n \rightarrow v_\infty$ in H and $w_n \rightarrow w_\infty$ in H .

Then

$$\|v_n\| \rightarrow \|v_\infty\| \quad \text{and} \quad \langle v_n, w_n \rangle \rightarrow \langle v_\infty, w_\infty \rangle$$

Lemma 2 (Convergence of Hilbert series)

Let $\{e_n\} \subseteq H$ be an orthonormal set

Let $\{a_n\}$ be a sequence of real numbers.

Then

$$\sum_{n=1}^{\infty} a_n e_n \text{ converges} \iff \sum_{n=1}^{\infty} a_n^2 \text{ converges}$$

$\uparrow$ series of vectors $\uparrow$ series of numbers

Lemma 3 (Bessel inequality)

Let $\{e_n\} \subseteq H$ be an orthonormal system. Let $v \in H$ be a vector.

Then

$$\sum_{n=1}^{\infty} \langle v, e_n \rangle^2 \leq \|v\|^2$$

Lemma 4 (Properties of Hilbert series)

Let $\{e_n\}_{n \geq 1} \subseteq H$ be an orthonormal system.

Let $\{a_n\} \subseteq \mathbb{R}$ be a sequence such that

$$\sum_{n=1}^{\infty} a_n^2 < +\infty$$

From Lemma 2 we know that $v := \sum_{n=1}^{\infty} a_n e_n \in H$ is well-def.

Then $\langle v, e_n \rangle = a_n \quad \forall n \geq 1$ and $\|v\|^2 = \sum_{n=1}^{\infty} a_n^2$

Lemma 5 (Characterization of the null vector)

Assume that $\{e_n\}_{n \geq 1} \subseteq H$ is a Hilbert basis. Let $v \in H$.

Assume that

$$\langle v, e_n \rangle = 0 \quad \forall n \geq 1$$

Then $v = 0$.

Proof of Lemma 1

$$| \|v_\infty\| - \|v_m\| | \leq \|v_\infty - v_m\| \quad (\text{triangular ineq.})$$

$$| \langle v_\infty, w_\infty \rangle - \langle v_m, w_m \rangle | \leq | \langle v_\infty, w_\infty \rangle - \langle v_\infty, w_m \rangle | + | \langle v_\infty, w_m \rangle - \langle v_m, w_m \rangle |$$

$$\leq \underbrace{\|v_\infty\|}_{\substack{\uparrow \\ \text{c.s.}}} \cdot \underbrace{\|w_\infty - w_m\|}_{\substack{\downarrow \\ 0}} + \underbrace{\|w_m\|}_{\substack{\downarrow \\ \text{unif. bounded, namely } \|w_m\| \leq M}} \cdot \underbrace{\|v_\infty - v_m\|}_{\substack{\downarrow \\ 0}}$$

unif. bounded, namely $\|w_m\| \leq M$

Sublemma: any convergent sequence is bounded.

Proof of Lemma 2

$$S_n := \sum_{k=1}^n a_k e_k \quad \hat{S}_n := \sum_{k=1}^n a_k^2$$

For every $m > n$

$$\|S_m - S_n\|^2 \underset{\substack{\uparrow \\ \text{defn. of } S_n}}{=} \left\| \sum_{k=n+1}^m a_k e_k \right\|^2 \underset{\substack{\uparrow \\ \{e_k\} \text{ orthonorm.}}}{=} \sum_{k=n+1}^m a_k^2 \underset{\substack{\uparrow \\ \text{defn. of } \hat{S}_n}}{=} \hat{S}_m - \hat{S}_n$$

$$\sum_{n=1}^{\infty} a_n e_n \text{ conv.} \Leftrightarrow \{S_n\} \text{ is a Cauchy seq. in } H$$

$$\Leftrightarrow \{\hat{S}_n\} \text{ is a Cauchy seq. in } \mathbb{R}$$

$$\Leftrightarrow \sum_{n=1}^{\infty} a_n^2 \text{ converges.}$$

Proof of Lemma 4

$$\langle v, e_k \rangle = \lim_{n \rightarrow +\infty} \underbrace{\langle S_n, e_k \rangle}_{= a_k \text{ for every } n \geq k} = a_k$$

$$\|v\|^2 = \lim_{n \rightarrow \infty} \|S_n\|^2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k^2 = \sum_{k=1}^{\infty} a_k^2.$$

Proof of Lemma 5 Let us fix $\varepsilon > 0$.

There exists $v_\varepsilon \in \text{Span}\{e_n\}$ s.t. $\|v - v_\varepsilon\|^2 \leq \varepsilon$
(because Span is dense).

$$\begin{aligned} \varepsilon &\geq \|v - v_\varepsilon\|^2 = \|v\|^2 + \|v_\varepsilon\|^2 - 2 \underbrace{\langle v, v_\varepsilon \rangle}_{\substack{= \\ 0}} \\ &\quad \text{because } v_\varepsilon \text{ is a FINITE} \\ &\quad \text{lin. comb. of } e_n\text{'s} \\ &\geq \|v\|^2 \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\|v\| = 0$, and hence $v = 0$.

(we need $\langle \cdot, \cdot \rangle$ to be pos. def.)

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LECTURE 10

Note Title

06/10/2021

Proof of Lemma 3 For every n let us write

$$v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n + r_n$$

$\uparrow$
remainder

claim: $\langle r_n, e_k \rangle = 0$ for $k = 1, \dots, n$

$$r_n = v - \langle v, e_1 \rangle e_1 - \dots - \langle v, e_n \rangle e_n$$

$$\begin{aligned} \langle r_n, e_k \rangle &= \langle v, e_k \rangle - \langle v, e_1 \rangle \langle e_1, e_k \rangle - \dots - \langle v, e_n \rangle \langle e_n, e_k \rangle \\ &= \langle v, e_k \rangle - \langle v, e_k \rangle = 0 \end{aligned}$$

Therefore

$$\begin{aligned} \|v\|^2 &= \langle v, e_1 \rangle^2 + \dots + \langle v, e_n \rangle^2 + \|r_n\|^2 \\ &\geq \sum_{k=1}^n \langle v, e_k \rangle^2 \end{aligned}$$

Since this is true for every $n \in \mathbb{N}$, the conclusion follows.

— 0 — 0 —

Proof of Theorem 2 The key point is proving that

$$v = \sum_{n=1}^{\infty} \langle v, e_n \rangle e_n$$

Convergence of the series. By Lemma 2

$$\sum_{n=1}^{\infty} \langle v, e_n \rangle e_n \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} \langle v, e_n \rangle^2 < +\infty, \text{ and this}$$

is true because of Lemma 3.

Let $\hat{v}$ be the sum of the series. By Lemma 4 we know that

$$\langle \hat{v}, e_n \rangle = \langle v, e_n \rangle \quad \forall n \geq 1 \quad \text{and therefore}$$

$$\langle \hat{v} - v, e_n \rangle = 0 \quad \forall n \geq 1.$$

Conclusion by Lemma 5.

Achtung! When preparing this exam, BEWARE of Thm 2.

Proof of Theorem 1

General case It is enough to consider a maximal orthonormal subset (Zorn lemma).

Separable case Let $\{v_n\} \subseteq H$ be a dense subset.

Then there exists a subsequence $\{v_{n_k}\}$ such that

$$\dim \text{Span} \{v_{n_1}, \dots, v_{n_k}\} = k \quad (\text{easy induction})$$

" $\text{Span} \{v_1, v_2, \dots, v_{n_k}\}$ [Added after video]

One can prove that $\text{Span} \{v_n\} = \text{Span} \{v_{n_k}\}$

Apply Gram-Schmidt process to $\{v_{n_k}\}$.

↑ same procedure as in $\mathbb{R}^d$.

Remark The process stops on the road if the algebraic dimension of H is finite.

— o — o —

WEAK CONVERGENCE

Def. Let H be a Hilbert space, and let $\{u_n\} \subseteq H$ be a sequence, and let $u_\infty \in H$.

We say that u_n converges weakly to u_∞ , and we write

$$u_n \rightharpoonup u_\infty$$

$$\text{if } \langle u_n, v \rangle \xrightarrow{\substack{\uparrow \\ \text{conv. of numbers}}} \langle u_\infty, v \rangle \quad \forall v \in H$$

Trivial properties

① Uniqueness of the limit

Assume $u_n \rightharpoonup u_\infty$ and $u_n \rightharpoonup \hat{u}_\infty$. Then

$$\langle u_\infty, v \rangle = \lim_{n \rightarrow \infty} \langle u_n, v \rangle = \langle \hat{u}_\infty, v \rangle \quad \forall v \in V$$

$$\leadsto \langle u_\infty - \hat{u}_\infty, v \rangle = 0 \quad \forall v \in V \quad \text{Use } v = u_\infty - \hat{u}_\infty \quad \ddot{\smile}$$

② Stability by subsequences : if $u_n \rightharpoonup u_\infty$, then $u_{n_k} \rightharpoonup u_\infty$

③ Linearity : if $u_n \rightharpoonup u_\infty$ and $w_n \rightharpoonup w_\infty$, then

$$\alpha u_n + \beta w_n \rightharpoonup \alpha u_\infty + \beta w_\infty \quad \forall (\alpha, \beta) \in \mathbb{R}^2$$

④ Strong implies weak : if $u_n \rightarrow u_\infty$, then $u_n \rightharpoonup u_\infty$

(continuity of scalar prod.)

⑤ If $\dim H < +\infty$ then $u_n \rightarrow u_\infty \Leftrightarrow u_n \rightarrow u_\infty$

Let $\{e_1, \dots, e_d\}$ be a orthonormal basis. Then

$$u_n \rightarrow u_\infty \Rightarrow \langle u_n, e_i \rangle \rightarrow \langle u_\infty, e_i \rangle \quad \forall i \in \{1, \dots, d\}$$

And in finite dim conv. of components is equiv. to conv. of vectors

$$u_n = \sum_{i=1}^d \langle u_n, e_i \rangle e_i \xrightarrow{n \rightarrow +\infty} \sum_{i=1}^d \langle u_\infty, e_i \rangle e_i = u_\infty$$

Achtung! The limit commutes with finite sums, not with series.

⑥ If $\dim H = +\infty$, then there are sequences that converge weakly but not strongly.

Example Let $\{e_n\}_{n \in \mathbb{N}}$ be a orthonormal system (it always exists if $\dim H = +\infty$)

Then

$$e_n \rightarrow 0$$

but of course $e_n \not\rightarrow 0$ ($\|e_n - 0\| = 1$)

In order to prove weak convergence we need to show that

$$\langle e_n, v \rangle \rightarrow \langle 0, v \rangle = 0 \quad \forall v \in H$$

This is true because $\sum_{n=1}^{\infty} \langle e_n, v \rangle^2 < +\infty$ (Bessel)

$$\text{Conv. series} \Rightarrow \langle e_n, v \rangle \rightarrow 0.$$

BIG ISSUE If $\dim H = +\infty$, then closed balls are NOT STRONGLY COMPACT

$\{e_n\} \subseteq \text{Ball of radius } 1$, but it admits no conv. subseq.

$$\|e_n - e_m\|^2 = \|e_n\|^2 + \|e_m\|^2 = 2 \quad \forall m \neq n$$

So $\{e_n\}$ does NOT ADMIT Cauchy subsequences.

— o — o —

In $\mathbb{R}^d$ convergence of vectors is equiv. to conv. of components

In a general Hilbert space

→ convergence of vectors is strong conv.

→ " of components is weak conv.

Prop. Let H be a Hilbert space

Let $\{e_n\}_{n \geq 1}$ be a Hilbert basis

Let $\{v_n\} \subseteq H$ be a sequence, and let $v_\infty \in H$

Let us assume that $\{v_n\}$ is bounded, namely

$$\exists M \in \mathbb{R} : \|v_n\| \leq M \quad \forall n \geq 1$$

Then $v_n \rightarrow v_\infty$ if and only if

$$\langle v_n, e_k \rangle \rightarrow \langle v_\infty, e_k \rangle \quad \forall k \geq 1.$$

Proof The nontrivial implication is $\Leftarrow$

Idea: $|\langle v_\infty, v \rangle - \langle v_m, v \rangle| \leq |\langle v_\infty, v \rangle - \langle v_\infty, v_\varepsilon \rangle|$ (1)

$$+ |\langle v_\infty, v_\varepsilon \rangle - \langle v_m, v_\varepsilon \rangle|$$
 (2)

$$+ |\langle v_m, v_\varepsilon \rangle - \langle v_m, v \rangle|$$
 (3)

$$(1) \leq \|v_\infty\| \cdot \|v - v_\varepsilon\| \leadsto \text{small if } v_\varepsilon \sim v$$

$$(2) \text{ is small if } m \text{ is large and } v_\varepsilon \in \text{Span}\{e_k\}$$

$$(3) \leq \|v_m\| \cdot \|v - v_\varepsilon\| \leadsto \text{small if } v_\varepsilon \sim v$$

$$\leq M$$

Formally

→ let $\varepsilon > 0$ be fixed

→ Choose $v_\varepsilon \in \text{Span}\{e_k\}$ (FINITE LIN COMB) s.t.

$$\|v_\infty\| \cdot \|v - v_\varepsilon\| \leq \frac{\varepsilon}{4}$$

$$M \cdot \|v - v_\varepsilon\| \leq \frac{\varepsilon}{4}$$

→ After choosing v_ε , choose m_0 s.t.

$$|\langle v_\infty, v_\varepsilon \rangle - \langle v_m, v_\varepsilon \rangle| \leq \frac{\varepsilon}{4} \quad \forall m \geq m_0$$

(this is possible because v_ε is FINITE lin. comb. of e_k 's)

→ As before conclude that

$$|\langle v_m, v \rangle - \langle v_\infty, v \rangle| \leq \frac{3}{4} \varepsilon \quad \forall m \geq m_0$$

This implies that $\langle v_m, v \rangle \rightarrow \langle v_\infty, v \rangle$.

It is essential that we assume $\{v_m\}$ to be bounded.

just consider $v_m = m e_1$. Claim (exercise)

$$\langle v_m, e_k \rangle \rightarrow 0 \quad \forall k \geq 1 \quad \text{but} \quad v_m \not\rightarrow 0.$$

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LECTURE 11

Note Title

07/10/2021

Prop. (weak conv. and scalar product)Let H be a Hilbert space. Let us assume that(i) $u_n \rightarrow u_\infty$ strongly(ii) $w_n \rightarrow w_\infty$ weakly(iii) $\{w_n\}$ is bounded (one day this will follow from (ii))

Then

$$\lim_{n \rightarrow \infty} \langle u_n, w_n \rangle = \langle u_\infty, w_\infty \rangle.$$

Proof

$$| \langle u_\infty, w_\infty \rangle - \langle u_n, w_n \rangle | \quad \begin{array}{c} \langle u_n, w_\infty - w_n \rangle \\ \parallel \\ \langle u_n, w_\infty \rangle - \langle u_n, w_n \rangle \end{array}$$

$$[\leq | \langle u_\infty, w_\infty \rangle - \langle u_n, w_\infty \rangle | + | \langle u_n, w_\infty \rangle - \langle u_n, w_n \rangle |]$$

(Bad choice: try it!!)

$$\leq \underbrace{| \langle u_\infty, w_\infty \rangle - \langle u_\infty, w_n \rangle |}_{\substack{\uparrow \\ \text{FIXED}}} + \underbrace{| \langle u_\infty, w_n \rangle - \langle u_n, w_n \rangle |}_{\substack{\leq \|w_n\| \cdot \|u_\infty - u_n\| \\ \uparrow \text{c.s.} \\ \leq M \|u_\infty - u_n\| \\ \downarrow 0}}$$

Therefore

$$\lim_{n \rightarrow \infty} | \langle u_\infty, w_\infty \rangle - \langle u_n, w_n \rangle | = 0$$

— 0 — 0 —

THINGS THAT DO NOT WORK IF $\dim H = +\infty$ ① $u_n \rightarrow u_\infty, w_n \rightarrow w_\infty \not\Rightarrow \langle u_n, w_n \rangle \rightarrow \langle u_\infty, w_\infty \rangle$

$$e_n \rightarrow 0, \quad e_n \rightarrow 0 \quad \langle e_n, e_n \rangle = \|e_n\|^2 = 1 \not\rightarrow 0$$

② The norm is NOT cont. wrt weak conv. $u_n \rightarrow u_\infty \not\Rightarrow \|u_n\| \rightarrow \|u_\infty\|$

$$u_n = e_n$$

③ Closed balls are not STRONGLY COMPACT

④ Spheres are NOT WEAKLY CLOSED

↑ $\{u \in H: \|u - u_0\|^2 = R^2\}$ Again $\|e_n\| = 1$ and $e_n \rightarrow 0$.
— o — o —

Theorem 1 (LSC of the norm) If $u_n \rightarrow u_\infty$ weakly in H , then

$$\liminf_{n \rightarrow \infty} \|u_n\| \geq \|u_\infty\|$$

Proof $\|u_n\|^2 = \|(u_n - u_\infty) + u_\infty\|^2$

$$= \|u_n - u_\infty\|^2 + \|u_\infty\|^2 + 2 \langle u_n - u_\infty, u_\infty \rangle$$

$$\geq \|u_\infty\|^2 + \underbrace{2 \langle u_n - u_\infty, u_\infty \rangle}_{\rightarrow 0}$$

Therefore

$$\liminf_{n \rightarrow \infty} \|u_n\|^2 \geq \|u_\infty\|^2$$

— o — o —

Theorem 2 (weak compactness of balls)

Assume that H is separable (actually we DO NOT NEED IT)

Let $\{u_n\} \subseteq H$ be a bounded sequence.

Then there exist an increasing sequence n_k of positive integers and $u_\infty \in H$ such that

$$u_{n_k} \longrightarrow u_\infty \text{ weakly in } H.$$

Proof. Let $M \in \mathbb{R}$ be such that $\|u_n\| \leq M$ for every $n \geq 1$.

Let $\{e_n\}$ be an orthonormal basis (Hilbert basis, complete orthonormal system).

Step 1 Let us consider e_1 . Then

$$|\langle v_m, e_1 \rangle| \leq \|v_m\| \leq M$$

Therefore $\{\langle v_m, e_1 \rangle\}$ admits a subsequence converging to some a_1 .
 $\uparrow$ sequence of numbers

Let us call N_1 the indices of the subseq.

Let us consider e_2 . Then

$$|\langle v_m, e_2 \rangle| \leq M \quad \forall m \in N_1$$

Therefore the PREVIOUS subsequence has a further subseq. that converges to some a_2 . Let $N_2 \subseteq N_1$ be the indices.

In this way we find $N \supseteq N_1 \supseteq N_2 \supseteq N_3 \supseteq \dots$

Key idea (diagonal argument): there exists an increasing sequence $\{m_k\}$ such that $m_k \in N_k$ for every $k \geq 1$.

(can be constructed by induction)

Claim:

$$\lim_{k \rightarrow \infty} \langle v_{m_k}, e_i \rangle = a_i \quad \forall i \geq 1$$

This is true because $m_k \in N_i \quad \forall k \geq i$

Step 2 The vector

$$v_\infty := \sum_{i=1}^{\infty} a_i e_i$$

is well defined. We need that $\sum_{i=1}^{\infty} a_i^2 < +\infty$

For every $m \in \mathbb{N}$ it is true that

$$\sum_{i=1}^m a_i^2 = \sum_{i=1}^m \lim_{k \rightarrow \infty} \langle v_{m_k}, e_i \rangle^2 \stackrel{\text{FINITE SUM}}{=} \lim_{k \rightarrow \infty} \sum_{i=1}^m \langle v_{m_k}, e_i \rangle^2$$

$$\leq \limsup_{k \rightarrow +\infty} \sum_{i=1}^{\infty} \langle u_{m_k}, e_i \rangle^2$$

$$= \limsup_{k \rightarrow +\infty} \|u_{m_k}\|^2 \leq \eta^2$$

Since m is arbitrary, the series converges.

Step 3 We have shown that

$$\lim_{k \rightarrow +\infty} \langle u_{m_k}, e_i \rangle = a_i = \langle u_{\infty}, e_i \rangle \quad \forall i \geq 1$$

This is enough to show that $u_{m_k} \rightarrow u_{\infty}$
 — o — o —

More general statement let $\{u_m\} \subseteq H$ be a bounded sequence.

Let us assume that

$$\langle u_m, v \rangle \rightarrow \langle u_{\infty}, v \rangle \quad \forall v \in W \subseteq H$$

$\uparrow$ subset

Then we can conclude that $u_m \rightarrow u_{\infty}$ provided that

$$\text{Clos}(\text{Span}(W)) = H$$

Proof Same as in the case where $W = \{e_n\}_{n \geq 1}$.
 — o — o —

Prop. (Normal convergence in Banach spaces)

Let V be a Banach space (complete normed vector space).

Let $\{u_m\} \subseteq V$ be a sequence.

Then

$$\sum_{n=1}^{\infty} \|u_n\| < +\infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} u_n \text{ converges in } V$$

Proof $S_m := v_1 + \dots + v_m$ $\hat{S}_m := \|v_1\| + \dots + \|v_m\|$

for every $m > n$ we have

$$\|S_m - S_n\| = \left\| \sum_{k=n+1}^m v_k \right\| \leq \sum_{k=n+1}^m \|v_k\| = \hat{S}_m - \hat{S}_n$$

$$\sum_{n=1}^{\infty} \|v_n\| < +\infty \iff \{\hat{S}_n\} \text{ is a Cauchy seq. in } \mathbb{R}$$

$$\implies \{S_n\} \text{ is a Cauchy seq. in } V$$

$$\iff \sum_{n=1}^{\infty} v_n \text{ converges in } V.$$

— o — o —

Istituzioni di Analisi

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LEZIONE 12

Note Title

07/10/2021

 ℓ^2 and L^2 spaces

$$\ell^2 = \{ \text{sequences } \{a_n\}_{n=1}^\infty \subseteq \mathbb{R} \text{ such that } \sum_{n=1}^\infty a_n^2 < +\infty \}$$

$$\Omega \subseteq \mathbb{R}^d \text{ is measurable, then } L^2(\Omega) = \{ f: \Omega \rightarrow \mathbb{R} \text{ s.t. } \int_{\Omega} f(x)^2 dx < +\infty \}$$

More generally we can define

$$L^2(X, \mathcal{M}, \mu)$$

$\uparrow$ set $\uparrow$ σ -algebra $\nwarrow$ measure ($\mu \geq 0$, $+\infty$ is allowed, countably additive)

$$\text{in } \ell^2 : \langle \{a_n\}, \{b_n\} \rangle := \sum_{n=1}^\infty a_n b_n$$

$$\text{in } L^2 : \langle f, g \rangle := \int_{\Omega} f(x) g(x) dx$$

Remark ℓ^2 is equal to $L^2(\mathbb{N})$ with respect to the measure

$$\mu(A) = \text{number of el. of } A \quad \forall A \subseteq \mathbb{N}.$$

If I is an uncountable set, then $L^2(I)$

is a Hilbert space which is NOT

separable

$\uparrow$ with the same measure as above

(Exercise: provide a Hilbert basis in this case)

Remark $|f(x)g(x)| \leq \frac{1}{2}|f(x)|^2 + \frac{1}{2}|g(x)|^2$ and this implies that $\langle f, g \rangle$ is well-defined

How to prove completeness of L^2

Let $\{f_n\} \subseteq L^2$ be a Cauchy sequence.

Step 1 Enough to prove existence of conv. subseq.

Step 2 (Acceleration Lemma) There exists a subseq (not relabeled) such that

$$\|f_{n+1} - f_n\| \leq \frac{1}{2^n}$$

Step 3 Observe that $f_n = f_1 + (f_2 - f_1) + (f_3 - f_2) + \dots + (f_n - f_{n-1})$ and therefore $\{f_n\}$ converges if and only if

$$\sum_{n=1}^{\infty} (f_{n+1} - f_n) \text{ converges}$$

DO NOT EVEN THINK OF USING THE LAST RESULT OF LECT. 11

Step 4 The functions

$$\hat{S}_n(x) := \sum_{k=1}^n |f_{k+1}(x) - f_k(x)|$$

are monotone w.r.t n and equi-bounded in L^2 (use acceleration)

Step 5 $\{\hat{S}_n(x)\}$ converges in the pointwise sense and the limit is in L^2 and dominated by a L^2 function (Beppo-Levi)

Step 6 Since the series with $|\cdot|$ converges, the series without absolute values converges.

— o — o —

Proof that ℓ^2 is complete I denote elements of ℓ^2 as functions $f: \mathbb{N} \rightarrow \mathbb{R}$.

Let $\{f_n(i)\}$ be a Cauchy sequence in ℓ^2 .

Claim $f_n(i) \rightarrow f_\infty(i)$ in ℓ^2 .

Step 1 There exists $f_\infty: \mathbb{N} \rightarrow \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} f_n(i) = f_\infty(i) \quad \forall i \in \mathbb{N}$$

For every $m > n$ and every $i \geq 1$ it turns out that

$$|f_m(i) - f_n(i)|^2 \leq \sum_{k=1}^{\infty} |f_m(k) - f_n(k)|^2 = \|f_m - f_n\|_{\ell^2}^2$$

$\Rightarrow \{f_n(i)\}$ is a Cauchy sequence of number for every fixed $i \in \mathbb{N}$

$\Rightarrow$ It has a limit which we call $f_\infty(i)$.

Step 2 We prove that $f_\infty \in \ell^2$.

For every $n \geq 1$:

$$\sum_{i=1}^n |f_\infty(i)|^2 = \sum_{i=1}^n \lim_{n \rightarrow \infty} |f_n(i)|^2$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n |f_n(i)|^2$$

$$\leq \limsup_{n \rightarrow \infty} \|f_n\|_{\ell^2}^2$$

$$\leq M \quad (\text{general fact: Cauchy sequences are bounded})$$

Step 3 (Uniform smallness of tails)

$$\forall \varepsilon > 0 \exists m \in \mathbb{N} \quad \forall n \in \mathbb{N} \quad \sum_{i=m+1}^{+\infty} |f_n(i)|^2 \leq \varepsilon$$

Since $\{f_n\}$ is a Cauchy sequence, there exists $m_0 \in \mathbb{N}$ s.t.

$$\|f_m - f_n\|_{\ell^2}^2 \leq \frac{1}{4} \varepsilon \quad \forall m \geq m_0 \quad \forall n \geq m_0$$

Let $m \in \mathbb{N}$ be such that

$$\sum_{i=m+1}^{+\infty} |f_m(i)|^2 \leq \frac{1}{4} \varepsilon \quad \forall m \in \underbrace{\{1, \dots, m_0\}}_{\text{only a FINITE number}}$$

Claim: this m is good for every $n \in \mathbb{N}$. Let $n > m_0$, then

$$\begin{aligned} \sum_{i=m+1}^{+\infty} |f_n(i)|^2 &= \sum_{i=m+1}^{+\infty} |f_n(i) - f_{m_0}(i) + f_{m_0}(i)|^2 \\ &\leq 2 \sum_{i=m+1}^{+\infty} |f_n(i) - f_{m_0}(i)|^2 + 2 \underbrace{\sum_{i=m+1}^{+\infty} |f_{m_0}(i)|^2}_{\leq \frac{1}{4} \varepsilon} \\ &\leq 2 \|f_n - f_{m_0}\|_{\ell^2}^2 \\ &\leq 2 \cdot \frac{1}{4} \varepsilon \\ &= \varepsilon. \end{aligned}$$

Step 3.5 The same holds true for f_∞ , namely

$$\sum_{i=m+1}^{+\infty} |f_\infty(i)|^2 \leq \varepsilon$$

$\uparrow$
 same m as before

Same proof of step 2.

Step 4 $f_n \rightarrow f_\infty$ in the L^2 sense

$$\begin{aligned} \|f_n - f_\infty\|^2 &= \sum_{i=1}^{\infty} |f_n(i) - f_\infty(i)|^2 \\ &= \underbrace{\sum_{i=1}^m |f_n(i) - f_\infty(i)|^2}_{\rightarrow 0 \text{ as } n \rightarrow +\infty \text{ because of pointwise convergence}} + \sum_{i=m+1}^{+\infty} |f_n(i) - f_\infty(i)|^2 \end{aligned}$$

$$\sum_{i=m+1}^{+\infty} |f_n(i) - f_\infty(i)|^2 \leq 2 \sum_{i=m+1}^{+\infty} |f_n(i)|^2 + 2 \sum_{i=m+1}^{\infty} |f_\infty(i)|^2$$

Usual conclusion

→ Fix $\varepsilon > 0$

→ Choose m is such a way that last 2 terms $\leq \frac{1}{2} \varepsilon$ for every n

→ Choose n_0 such that first term $\leq \frac{1}{2} \varepsilon$ for every $n \geq n_0$.

— 0 — 0 —

Example Consider $H = L^2((0, 2\pi))$

Claim: the sequence $\{\sin(mx)\}$ converges weakly to 0.

It is enough to prove that $\int_0^{2\pi} f(x) \sin(mx) dx \rightarrow 0$

$\forall f \in W \leftarrow$ span a dense subset of $L^2((0, 2\pi))$.

$W =$ charact. funct. of intervals !!! $\int_a^b \sin(mx) dx$

$$= -\frac{1}{m} (\cos(mb) - \cos(ma)) \rightarrow 0.$$

Istituzioni di Analisi

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LECTURE 13

Note Title

11/10/2021

Weak derivatives and Sobolev spacesBasic case

- dim 1
- open set $(a,b) \subseteq \mathbb{R}$
- deriv. of order 1

General case

- dim $d \geq 1$
- any open set $\Omega \subseteq \mathbb{R}^d$
- deriv. of any order

Two main approaches

- definition W → weak (good for passing to the limit)
- definition H (good for "formulae" and algebraic things)

Weak derivatives - definition W

Let u and v be in $L^1_{loc}((a,b))$. We say that v is W-weak deriv. of u if

$$\int_a^b u(x) \varphi'(x) dx = - \int_a^b v(x) \varphi(x) dx$$

$\forall \varphi \in C_c^\infty((a,b))$
 $\uparrow$ test function

Remark The weak deriv. of u plays the role of u' in the integration by parts.

Basic properties

① Uniqueness Assume v_1 and v_2 a w.d. of μ . Then

$$\int_a^b v_1(x) \varphi(x) dx = - \int_a^b \mu(x) \varphi'(x) dx = \int_a^b v_2(x) \varphi(x) dx$$

$$\Rightarrow \int_a^b (v_1(x) - v_2(x)) \varphi(x) dx = 0 \quad \text{FLCV} \Rightarrow v_1(x) = v_2(x) \text{ almost everywhere}$$

② Linearity $(\alpha \mu_1 + \beta \mu_2)' = \alpha v_1 + \beta v_2$
if v_1 is w.d. of μ_1 and v_2 is w.d. of μ_2

③ Trivial classical compatibility If $\mu \in C^1((a,b))$, then $\mu'(x)$ is the w-weak deriv.

④ less trivial classical comp. Assume that $\mu \in L^1_{loc}$ admits a weak der. $v \in C^0((a,b))$
Then $\mu \in C^1((a,b))$

Proof. Let $U: (a,b) \rightarrow \mathbb{R}$ be such that $U'(x) = v(x)$
Then

$$- \int_a^b U(x) \varphi'(x) dx = \int_a^b v(x) \varphi(x) dx = - \int_a^b \mu(x) \varphi'(x) dx$$

$\uparrow$ classical int. by part. $\uparrow$ v is w.d. of μ

$$\Rightarrow \int_a^b (U(x) - \mu(x)) \varphi'(x) dx = 0 \quad \forall \varphi \in C_c^\infty((a,b))$$

(DBR)

$$\Rightarrow U(x) - \mu(x) \equiv \text{constant}$$

$$\Rightarrow \mu \in C^1$$

(coincides a.e. with C^1 function)

Example 1 Assume $c \in (a, b)$, $u_1 \in C^1([a, c])$, $u_2 \in C^1([c, b])$

Assume that $u_1(c) = u_2(c)$. Then

$$u(x) := \begin{cases} u_1(x) & x \in [a, c] \\ u_2(x) & x \in [c, b] \end{cases}$$

satisfies the w-weak deriv. that we expect.

$$\begin{aligned} \int_a^b u(x) \varphi'(x) dx &= \int_a^c u_1(x) \varphi'(x) dx + \int_c^b u_2(x) \varphi'(x) dx \\ &= \cancel{u_1(c)\varphi(c)} - \int_a^c u_1'(x) \varphi(x) dx \\ &\quad - \cancel{u_2(c)\varphi(c)} - \int_c^b u_2'(x) \varphi(x) dx \end{aligned}$$

Example 2 The Heaviside function

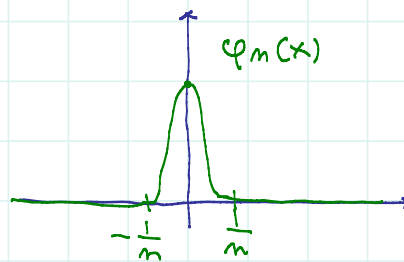
$$u(x) = \begin{cases} 0 & \text{if } x \in (-1, 0) \\ 1 & \text{if } x \in (0, 1) \end{cases}$$

does not admit a weak deriv. in $L^1_{loc}((a, b))$

Proof $\int_{-1}^1 u(x) \varphi'(x) dx = \int_0^1 \varphi'(x) dx = -\varphi(0) \quad \forall \varphi \in C_c^\infty((-1, 1))$

If v is the w.d. of u then

$$\boxed{\int_{-1}^1 v(x) \varphi(x) dx = \varphi(0)}$$



If we apply with $\varphi_n(x)$ as above we see that

$$\text{RHS} \equiv 1$$

LHS $\rightarrow 0$ by dominated convergence
(pointwise conv. to 0 + domination by $|v(x)|$)

Stability when passing to the limit

Let $\{u_m\}$ and $\{v_m\}$ be two seq. in $L^1_{loc}(a,b)$.

Assume that

(i) v_m is the w weak deriv. of u_m for every $m \in \mathbb{N}$

(ii) $u_m \rightarrow u_\infty$ in $L^1_{loc}(a,b)$

(iii) $v_m \rightarrow v_\infty$ in $L^1_{loc}(a,b)$ } NOT REALLY NECESSARY

Then v_∞ is the w.d. of u_∞ .

Proof. For every m and every $\varphi \in C_c^\infty(a,b)$:

$$\int_a^b u_m(x) \varphi'(x) dx = - \int_a^b v_m(x) \varphi(x) dx$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\int_a^b u_\infty(x) \varphi'(x) dx = - \int_a^b v_\infty(x) \varphi(x) dx \quad \text{☺}$$

Remark Weak convergence in $L^2(a,b)$ or even in $L^p(a,b)$ is enough for the previous result.

Theorem Assume that $u \in L^1_{loc}(a,b)$ admits a weak deriv. $v \in L^1_{loc}(a,b)$.

Then for every $x_0 \in (a,b)$ it turns out that

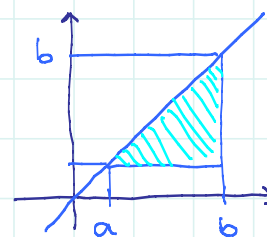
$$u(x) = u(x_0) + \int_{x_0}^x v(t) dt \quad \forall x \in (a,b)$$

Proof Let us assume for simplicity that $v \in L^1(a,b)$

$$U(x) = \int_a^x v(t) dt$$

$$\begin{aligned}
 \int_a^b U(x) \varphi'(x) dx &= \int_a^b \int_a^x v(t) dt \varphi'(x) dx \\
 &= \int_a^b \int_a^x v(t) \varphi'(x) dt dx \\
 &\quad \text{over } T \\
 &= \int_a^b dt \int_t^b v(t) \varphi'(x) dx \\
 &= \int_a^b v(t) dt \int_t^b \varphi'(x) dx \\
 &= - \int_a^b v(t) \varphi(t) dt \\
 &= - \int_a^b v(x) \varphi(x) dx \\
 &\stackrel{\nearrow}{=} \int_a^b u(x) \varphi'(x) dx
 \end{aligned}$$

u is
weak deriv.



$$\Rightarrow \int_a^b (U(x) - u(x)) \varphi'(x) dx = 0 \quad \text{DBR} \Rightarrow U(x) = u(x) + \text{const.}$$

If $v \in L^1(a,b)$ we have to set $U(x) = \int_{x_0}^x v(t) dt$

and distinguish two triangles

— o — o —

Reek This proves that $u(x)$ "is" continuous.

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LECTURE 14

Note Title

11/10/2021

Weak derivatives - Definition 1

Defn Let u and v be in $L^1_{loc}((a,b))$. We say that v is the H -weak deriv. of u if there exists a sequence $\{u_n\} \subseteq C^\infty((a,b))$ such that (asking $u_n \in C^1(a,b)$ is the same)
 $\uparrow$
 no cpt. supp. requirement

$$u_n \rightarrow u \quad \text{and} \quad u'_n \rightarrow v \quad \text{in } L^1_{loc}((a,b))$$

$\uparrow$
classical

Exercise Prove uniqueness of the H -weak derivative.

Prop. ($H \Rightarrow W$) The H -deriv. is also a W -deriv.

Proof

$$\int_a^b u_n(x) \varphi'(x) dx = - \int_a^b u'_n(x) \varphi(x) dx$$

$$\downarrow$$

$$\int_a^b u(x) \varphi'(x) dx = - \int_a^b v(x) \varphi(x) dx \quad \square$$

SOBOLEV SPACES $p \in [1, +\infty]$ (endpoints included)

Defn $W^{1,p}((a,b)) = \{ f \in L^p((a,b)) : f' \in L^p((a,b)) \}$
 $\uparrow$
 W weak deriv. of f

$H^{1,p}((a,b)) = \dots$ the same with $f' = H$ weak deriv.

Remark By the previous result $H^{1,p}((a,b)) \subseteq W^{1,p}((a,b))$

Regularity of $W^{1,p}$ functions

Theorem (Hölder continuity) Assume that $p > 1$ and let q denote the conjugate exponent $\frac{1}{p} + \frac{1}{q} = 1$.

Then any function $u \in W^{1,p}((a,b))$ is $\frac{1}{q}$ -Hölder continuous.
(lip. cont. if $p = +\infty$).

Proof Let v denote the w.d. of u . Then for every $a \leq x < y \leq b$:

$$|u(y) - u(x)| = \left| \int_x^y v(t) dt \right| = \left| \int_a^b v(t) \underbrace{\mathbb{1}_{(x,y)}}_{\frac{1}{p}}(t) dt \right|$$

$$\leq \underbrace{\left\{ \int_a^b |v(t)|^p dt \right\}^{1/p}}_{\text{number}} \cdot |y-x|^{1/q}$$

↑
Hölder inequality

The case $p = +\infty$ is even simpler.

Important remark If $u \in W^{1,p}((a,b))$ (without "loc") then " u is defined and continuous up to the boundary" and therefore it makes sense to consider pointwise values

$$u(x) \quad \forall x \in [a,b]$$

Remark In the case $p=1$ we have in general no Hölder cont., but continuity up to the boundary, and therefore uniform continuity.

Theorem $W^{1,p}((a,b)) \subseteq H^{1,p}((a,b))$ (and therefore they are equal)

and W -weak deriv. are H -weak derivatives

TRUE ONLY
IF $p < +\infty$

Proof Let v be the W -derivative of u .

Idea: approximate v by a sequence $\{v_m\} \subseteq C^\infty((a,b))$.

Define, for some $x_0 \in (a,b)$,

↑ HERE WE NEED
 $p < +\infty$

$$u_m(x) := u(x_0) + \int_{x_0}^x v_m(t) dt \quad \forall x \in [a,b]$$

↑
pointwise values
are well defined

It is trivial that $u_m \in C^\infty((a,b))$, and that we can choose $v_m \rightarrow v$ in the same space where v lies

We need to prove that

$u_m \rightarrow u$ in the sense we need ($L^1_{loc}((a,b))$ or $L^p((a,b))$).

The idea is that convergence is better

$$|u(x) - u_m(x)| = \left| \int_{x_0}^x (v(t) - v_m(t)) dt \right| \leq \int_{x_0}^x |v(t) - v_m(t)| dt$$

↑
at least if $x \geq x_0$

Depending on the assumptions on v , we can conclude that $u_m(x) \rightarrow u(x)$ (if $v_m \rightarrow v$ in L^p , we need Hölder inequality)

If there is no "loc", the convergence is even uniform.

Remark $H^{1,\infty}((a,b)) \subsetneq W^{1,\infty}((a,b))$

↑
 $C^1((a,b))$ with

↑ equal to the set of
Lipschitz functions

u and u' bounded
(L^∞ limit is uniform limit
and unif. limits preserve continuity)

Integration by parts

Assume that u and v are in $W^{1,1}(a,b)$
 (note that $W^{1,p}(a,b) \subseteq W^{1,1}(a,b)$)

Then

$$\int_a^b u(x) v'(x) dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x) v(x) dx$$

where u' and v' are the weak derivatives.

Proof Use H-approx.

$$\begin{array}{ll} u_n \xrightarrow{L^1} u & u_n'(x) \xrightarrow{L^1} u'(x) \\ v_n \xrightarrow{L^1} v & v_n'(x) \xrightarrow{L^1} v'(x) \end{array}$$

actually
uniform, and this
is enough

— o — o —

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LECTURE 15

Note Title

13/10/2021

ROAD MAP OF THE DIRECT METHODSetting: S is a set, and $F: S \rightarrow \mathbb{R}$ is a functionGoal: Proving that

$$\min \{ F(s) : s \in S \}$$

exists.

- Road map
- ① Weak formulation
 - ② Compactness
 - ③ Lower semicontinuity
 - ④ Regularity

Weak formulation We introduce a set $\hat{S} \supseteq S$ and

$$\hat{F}: \hat{S} \rightarrow \mathbb{R} \cup \{+\infty\}$$

↖ let us allow it, it is our friend

such that $\hat{F}(s) = F(s) \quad \forall s \in S$

$\leq \leftarrow$ this is enough

Compactness We look for a notion of convergence in $\hat{S}$ with the following property:

if $\{s_n\} \subseteq \hat{S}$ and $\hat{F}(s_n) \leq M$, then there exists $s_{n_k} \rightarrow s_\infty$

LSC We hope that $\hat{F}$ is LSC w.r.t. the same notion of conv.
 more precisely, if $s_n \rightarrow s_\infty$ in $\hat{S}$ and $\hat{F}(s_n) \leq M$, then

$$\liminf_{n \rightarrow +\infty} \hat{F}(s_n) \geq \hat{F}(s_\infty)$$

At this point we already know that

$$\min \{ \hat{F}(s) : s \in \hat{\mathcal{S}} \}$$

exists (Weierstrass theorem) [De Giorgi : every pbm. has a solution, if we relax enough the notion of solution].

Regularity Let $s_0 \in \hat{\mathcal{S}}$ be a min. point. We want to prove that $s_0 \in \mathcal{S}$ and $\hat{F}(s_0) = F(s_0)$.

If we can do this, then s_0 is a minimizer for the original pbm

$$F(s_0) = \hat{F}(s_0) \leq \hat{F}(s) \leq F(s) \quad \forall s \in \mathcal{S}$$

— o — o —

Model case $\mathcal{S} := C^1([a, b])$ with $u(a) = 7$ and $u(b) = 13$

$$F(u) := \int_a^b [u'(x)^2 + \cos(u(x)^3)] dx \quad \forall u \in \mathcal{S}$$

1 Weak formulation We consider [Notion : $H^1 = H^{1,2} = W^{1,2}$]

$$\hat{\mathcal{S}} := \{ u \in H^1(a, b) : u(a) = 7, u(b) = 13 \}$$

$$\hat{F}(u) = \int_a^b [\overset{\uparrow \text{weak derivative}}{u'(x)^2} + \cos(u(x)^3)] dx \quad \forall u \in \hat{\mathcal{S}}$$

Remarks :

- boundary values of $u \in H^1$ are well defined
- also $\cos(u(x)^3)$ is integrable because u is cont.
- in this case $\hat{F}$ has real values

2 Compactness let $\{u_n\} \subseteq \hat{\mathcal{S}}$ be such that $\hat{F}(u_n) \leq M$

$$M \geq \int_a^b [u_n(x)^2 + \cos(u_n(x)^3)] dx$$

$$\geq \int_a^b [u_n(x)^2] dx - (b-a)$$

$$\leadsto \int_a^b u_n(x)^2 dx \leq M + (b-a)$$

Weak compactness of balls $\Rightarrow \exists u_{n_k}(x) \xrightarrow{\uparrow} v_\infty(x)$
weak convergence in $L^2(a,b)$

We recall that $u_n(x)$ are $\frac{1}{2}$ -Hölder continuous, and in particular

$$\begin{aligned} |u_n(x) - u_n(a)| &\leq \int_a^x |u_n(t)| dt \\ &\leq \int_a^b |u_n(t)| dt \\ &\leq |b-a|^{1/2} \left\{ \int_a^b |u_n(t)|^2 dt \right\}^{1/2} \\ &\leq |b-a|^{1/2} \cdot M' \end{aligned}$$

In particular

$$|u_n(x)| \leq |u_n(a)| + |b-a|^{1/2} \cdot M' \quad \forall x \in [a,b]$$

$\uparrow$ Fixed because of DBC $\uparrow$ number

$$\leadsto |u_n(x)| \leq M'' \quad \forall x \in [a,b] \text{ and for every } n.$$

So we have

- a uniform bound on $|u_n(x)|$
- a uniform bound on the $\frac{1}{2}$ -Hölder constant of u_n

Ascoli - Arzelà thm $\Rightarrow \{u_n(x)\}$ has a subsequence that converges uniformly in $[a, b]$.

Important point: if we apply Ascoli - Arzelà to $\{u_{n_{k_i}}(x)\}$ that we found before, we obtain a further subsequence $\{u_{m_{k_i}}(x)\}$ such that

$$u'_{m_{k_i}}(x) \rightarrow v_\infty(x) \quad \text{weak in } L^2([a, b])$$

$$u_{m_{k_i}}(x) \rightarrow u_\infty(x) \quad \text{uniform in } [a, b]$$

Fortunately $v_\infty(x) = u'_\infty(x)$ weak derivative

$$\int_a^b u_{m_{k_i}}(x) \varphi'(x) dx = - \int_a^b u'_{m_{k_i}}(x) \varphi(x) dx$$

$$\int_a^b u_\infty(x) \varphi'(x) dx = - \int_a^b v_\infty(x) \varphi(x) dx$$

$$\forall \varphi \in C_c^\infty([a, b])$$

Note that DBC pass to the limit due to unif. conv.

③ LSC Assume that $u_n \rightarrow u_\infty$ unif.
 $u'_n \rightarrow u'_\infty$ weakly

Can we conclude that

$$\liminf_{n \rightarrow \infty} \hat{F}(u_n) \geq F(u_\infty)$$

This follows from

$$\liminf_{n \rightarrow +\infty} \int_a^b u_n(x)^2 dx \geq \int_a^b u_\infty(x)^2 dx \quad \text{LSC of the norm}$$

$$\lim_{n \rightarrow +\infty} \int_a^b \cos(u_n(x)^3) dx = \int_a^b \cos(u_\infty(x)^3) dx \quad \text{unif. conv.}$$

[4] Now we know that a minimizer $u_0(x)$ exists in $H^1(a,b)$

[4.1] weak form of ELE

[4.2] from H^1 to C^1

[4.3] from C^1 to C^∞

[4.1] We consider $\varphi(t) := \hat{F}(u_0 + t v)$ $\downarrow C_c^\infty(a,b)$

$$= \int_a^b [u_0(x) + t v(x)]^2 + \cos((u_0 + t v)^3)$$

$$\varphi'(t) = \int_a^b \{ 2[u_0(x) + t v(x)] v(x) - \sin(u_0(x) + t v(x))^3 \cdot 3(u_0(x) + t v(x))^2 \cdot v(x) \} dx$$

$$0 = \varphi'(0) = \int_a^b \{ 2 u_0(x) \overset{\text{weak derivative}}{\dot{v}(x)} - \sin(u_0(x)^3) 3 u_0(x)^2 v(x) \} dx$$

$$\int_a^b \underbrace{2 u_0(x)}_{f(x)} \underbrace{\dot{v}(x)}_{\dot{\varphi}(x)} dx = - \int_a^b \underbrace{\{ -\sin(u_0(x)^3) 3 u_0(x)^2 \}}_{g(x)} \underbrace{v(x)}_{\varphi(x)} dx$$

This implies that $2u_0(x)$ admits a weak derivative and

$$\boxed{(2u_0(x))'} = -\sin(u_0(x)^3) \cdot 3u_0(x)^2$$

$\uparrow$
 weak derivative
 of weak derivative

$$\boxed{(2\dot{u}_0(x))' = -\sin(u_0(x)^3) \cdot 3u_0(x)^2}$$

↑
weak derivative
of weak derivative

4.2 From H^1 to C^1 .

$$u_0 \in H^1([a,b]) \Rightarrow u_0 \in C^0([a,b]) \Rightarrow \text{RHS} \in C^0([a,b])$$

$$\Rightarrow 2\dot{u}_0(x) \in C^1([a,b]) \quad (\text{classical compatibility: if } f' \in C^0, \text{ then } f \in C^1)$$

$$\Rightarrow \dot{u}_0(x) \in C^0([a,b]) \Rightarrow u_0(x) \in C^1([a,b])$$

If we exploit that $u_0(x)$ is the antiderivative of $\dot{u}_0(x)$ we obtain also that $u_0(x) \in C^2([a,b])$.

At this point ELE is

$$2\dot{u}_0''(x) = -3u_0(x)^2 \sin(u_0(x)^3)$$

4.3 From C^2 to C^∞ is the usual BOOTSTRAP ARGUMENT

$$u_0 \in C^2 \Rightarrow \text{RHS} \in C^2 \Rightarrow \dot{u}_0'' \in C^2 \Rightarrow u_0 \in C^4 \Rightarrow \text{RHS} \in C^4$$

and so on.

In order to justify the computation of $\varphi'(t)$, we need a domination for the integrand in

$$\int_a^b \{ 2[u_0(x) + t v(x)] v(x) - \sin(u_0(x) + t v(x))^3 \cdot 3(u_0(x) + t v(x))^2 \cdot v(x) \} dx$$

$\uparrow$ in L^1 because it is in L^2 $\uparrow$ bounded $\uparrow$ bounded

Example 1 $F(u) = \int_a^b \{ \dot{u}(x)^2 + u(x) \cos u(x) \} dx + DBC$

Weak formulation in H^1 .

Compactness $M \geq \int_a^b \{ \dot{u}_m(x)^2 + u_m(x) \cos(u_m(x)) \} dx$

$$\geq \int_a^b \{ \dot{u}_m(x)^2 - |u_m(x)| \} dx = (*)$$

As in the previous example

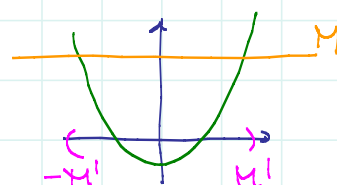
$$|u_m(x)| \leq |u_m(a)| + \int_a^x |\dot{u}_m(t)| dt$$

$$\leq \underbrace{|u_m(a)|}_A + |b-a|^{1/2} \left\{ \int_a^b |\dot{u}_m(t)|^2 dt \right\}^{1/2}$$

In other words $|u_m(x)| \leq A + |b-a|^{1/2} \|\dot{u}_m\|_{L^2(a,b)}$
and therefore

$$M \geq \|\dot{u}_m\|_{L^2(a,b)}^2 - (b-a) [A + |b-a|^{1/2} \|\dot{u}_m\|_{L^2(a,b)}]$$

= second order polynomial in $\|\dot{u}_m\|_{L^2(a,b)}$



From this inequality we deduce that

$$\|u_m\|_{L^2(a,b)} \leq M'$$

From now on we proceed as before.

GENERAL FACT Assume that

$$L(x, s, p) \geq A p^2 - B |s|^{\alpha} \quad \text{with } A > 0 \text{ and } \alpha < 2$$

and that we have at least one DBC. Then

$$F(u_m) \leq M \quad \Rightarrow \quad \|u_m\|_{L^2(a,b)} \leq M'$$



Example 1 bis $\min \left\{ \int_a^b (u^2 + u \cos u) dx : u \in C^1 \right\}$

Inf $= -\infty !!$ u very big and constant with $\cos u = -1$

Example 1 ter $\min \left\{ \int_a^b (u^2 + u^3) dx : u \in C^1, u(a) = u(b) = 0 \right\}$

Consider $u_0(x) \geq 0$ and satisfies the DBC. Then consider

$$u_n(x) = -n u_0(x)$$

$$F(u_n) = n^2 \int_a^b u_0(x)^2 dx - n^3 \int_a^b u_0(x)^3 dx$$

Example 2 $\min \left\{ \int_a^b (u^2 + (u-x)^{20}) dx : u \in C^1 \text{ NO DBC} \right\}$

... Compactness $M \geq F(u_n) \geq \int_a^b u_n(x)^2 dx$

$\leadsto u_{n_k} \rightarrow v_{\infty}$

Now we need a unif bound on $|u_n(x)|$ in order to apply A.A.
We know that

$$M \geq \int_a^b (u(x) - x)^{20} dx = (b-a) \underbrace{(u(x_n) - x_n)^{20}}_{\substack{\text{mean value} \\ \text{thm for integrals}}} \quad \begin{matrix} \uparrow & \uparrow \\ \text{mysterious point} \\ \text{in } [a,b] \end{matrix}$$

$$\leadsto (u_n(x_n) - x_n)^{20} \leq \frac{M}{b-a}$$

$$|u_n(x_n) - x_n| \leq M' \leadsto |u_n(x_n)| \leq |x_n| + M' \leq M''$$

At this point, as before

$$\begin{aligned} |u_n(x)| &\leq |u_n(x_n)| + \left| \int_{x_n}^x u_n'(t) dt \right| \\ &\leq M'' + |b-a|^{1/2} \|u_n'\|_{L^2(a,b)} \leq M''' \end{aligned}$$

GENERAL FACT Assume that

$$L(x, s, p) \geq Ap^2 + \psi(s) - B$$

where $A > 0$ and $\lim_{s \rightarrow \pm\infty} \psi(s) = +\infty$ without DBCs.

$\rightarrow$ This implies that $\psi(s) \geq -C$

Then $F(u_n) \leq M$ implies $\|u_n'\|_{L^2} \leq M'$ and $|u_n|_{L^\infty} \leq M''$

Another possibility for LSC is the following result

$$\liminf_{n \rightarrow +\infty} \int_a^b \psi(x, u_n(x)) dx \geq \int_a^b \psi(x, u_\infty(x)) dx$$

is true provided that $u_n(x) \rightarrow u_\infty(x)$ (it is enough in a pointwise sense) and that $\psi(x, s)$ is bounded from below

This is FATOU lemma. (Classical example: $\psi(x, s) = \frac{1}{s^2}$).

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Istituzioni di Analisi -

LECTURE 17

Note Title

14/10/2021

VARIATIONAL APPROACH TO BOUNDARY VALUE PROBLEMS

Example
$$\begin{cases} u'' = \arctan u \\ u(0) = 3, u(1) = 6 \end{cases}$$

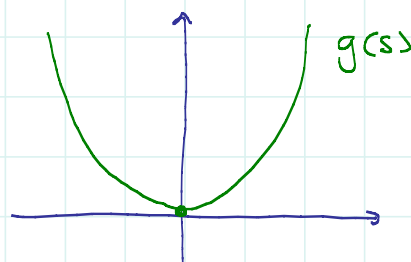
Goal : discuss existence / uniqueness / regularity of solutions
 $u: [0, 1] \rightarrow \mathbb{R}$

Strategy : look for a variational problem for which the given pbm. is ELE and then apply the direct method.

$$(L_p)' = L_s \quad L(x, s, p) = \frac{1}{2} p^2 + g(s) \quad \text{where } g'(s) = \arctan s$$

Consider the min. pbm.

$$\min \left\{ \int_0^1 \left[\frac{1}{2} u'^2 + g(u) \right] dx : \text{DBC} \right\}$$



1. Weak formulation

$$F(u) = \int_0^1 \left[\frac{1}{2} u'^2 + g(u) \right] dx \quad \forall u \in H^1((0, 1))$$

$$F: H^1((0, 1)) \rightarrow \mathbb{R} \quad \text{Also DBCs make sense in } H^1$$

2. Compactness $M \geq F(u_m) \geq \frac{1}{2} \int_0^1 u_m'^2 dx$

$$\Rightarrow \|u_m\|_{L^2}^2 \leq 2M \quad L^2 \text{ bound on derivatives}$$

$$\Rightarrow \text{unif. estimate on } \frac{1}{2} \text{-Hölder constant of } u_m$$

Together with DBC, we obtain

$$|u_n(x)| \leq M' \quad \forall x \in [a, b] \quad L^\infty \text{ bound on functions}$$

In conclusion there exists a subsequence s.t.

$$u_{n_k}(x) \rightarrow u_\infty(x) \quad \text{uniformly} \quad (\text{Ascoli - Arzelà})$$

$$i_{n_k}(x) \rightarrow v_\infty(x) \quad \text{weakly } L^2 \quad (\text{weak cpt. of balls})$$

With the usual trick $v_\infty(x) = i_\infty(x)$

We observe that DBC pass to the limit.

3. LSC If $u_n \rightarrow u_\infty$ unif. then

$$\lim_{n \rightarrow +\infty} \int_0^1 g(u_n(x)) dx = \int_0^1 g(u_\infty(x)) dx$$

If $i_n \rightarrow i_\infty$ weakly L^2 , then

$$\liminf_{n \rightarrow +\infty} \int_0^1 i_n(x)^2 dx \geq \int_0^1 i_\infty(x)^2 dx$$

4. Regularity Let $u_0 \in H^1((0,1))$ be a minimizer.

For every $v \in C_c^\infty((0,1))$ and every $t \in \mathbb{R}$ we define

$$\varphi(t) := F(u_0 + tv) = \int_0^1 (i_0 + t\dot{v})^2 + g(u_0 + tv) dx$$

and we hope that

$$\varphi'(t) = \int_0^1 \{ 2(i_0 + t\dot{v})\dot{v} + \arctan(u_0 + tv) \cdot v \} dx$$

let us check the domination

$$2(|\dot{u}_0(x)| + |t| \cdot |\dot{v}(x)|) |\dot{v}(x)| + |\arctan(\dots)| \cdot |v(x)|$$

$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
 $\in L^1(0,1)$ because $\in L^2(0,1)$ $t \in [-1,1]$ bounded bounded bounded

Since u_0 is a minimizer, then

$$0 = \varphi'(0) = \int_0^2 \{ 2u_0 \dot{v} + \arctan(u_0) \cdot v \} dx$$

which is equivalent to

$$\int_0^2 \underbrace{2u_0}_{f} \underbrace{\dot{v}}_{\varphi} dx = - \int_0^2 \underbrace{\arctan(u_0)}_{g} \cdot \underbrace{v}_{\varphi} dx$$

$\leadsto g$ is the weak derivative of f , namely

$$\underbrace{(2u_0)'}_{\substack{\uparrow \text{weak deriv. of} \\ \text{weak deriv.}}} = \arctan(u_0)$$

$$\boxed{\text{From } H^1 \text{ to } C^2} \quad u_0 \in H^1 \Rightarrow u_0 \in C^0 \Rightarrow \text{RHS} \in C^0$$

$$\Rightarrow 2u_0 \in C^1 \Rightarrow u_0 \in C^2$$

$\uparrow$
classical
compatibility

In this moment, $u_0(x)$ is a solution of the equation in the classical sense

Bootstrap $\leadsto u_0 \in C^\infty([0,1])$.

We proved existence and regularity.

UNIQUENESS Two possible approaches

- use the functional
- use the equation

VIA FUNCTIONAL This requires two steps

1. Proving that every solution to the equ. is a minimizer
2. " " the minimizer is unique

Both points follow in some way from the convexity of the Lagrangian.

Every competitor u can be written as $u = u_0 + v$, where $v \in H^1((0,1))$ with $v(0) = v(1) = 0$.

$$F(u_0 + v) = \underbrace{\int_0^1 u_0^2 dx}_{\text{orange}} + 2 \int_0^1 u_0 \dot{v} dx + \int_0^1 \dot{v}^2 dx + \int_0^1 g(u_0 + v) dx$$

$$\geq \underbrace{\int_0^1 g(u_0) dx}_{\text{orange}} + \int_0^1 \arctan u_0 \cdot v dx$$

$g(a+b) \geq g(a) + b g'(a) \nearrow$

$$= F(u_0) + \int_0^1 \dot{v}^2 dx + \underbrace{\int_0^1 \{2 u_0 \dot{v} + \arctan u_0 \cdot v\} dx}_{=0 \text{ if } u_0 \text{ satisfies ELE even in weak sense (we need int. by parts in } H^1)}$$

$$\geq F(u_0)$$

$\Rightarrow u_0$ is a minimizer

Moreover we have equality $\Leftrightarrow \dot{v} \equiv 0 \Leftrightarrow v \equiv \text{constant}$
 $\Leftrightarrow v \equiv 0$

$\Rightarrow$ the minimizer is unique.

VIA EQUATION Assume we have two solutions

$$v'' = \arctan v$$

$$w'' = \arctan w$$

$$(v-w)'' = \arctan v - \arctan w$$

Multiply by $v-w$ and integrate:

$$\int_0^1 (v-w)(v-w)'' dx = \int_0^1 \underbrace{(v-w)(\arctan v - \arctan w)}_{\geq 0} dx$$

" ≥ 0

$$-\int_0^1 (v-w)^2 dx + \underbrace{[(v-w)(v-w)']}_{\uparrow 0} \Big|_{x=0}^{x=1}$$

$$\Rightarrow \int_0^1 (v-w)^2 dx \leq 0 \Rightarrow v-w \equiv 0 \Rightarrow \dots \Rightarrow v \equiv w$$

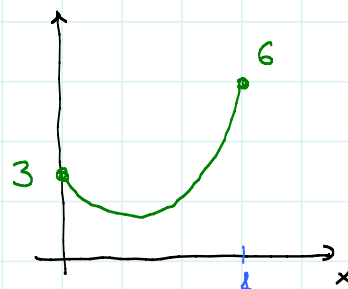
Remark The convexity of $g(s)$ is equivalent to the monotonicity of $\arctan s$.

Qualitative properties of the solution

$$\begin{cases} u'' = \arctan u \\ u(0) = 3, u(1) = 6 \end{cases}$$

The solution is strictly positive and (therefore) convex

u_0 is ≥ 0 : truncation argument!!
(we used that g has a min in 0)



Want to exclude that $u_0(x_0) = 0$ at some $x_0 \in (0, 1)$

In this case, x_0 would be a min point, therefore $u_0'(x_0) = 0$

But in this case u_0 solves

$$\begin{cases} u_0'' = \arctan u_0 \\ u_0(x_0) = 0 \\ u_0'(x_0) = 0 \end{cases}$$

Cauchy problem with unique soln
 $u_0(x) \equiv 0$.

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Istituzioni di Analisi —

LECTURE 18

Note Title

14/10/2021

How to obtain different BC

Example 1

$$\begin{cases} \ddot{u} = u^3 \\ u(0) = 4 & \text{DBC} \\ \dot{u}(1) = 0 & \text{NBC} \end{cases}$$

$$F(u) = \int_0^1 \left\{ \frac{1}{2} \dot{u}^2 + \frac{1}{4} u^4 \right\} dx \quad L(x, s, p) = \frac{1}{2} p^2 + \frac{1}{4} s^4$$

Variational formulation

$$\min \{ F(u) : u \in H^1((0,1)) : u(0) = 4, \dot{u}(1) = 0 \}$$

↑
this
makes sense

~~ABSOLUTELY NOT~~
This condition makes
NO SENSE

Compactness and LSC are standard

$$M \geq F(u_m) \Rightarrow \|\dot{u}_m\|_{L^2}^2 \leq 2M$$

The DBC we have is enough to deduce $\|u_m\|_{L^\infty} \leq M'$

Regularity and NBC

As usual

$$\varphi(t) := F(u_0 + t v) \quad \begin{matrix} \uparrow \\ \text{minimizer} \end{matrix} \quad \begin{matrix} \uparrow \\ \in C_c^\infty \end{matrix}$$

$$\varphi'(0) = \int_0^1 (\dot{u}_0 \dot{v} + u_0^3 v) dx = 0$$

$$\leadsto \int_0^1 \dot{u}_0 \dot{v} dx = - \int_0^1 u_0^3 v dx$$

$\neq 0$ $g \varphi$

$$\leadsto (\ddot{u})' = u^3 \quad \text{weak form of ELE}$$

From H^1 to C^2 and then C^∞ in the usual way

Now we consider $v \in C^1([0,1])$ s.t. $v(0) = 0$ and $v(1) \neq 0$.
Then

$$\begin{aligned} 0 - \varphi'(0) &= \int_0^1 \{ \ddot{u} v + u^3 v \} dx = \\ &= \int_0^1 \{ \underbrace{-\ddot{u}}_0 + u^3 \} v dx + \underbrace{\ddot{u}(1)}_{\neq 0} v(1) - \underbrace{\ddot{u}(0)}_0 v(0) \end{aligned}$$

Necessarily $\ddot{u}(1) = 0$.

Uniqueness as in the previous lecture.
— o — o —

Example 2

$$\begin{cases} \ddot{u} = u^3 \\ u(0) = 4 \\ \ddot{u}(1) = 7 \end{cases}$$

Let us recall that NBCs are of the form $\underset{p=7}{L_p} = 0$

$$\leadsto L(x, s, p) = \frac{1}{2} p^2 - 7p + \frac{1}{4} s^4$$

the weak formulation becomes

$$\min \left\{ \int_0^1 \left\{ \frac{1}{2} \dot{u}^2 - 7\dot{u} + \frac{1}{4} u^4 \right\} dx : u \in H^1((0,1)), u(0) = 4 \right\}$$

Compactness The key point is that

$$L(x, s, p) \geq \frac{1}{4} p^2 + \frac{1}{4} s^4 - A$$

use that

$$\frac{1}{2} p^2 - 7p \geq \frac{1}{4} p^2 - A \quad \forall p \in \mathbb{R}$$

$$M \geq F(u_n) \Rightarrow \|u_n\|_{L^2(0,1)}^2 \leq M' = 4M + A \cdot \text{length of interval}$$

The rest is as usual.

LSC If $u_n \rightarrow u_\infty$ then

$$\liminf_{n \rightarrow \infty} \int_0^1 \frac{1}{2} u_n^2 dx \geq \frac{1}{2} \int_0^1 u_\infty^2 dx$$

$$\lim_{n \rightarrow \infty} \int_0^1 u_n(x) dx = \int_0^1 u_\infty(x) dx$$

↑
scalar product of u_n and 1 in L^2

Genesis of NBC $\varphi(t) \dots$

$$\begin{aligned} 0 = \varphi'(0) &= \int_0^1 \{ (u_0 - 7) \dot{v} + u_0^3 v \} dx \\ &= \int_0^1 \{ \underbrace{-u_0}_{=0} + u_0^3 \} v dx + [(u_0 - 7)v]_{x=0}^{x=1} \end{aligned}$$

when we choose $v \in C^1$ with $v(1) \neq 0$ we get $u_0(1) = 7$.

Example 3
$$\begin{cases} u'' = u^3 \\ \dot{u}(0) = 4 \\ \dot{u}(1) = 7 \end{cases}$$

Two possible approaches

→ variable change

→ modified Lagrangian

1 Variable change Choose u_0 satisfying the 2 strange NBC

$$\dot{u}_0(x) = 4 + 3x \quad u_0(x) = \frac{3}{2}x^2 + 4x$$

Look for a solution $u(x)$ of the form

$$u(x) = u_0(x) + v(x)$$

What is the equ. satisfied by v : $\dot{v}(0) = \dot{v}(1) = 0$

$$\ddot{v} = (u - u_0)''' = \ddot{u} - \ddot{u}_0 = u^3 - 3 = \left(\frac{3}{2}x^2 + 4x + v\right)^3 - 3$$

So we need to solve

$$\ddot{v} = \left(\frac{3}{2}x^2 + 4x + v\right)^3 - 3 \quad \text{with } \dot{v}(0) = \dot{v}(1) = 0$$

$$F(v) = \int_0^1 \left\{ \frac{1}{2} \dot{v}^2 + \frac{1}{4} \left(\frac{3}{2}x^2 + 4x + v\right)^4 - 3v \right\} dx$$

$$L(x, s, p) = \frac{1}{2} p^2 + \frac{1}{4} \left(\frac{3}{2}x^2 + 4x + s\right)^4 - 3s$$

$$\min \{ F(v) : v \in H^1((0, 1)) \} \quad \text{without BCs.}$$

Compactness The key point is that

$$L(x, s, p) \geq \frac{1}{2} p^2 + \frac{1}{27} s^4 - A$$

$$\leadsto F(u_m) \leq M \Rightarrow \|u_m\|_{L^2} \leq M^1$$

$$\Leftrightarrow \int_0^1 u_m(x)^4 dx \leq M''$$

$$\Rightarrow \exists x_m \in [0, 1] \text{ s.t. } |u_m(x_m)|^4 \leq M'' \Rightarrow L^\infty \text{ estimate}$$

The rest is standard.

② Modified Lagrangian

$$\dot{u}(0) = 4$$

$$\dot{u}(1) = 7$$

$$L_p = 0$$

$$L_p = p - 4 - 3x$$

$$L_s = s^3$$

$$L(x, s, p) = \frac{1}{2} p^2 - (4 + 3x)p + \frac{1}{4} s^4$$

$$(L_p)' = L_s \quad [\dot{u} - 4 - 3x]' = u^3 \leadsto \ddot{u} - 3 = u^3 \quad \text{☹}$$

The correct choice is

$$L(x, s, p) = \frac{1}{2} p^2 - (4 + 3x)p + \frac{1}{4} s^4 - 3s$$

and we minimize without boundary cond.

$$\text{Compactness} \leadsto L(x, s, p) \geq \frac{1}{27} p^2 + \frac{1}{35} s^4 - A$$

LSC $\leadsto$ if $u_m \rightarrow u_\infty$, then

$$\lim_{m \rightarrow \infty} \int_0^1 (4 + 3x) u_m(x) dx = \int_0^1 (4 + 3x) u_\infty(x) dx$$

↑
scalar prod. of
 u_m times $4 + 3x$

— o — o —

Istituzioni di Analisi

LECTURE 19

Note Title

18/10/2021

weak convergence in L^p Setting $L^p(X, m, \mu)$ with $p \in [1, +\infty]$.

Defn Let $\{f_n\} \subseteq L^p(X)$ and let $f_\infty \in L^p$. Let q denote the conjugate exponent $\frac{1}{p} + \frac{1}{q} = 1$.
we say that

$$\begin{aligned} f_n &\rightharpoonup f_\infty \quad \text{weakly in } L^p \quad \text{if } p < +\infty \\ f_n &\xrightarrow{*} f_\infty \quad \text{weakly } * \text{ in } L^\infty \quad \text{if } p = +\infty \end{aligned}$$

if

$$\lim_{n \rightarrow +\infty} \int_X f_n(x) g(x) d\mu = \int_X f_\infty(x) g(x) d\mu$$

for every $g \in L^q(X)$.

Remark It is the generalization of the usual case $p=q=2$.

Special case Assume that $\mu(X)$ is finite. For every $1 \leq p_1 < p_2 \leq +\infty$ it turns out that

$$L^{p_2}(X) \subseteq L^{p_1}(X)$$

and as a consequence

$$f_n \rightharpoonup f_\infty \text{ weakly in } L^{p_2}(X) \Rightarrow f_n \rightharpoonup f_\infty \text{ weakly in } L^{p_1}(X)$$

(this is because $L^{q_2}(X) \supseteq L^{q_1}(X)$)

Remark If $\mu(X)$ is **finite**, then weak conv. in $L^1(X)$ is the "weakest" possible.

On the contrary, the strongest weak conv. is weak $*$ convergence in $L^\infty(X)$

Prop Let us assume that $\{f_n\} \subseteq L^p(X)$ is **BOUNDED**.
Let $W \subseteq L^q(X)$ be a subset such that $\text{Span}(W)$ is dense.
Let us assume that

$$\lim_{n \rightarrow +\infty} \int_X f_n(x) g(x) d\mu = \int_X f_\infty(x) g(x) d\mu \quad \forall g \in W$$

Then $f_n \rightarrow f_\infty$ weakly in L^p (weakly $*$ if $p = +\infty$).

Proof Consider

$$\begin{aligned} \left| \int_X f_n g - \int_X f_\infty g \right| &\leq \left| \int_X f_n g - \int_X f_n g_\varepsilon \right| \\ &\quad + \left| \int_X f_n g_\varepsilon - \int_X f_\infty g_\varepsilon \right| \\ &\quad + \left| \int_X f_\infty g_\varepsilon - \int_X f_\infty g \right| \end{aligned}$$

$$\leq \underbrace{\|f_n\|_{L^p(X)}}_{\leq M} \|g - g_\varepsilon\|_{L^q(X)} + \left| \int_X f_n g_\varepsilon - \int_X f_\infty g_\varepsilon \right| + \|f_\infty\|_{L^p} \cdot \|g - g_\varepsilon\|_{L^q}$$

If $g_\varepsilon \in \text{Span}(W)$ then the second term goes to 0 as $n \rightarrow +\infty$.
Therefore

$$\limsup_{n \rightarrow +\infty} \left| \int_X f_n g - \int_X f_\infty g \right| \leq \{M + \|f_\infty\|_{L^p(X)}\} \|g - g_\varepsilon\|_{L^q(X)}.$$

Since $\text{Span}(W)$ is dense, and g_ε is arbitrary, then the LHS is 0.
— 0 — 0 —

Corollary Assume again that $\mu(X)$ is bounded.

Assume that $\{f_n\} \subseteq L^p(X)$ is bounded and that

$$f_n \rightarrow f_\infty \text{ in } L^1(X) \quad (\text{the weakest possible})$$

Then

$$f_n \rightarrow f_\infty \text{ in } L^p(X) \quad (\text{with } * \text{ if } p = +\infty)$$

Proof $L^\infty(X)$ is dense in $L^q(X)$.

— o — o —

Theorem (LSC of convex integrands)

Let us assume that $\mu(X) < +\infty$.

Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function that is bounded from below

Let us assume that

$$f_n \rightarrow f_\infty \text{ weakly in } L^1(X)$$

Then

$$\liminf_{n \rightarrow +\infty} \int_X \varphi(f_n(x)) d\mu \geq \int_X \varphi(f_\infty(x)) d\mu$$

Proof Let us consider the special case where

$$\rightarrow \varphi \in C^1(\mathbb{R})$$

$$\rightarrow \varphi' \text{ is bounded}$$

Let us consider the usual inequality

$$\varphi(a+b) \geq \varphi(a) + \varphi'(a)b \quad \forall (a,b) \in \mathbb{R}^2$$

Let us apply it with $a = f_\infty(x)$ and $b = f_n(x) - f_\infty(x)$.

We obtain that

$$\int_X \varphi(f_n(x)) d\mu \geq \int_X \varphi(f_\infty(x)) d\mu + \int_X \underbrace{\varphi'(f_\infty(x))}_{\in L^\infty(X) \text{ because } \varphi' \text{ is bounded}} (f_n(x) - f_\infty(x)) d\mu$$

If we let $n \rightarrow \infty$ we obtain what we need.

There are two ways to approach the general case

- by the general principle "STRONG + CONVEX $\Rightarrow$ WEAK"
- by hand with an approximation result.

Lemma Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function.

Then there exists a family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ with the following properties

- (i) $\varphi_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$ is convex for every $\varepsilon>0$,
- (ii) $\varphi_\varepsilon \in C^1(\mathbb{R})$ and φ_ε' is bounded for every $\varepsilon>0$ (with a bound that depends on ε)
- (iii) $\varphi_\varepsilon(x) \nearrow \varphi(x)$ as $\varepsilon \rightarrow 0^+$, namely for every fixed $x \in \mathbb{R}$ the function $\varepsilon \rightarrow \varphi_\varepsilon(x)$ is nonincreasing w.r.t ε .

Example $\varphi(x) = x^4$

Take $\varphi_\varepsilon(x) = \varphi(x)$ in $[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$

and then the tangent lines.



Proof of LSC given the Lemma For every $\varepsilon>0$ we observe that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\mathbb{X}} \varphi(f_n(x)) d\mu &\geq \liminf_{n \rightarrow \infty} \int_{\mathbb{X}} \varphi_\varepsilon(f_n(x)) d\mu \\ &\quad \uparrow \\ &\quad \varphi \geq \varphi_\varepsilon \\ &\geq \int_{\mathbb{X}} \varphi_\varepsilon(f_\infty(x)) d\mu \end{aligned}$$

We conclude by considering the supremum with respect to $\varepsilon>0$ (this requires monotone convergence thm (Beppo Levi thm), and hence some boundedness from below).

Example 1 Consider $f_n(x) = \cos(nx)$

This converges weakly to 0 in every $L^p((a,b))$

Proof It is enough to prove that

$$f_n \xrightarrow{*} 0 \quad \text{weakly in } L^\infty((a,b))$$

Need to check $\int_a^b f_n(x) g(x) dx \rightarrow 0 \quad \forall g \in L^1((a,b))$

This is almost trivial if $g(x) = \mathbb{1}_{(c,d)}(x)$ with $(c,d) \subseteq (a,b)$
 but this class spans a dense subset of L^1 .

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Istituzioni di Analisi

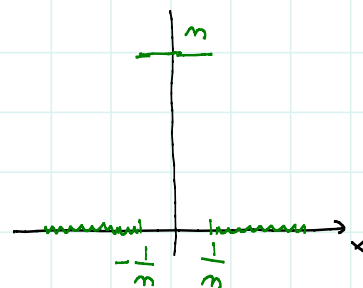
LECTURE 20

Note Title

18/10/2021

Weak compactnessMysterious theoremLet us assume that $p > 1$.Assume that $\{f_n\} \subseteq L^p(\mathbb{R})$ is BOUNDED.Then there exists a subsequence that converges weakly to some $f_0 \in L^p(\mathbb{R})$
(weakly $\neq$ if $p = +\infty$)Example The statement is false if $p = 1$.Let us consider $L^1((-1, 1))$ Let $\varphi_n(x)$ be as in the figure

$$\varphi_n(x) = n \mathbb{1}_{(-\frac{1}{n}, \frac{1}{n})}(x)$$

It is clear that $\|\varphi_n\|_{L^1((-1, 1))} = 2$ for every $n \geq 1$.

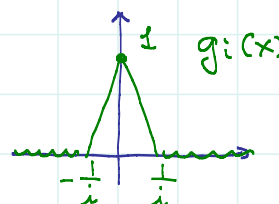
Claim: there exists no converging subsequence in the weak sense.

It is easy to see that

$$\lim_{n \rightarrow +\infty} \int_{-1}^1 \varphi_n(x) g(x) dx = 2g(0) \quad \forall g \in C^0([-1, 1])$$

Assume that $\varphi_{n_k} \rightarrow \varphi_0$. Then this means that

$$\int_{-1}^1 \varphi_0(x) g(x) dx = 2g(0) \quad \forall g \in C^0([-1, 1])$$

Consider $g_i(x)$ as in the figure

$$\int_{-1}^1 \varphi_n(x) g_i(x) dx = 2 g_i(0) = 2$$

$\downarrow$ pointwise conv.
 $\downarrow$ dominated by $|\varphi_n(x)|$.

More mysterious theorem

Assume that $\mu(X) < +\infty$.

Assume that $\{f_n\} \subseteq L^1(X)$ is BOUNDED

Assume that there exists a continuous function

$$\varphi: \mathbb{R} \rightarrow [0, +\infty)$$

with superlinear growth at ∞ , namely

$$\lim_{p \rightarrow \pm\infty} \frac{\varphi(p)}{|p|} = +\infty$$

such that $\int_X \varphi(f_n(x)) d\mu$ is BOUNDED

Then there exists a subsequence that converges weakly in $L^1(X)$.

Proof of myst. theorem in the case $\mu(X) < +\infty$ and $p \geq 2$

Step 1 Bound in $L^p(X) \Rightarrow$ Bound in $L^2(X)$

Step 2 $\exists f_{n_k} \rightharpoonup f_\infty$ weakly in $L^2(X)$ and also weakly in $L^1(X)$

Step 3 weak conv. in $L^1(X)$ + bound in $L^p(X) \Rightarrow$ weak conv. in L^p .

Road map to a proof in the general case

Step 1 Consider a countable dense subset $\{g_i\} \subseteq L^q(X)$

With a diagonal argument we find a subsequence $\{f_{n_k}\}$ s.t.

$$\lim_{k \rightarrow \infty} \int_X f_{n_k}(x) g_i(x) d\mu \text{ exists for every } i \in \mathbb{N}$$

Step 2 By density and using again the boundedness of f_n we obtain that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{X}} f_{n_k}(x) g(x) d\mu = L(g) \quad \forall g \in L^1(\mathbb{X})$$

↑ the limit exists
and depends linearly on g

Step 3 we would like very much to know that

$$L(g) = \int_{\mathbb{X}} g(x) f_{\infty}(x) d\mu \quad \text{for some } f_{\infty} \in L^1(\mathbb{X}).$$

This requires a REPRESENTATION THEOREM

Example $F(u) = \int_0^1 (u^4 + \arctan u) dx$

$$\min \{ F(u) : u \in C^1([0,1]), u(0) = u(1) = 0 \}$$

Weak Formulation

- $u \in W^{1,4}((0,1))$ in this case $\hat{F}: W^{1,4} \rightarrow \mathbb{R}$
- $u \in H^1((0,1))$ " $\hat{F}: H^1 \rightarrow \mathbb{R} \cup \{+\infty\}$

Compactness $M \geq F(u_n) \Rightarrow \|u_n\|_{L^4((0,1))} \leq M + \frac{7}{2}$

$$\Rightarrow \exists u_{n_k} \rightharpoonup v_{\infty} \text{ weakly in } L^p((0,1)) \quad \forall p \in [1,4]$$

In the usual way $\|u_n(x)\|_{L^{\infty}((0,1))} \leq M'$

A.A. At the end of day

$$\begin{aligned} u_{n_k}(x) &\rightarrow u_{\infty}(x) & \forall p \in [1,4] \\ u_{n_k}(x) &\rightarrow u_{\infty}(x) & \text{unif. in } [0,1] \end{aligned}$$

LSC

$$\int_0^1 \arctan(u_n(x)) dx \rightarrow \int_0^1 \arctan(u_\infty(x)) dx$$

↑
unif. conv.

$$\liminf_{n \rightarrow \infty} \int_0^1 u_n(x)^4 dx \geq \int_0^1 u_\infty(x)^4 dx$$

↑
 L^1 weak conv.
+ convexity of p^4

Regularity $\varphi(t) = F(u+tv)$ for $v \in C_c^\infty((0,1))$

$$\varphi'(t) = \int_0^1 \left\{ 4(u+tv)^3 \dot{v} + \frac{1}{1+(u+tv)^2} v \right\} dx$$

To this end, we need a unif. domination. The essential term is

$$(|\dot{u}(x)| + t|\dot{v}(x)|)^3 |\dot{v}(x)|$$

↑ ↑ ↑
 $\in L^3$ because bounded bounded
it is in L^4

$$\int_0^1 \left\{ 4\dot{u}^3 \dot{v} + \frac{1}{1+u^2} v \right\} dx = 0 \quad \forall v \in C_c^\infty((0,1))$$

if u is a minimizer in $W^{1,4}((0,1))$. As usual

$$(4\dot{u}_0^3)' = \frac{1}{1+u_0^2} \quad \text{weak form of ELE}$$

$$u_0 \in W^{1,4} \Rightarrow u_0 \in C^0 \Rightarrow \text{RHS} \in C^0 \Rightarrow \dot{u}_0^3 \in C^1$$

$$\Rightarrow \dot{u}_0^3 \in C^0 \Rightarrow \dot{u}_0 \in C^0 \Rightarrow u_0 \in C^1$$

↑
 $\sqrt[3]{\cdot}$ is continuous

we CANNOT say that $\dot{u}_0 \in C^1$, because $\sqrt[3]{C^1} \not\subset C^1$

But on the other hand

$$\left. \begin{array}{l} u_0 \in C^0 \\ u_0^3 \in \text{Lip} \end{array} \right\} \Rightarrow u_0 \text{ is } \frac{1}{3} - \text{Hölder continuous}$$

and therefore $u_0 \in C^{1,1/3}$

(C^1 with deriv. $\frac{1}{3}$ Hölder cont.)

In this case the solution is exactly $C^{1,1/3}$

The solution admits at least one min. point x_0



Claim $\lim_{x \rightarrow x_0} \frac{u_0(x)^3}{x - x_0} \neq 0$

$$= \lim_{x \rightarrow x_0} \frac{u_0(x)^3 - u_0(x_0)^3}{x - x_0} \underset{\text{Höf}}{=} \lim_{x \rightarrow x_0} \frac{[u_0(x)^3]'}{1} \underset{\text{use ELE}}{\neq 0}$$

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Istituzioni di Analisi

LECTURE 21

Note Title

20/10/2021

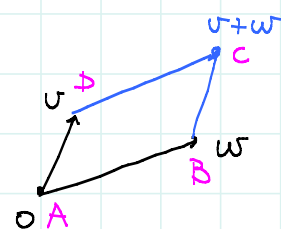
Parallelogram identity] Let H be a Hilbert (it's enough a vector space with a scalar product)

Then

$$\|u+w\|^2 + \|u-w\|^2 = 2\|u\|^2 + 2\|w\|^2 \quad \forall (u,w) \in H^2$$

Proof: trivial

Geom. interpretation



$$AC^2 + BD^2 = 2AD^2 + 2AB^2$$

(sum of $\square$ of diagonals = sum of $\square$ of sides)

Prop. (Jordan - Fréchet - von Neumann) Let V a vector space with a norm that satisfies the parallelogram identity. Then the norm originates from a scalar product.

Proof] Let us consider

$$p(x,y) := \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

$$\left[\frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2} \right]$$

Claim: this is a scalar product

Step 0] Trivial facts:

$$\|x\|^2 = p(x,x)$$

$$p(x,0) = 0$$

$$p(x,y) = p(y,x)$$

$$p(x-y) = -p(x,y)$$

Step 1 [Remember: we need a factor 2]

$$p(2x, y+z) = 2p(x, y) + 2p(x, z)$$

$$\text{LHS} = \frac{1}{4} \{ \|2x+y+z\|^2 - \|2x-y-z\|^2 \}$$

$$= \frac{1}{4} \{ \|(x+y)+(x+z)\|^2 - \|(x-y)+(x-z)\|^2 \}$$

$$\begin{aligned} &= \frac{1}{4} \{ 2\|x+y\|^2 + 2\|x+z\|^2 - \cancel{\|y-z\|^2} \\ &\quad - 2\|x-y\|^2 - 2\|x-z\|^2 + \cancel{\|y-z\|^2} \} \end{aligned}$$

parall. identity

$$= \text{RHS} \quad \square$$

Step 2 We prove true additivity

From the previous step with $z=0$ we obtain that

$$p(2x, y) = 2p(x, y)$$

At this point

$$2p(x, y+z) = p(2x, y+z) = 2p(x, y) + 2p(x, z)$$

$\Rightarrow$ additivity

Step 3 It remains to show that $p(\lambda x, y) = \lambda p(x, y)$

- $\lambda \in \mathbb{N} \rightsquigarrow$ additivity + involution
- $\lambda \in \mathbb{Z} \rightsquigarrow \lambda \in \mathbb{N} + \quad p(x, -y) = -p(x, y)$
- $\lambda \in \mathbb{Q} \rightsquigarrow \lambda = \frac{m}{n}$

$$n p\left(\frac{m}{n} x, y\right) = p(mx, y) = m p(x, y) \rightsquigarrow \text{divide by } n$$

- $\lambda \in \mathbb{R}$ This requires some work. We consider two functions

$$\lambda \rightarrow \lambda p(x, y)$$

$$\lambda \rightarrow p(\lambda x, y)$$

They coincide $\forall \lambda \in \mathbb{Q}$. The first one is continuous.
 It remains to show that also the second one is cont.
 It is enough to show that

$\lambda \rightarrow \|\lambda x + y\|$
 is cont. wrt λ , and this is true because

$$\left| \|\lambda_1 x + y\| - \|\lambda_2 x + y\| \right| \leq \|(\lambda_1 - \lambda_2)x\| \leq |\lambda_1 - \lambda_2| \cdot \|x\|$$

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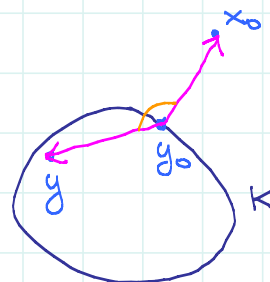
PROJECTION ONTO A CLOSED CONVEX SET

Setting : H is a Hilbert space

$K \subseteq H$ is convex and closed (wrt strong convergence)

For every $x_0 \in H$ we consider

$$\min \{ \|x_0 - y\| : y \in K \}$$



THEOREM Setting as above

(1) The minimum exists for every $x_0 \in H$

(2) The minimizer is a unique point $y_0 \in K$. Let us call it $P_K x_0$

(3) The minimizer y_0 is the unique point $y_0 \in K$ such that

$$\langle x_0 - y_0, y - y_0 \rangle \leq 0 \quad \forall y \in K$$

angle $\in [90^\circ, 180^\circ]$

(4) $\|P_K x_1 - P_K x_2\| \leq \|x_1 - x_2\| \quad \forall (x_1, x_2) \in H^2$

(1-Lip. continuity)

Proof (i) Let $\{y_n\}$ be an inf-izing sequence, and let

$$I := \inf \{ \|x_0 - y\|^2 : y \in K \}$$

We show it is a Cauchy sequence

$$\begin{aligned} \|y_m - y_m\|^2 &= \|(x_0 - y_m) - (x_0 - y_m)\|^2 \\ &= 2\|x_0 - y_m\|^2 + 2\|x_0 - y_m\|^2 - \|2x_0 - (y_m + y_m)\|^2 \\ &= 2\|x_0 - y_m\|^2 + 2\|x_0 - y_m\|^2 - 4\left\|x_0 - \frac{y_m + y_m}{2}\right\|^2 \\ &\quad \in K \end{aligned}$$

let $\varepsilon > 0$ be fixed. When n and m are large enough, then

$$\|x_0 - y_m\|^2 \leq I + \varepsilon \qquad \|x_0 - y_m\|^2 \leq I + \varepsilon$$

$$\|x_0 - \frac{y_m + y_m}{2}\|^2 \geq I$$

We obtain $\text{LHS} \leq 2I + 2\varepsilon + 2I + 2\varepsilon - 4I = 4\varepsilon$

$\Rightarrow \{y_n\}$ is a Cauchy sequence $\Rightarrow y_n \rightarrow y_\infty \in K$
 $\uparrow$ strongly closed

$\Rightarrow y_0$ is a minimizer.

② **Uniqueness** Assume that y_1 and y_2 are two minimizers.

Then $y_1, y_2, y_1, y_2, y_1, y_2, \dots$

is a minimizing sequence

$\Rightarrow$ it is a Cauchy sequence

$\Rightarrow \eta$ is constant.

Alternative: argue as in Step 1 with $m=1$ and $m=2$.

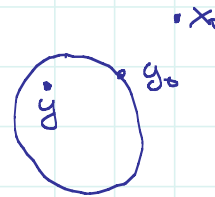
③ Characterization let y_0 be the unique minimizer.

We show that $\langle y - y_0, x_0 - y_0 \rangle \leq 0 \quad \forall y \in K$.

We observe that (remember the place of t)

$$ty + (1-t)y_0$$

is a competitor for every $t \in [0, 1]$, therefore



$$\|x_0 - ty - (1-t)y_0\|^2 \geq \|x_0 - y_0\|^2$$

$$\|(x_0 - y_0) - t(y - y_0)\|^2 \geq \|x_0 - y_0\|^2$$

$$\cancel{\|x_0 - y_0\|^2} + t^2 \|y - y_0\|^2 - 2t \langle x_0 - y_0, y - y_0 \rangle \geq \cancel{\|x_0 - y_0\|^2}$$

Now let $t \rightarrow 0^+$.

We need also the converse. Assume now that

$$\langle x_0 - y_0, y - y_0 \rangle \leq 0 \quad \forall y \in K$$

Claim: y_0 is a minimizer

$$\|x_0 - y_0\|^2 \stackrel{?}{\leq} \|x_0 - y\|^2$$

$$\begin{aligned} \cancel{\|x_0 - y_0\|^2} &\stackrel{?}{\leq} \|x_0 - y_0 + y_0 - y\|^2 \\ &= \cancel{\|x_0 - y_0\|^2} + \|y_0 - y\|^2 - 2 \underbrace{\langle x_0 - y_0, y - y_0 \rangle}_{\geq 0} \end{aligned}$$

④ Lipschitz continuity

$$x_1 \rightsquigarrow P_K x_1$$

$$x_2 \rightsquigarrow P_K x_2$$

$$\langle x_1 - P_K x_1, P_K x_2 - P_K x_1 \rangle \leq 0$$

$$\langle x_2 - P_K x_2, P_K x_1 - P_K x_2 \rangle \leq 0$$

$$\langle P_K x_2 - x_2, P_K x_2 - P_K x_1 \rangle \leq 0 \quad \uparrow \text{the same}$$

Add 1 and 3 and obtain

$$\langle (x_1 - x_2) - (P_K x_1 - P_K x_2), P_K x_2 - P_K x_1 \rangle \leq 0$$

$$- \langle x_1 - x_2, P_K x_1 - P_K x_2 \rangle + \|P_K x_1 - P_K x_2\|^2 \leq 0$$

$$\Rightarrow \|P_K x_1 - P_K x_2\|^2 \leq \langle x_1 - x_2, P_K x_1 - P_K x_2 \rangle$$

$$\leq \|x_1 - x_2\| \cdot \|P_K x_1 - P_K x_2\|$$

$\Rightarrow$ conclusion.

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Istituzioni di Analisi -

LECTURE 22

Note Title

20/10/2021

ORTHOGONAL OF A SUBSPACE

Theorem Let H be a Hilbert space, and let $V \subseteq H$ be a closed vector subspace.

(1) The projection $P_V : H \rightarrow V$ is well-defined

(2) P_V is linear

(3) For every $x_0 \in H$ the projection $y_0 \in V$ is the unique element of V such that

$$\langle x_0 - y_0, v \rangle = 0 \quad \forall v \in V$$

(4) The ker of P_V is a closed subspace of H equal to

$$V^\perp = \{ w \in H : \langle w, v \rangle = 0 \quad \forall v \in V \}$$

and

$$H = V \oplus V^\perp$$

↑ orthogonal complement

Proof (1) is trivial (just observe that V is convex)

(3) From the general theory we know that y_0 is characterized by

$$\langle x_0 - y_0, y - y_0 \rangle \leq 0 \quad \forall y \in V$$

Given $v \in V$, apply the ineq. with $y = v + y_0$ and $y = -v + y_0$
 we obtain

$$\langle x_0 - y_0, v \rangle \leq 0 \quad \text{and} \quad \langle x_0 - y_0, -v \rangle \leq 0$$

CONVEX + STRONG $\Rightarrow$ WEAK

Theorem 1 Let $K \subseteq H$. Let us assume that

(1) K is convex

(2) K is strongly closed (if $x_n \rightarrow x_\infty$, and $x_n \in K$ for every n , then $x_\infty \in K$).

Then K is weakly closed (if $x_n \rightarrow x_\infty$, and $x_n \in K$ for every n , then $x_\infty \in K$).

$\nearrow$ we can also allow x_∞

Theorem 2 Let $F: H \rightarrow \mathbb{R}$ be a function. Let us assume that

(1) F is convex

(2) F is strongly LSC (if $x_n \rightarrow x_\infty$ then $\liminf F(x_n) \geq F(x_\infty)$)

Then F is weakly LSC (if $x_n \rightarrow x_\infty$, then the same)

Strong separation POINT 2 CONVEX

Let $K \subseteq H$ be a closed convex set. Let $x_0 \notin K$.

Then there exist $U \in H$ and $\delta_1 < \delta_2$ such that

$$\langle U, y \rangle \leq \delta_1 < \delta_2 = \langle U, x_0 \rangle \quad \forall y \in K$$

Proof Define

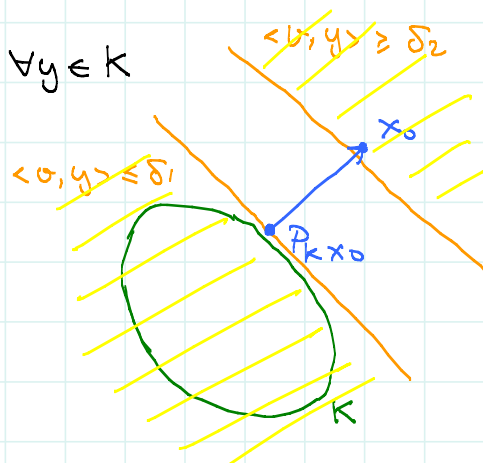
$$U := x_0 - P_K x_0$$

$$\delta_2 := \langle x_0 - P_K x_0, x_0 \rangle$$

$$\delta_1 := \langle x_0 - P_K x_0, P_K x_0 \rangle$$

$$\delta_2 - \delta_1 = \|x_0 - P_K x_0\|^2 > 0$$

It remains to prove that $\langle U, y \rangle \leq \delta_1 \quad \forall y \in K$



namely

$$\langle x_0 - P_K x_0, y \rangle \leq \langle x_0 - P_K x_0, P_K x_0 \rangle$$

namely

$$\langle x_0 - P_K x_0, y - P_K x_0 \rangle \leq 0 \quad \forall y \in K$$

This is the characterization of the projection.
— o — o —

Proof of Theorem 1

Let us consider a sequence $x_n \rightarrow x_\infty$
with $x_n \in K$ for every $n \in \mathbb{N}$.

Claim: $x_\infty \in K$.

Assume this is not the case, namely $x_\infty \notin K$.

Consider a strong separation, namely $v \in H$ s.t.

$$\langle y, v \rangle \leq \delta_2 < \delta_1 = \langle x_\infty, v \rangle \quad \forall y \in K$$

Apply with $y = x_n$:

$$\langle x_n, v \rangle \leq \delta_2 < \delta_1 = \langle x_\infty, v \rangle$$

↓

$$\langle x_\infty, v \rangle$$

Remark A weak separation would NOT be enough

$$\langle y, v \rangle < \langle x_\infty, v \rangle \quad \forall y \in K$$

$$\langle x_n, v \rangle < \langle x_\infty, v \rangle$$

STRICT INEQUALITIES DO NOT PASS TO THE LIMIT.

• x_∞



Proof of Theorem 2 let us consider any $x_n \rightarrow x_\infty$.
We need to show that

$$l = \liminf_{n \rightarrow \infty} F(x_n) \geq F(x_\infty).$$

Up to subsequences, we can assume that

$$l = \lim_{n \rightarrow \infty} F(x_n)$$

Consider the sublevels of F :

$$K_\varepsilon := \{x \in H : F(x) \leq l + \varepsilon\}$$

These sets are strongly closed because F is strongly LSC. They are also convex because F is convex.

By theorem 1 they are weakly closed. Since $x_n \in K_\varepsilon$ for every n large enough, we deduce that $x_\infty \in K_\varepsilon$.

This proves that

$$F(x_\infty) \leq l + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows.

Note: what about the case $l = -\infty$. It can be treated in the same way by considering all sublevels K_M with $M \in \mathbb{R}$

Standard application

Consider

$$F(u) = \int_a^b \varphi(v(x)) dx$$

$u \in L^2$
 $\nearrow$ strongly LSC (Fatou's Lemma)
 $\searrow$ convex

Assume that φ is convex with some growth assumption at $\pm \infty$.

Then F is weakly LSC.

Lagrange Multipliers

Classical setting : open set $\Omega \subseteq \mathbb{R}^d$

$f: \Omega \rightarrow \mathbb{R}$ and $g: \Omega \rightarrow \mathbb{R}$ functions of class C^1

$x_0 \in \Omega$

$$Z_g := \{x \in \Omega : g(x) = 0\}$$

Theorem Suppose that $x_0 \in Z_g$ is a min. point of $f(x)$ restricted to Z_g , namely

$$f(x_0) \leq f(x) \quad \forall x \in Z_g.$$

Then at least one of the following two cond. is satisfied :

(1) $\nabla g(x_0) = 0$

(2) $\exists \lambda \in \mathbb{R}$ such that $\nabla f(x_0) = \lambda \nabla g(x_0)$

$\uparrow$ Lagrange multiplier

Several proofs: $\rightarrow$ explicit function thm

$\rightarrow$ inverse function thm

$\rightarrow$ penalization technique

} lectures of Analisi 2

Theorem Let $\mathbb{X}$ be an affine space with corresponding vect. sp. V

Let $F: \mathbb{X} \rightarrow \mathbb{R}$ and $G: \mathbb{X} \rightarrow \mathbb{R}$ be two functions.

Let $Z_G := \{x \in \mathbb{X} : G(x) = 0\}$. Let $x_0 \in \mathbb{X}$.

Let us assume that for every $v \in V$ and $w \in V$, the functions

$$\mathbb{R}^2 \ni (t, s) \rightarrow F(x_0 + tv + sw) \in \mathbb{R}$$

$$\mathbb{R}^2 \ni (t, s) \rightarrow G(x_0 + tv + sw) \in \mathbb{R}$$

are of class C^1 (enough in a neigh. of $(0, 0)$)

Assume that $x_0 \in Z_G$ is a min point for the restriction of $F(x)$ to Z_G .

Then at least one of the following is true:

(1) $\delta G(x_0, w) = 0$ for every $w \in V$

(2) $\exists \lambda \in \mathbb{R}$ s.t. $\delta F(x_0, v) = \lambda \delta G(x_0, v) \quad \forall v \in V$.

$\uparrow$
does NOT depend on v .

Proof (Based on the inverse function theorem).

Let us consider

$$f(t, s) := F(x_0 + tv + sw)$$

$$g(t, s) := G(x_0 + tv + sw)$$

We observe that

$$g_t(0, 0) = \delta G(x_0, v)$$

$$g_s(0, 0) = \delta G(x_0, w)$$

and analogously for $f_t(0, 0)$ and $f_s(0, 0)$.

Consider the function

$$\mathbb{R}^2 \ni \Phi(t, s) := (f(t, s) - f(0, 0), g(t, s)) \in \mathbb{R}^2$$

$$J\Phi(0, 0) = \begin{pmatrix} \delta F(x_0, v) & \delta F(x_0, w) \\ \delta G(x_0, v) & \delta G(x_0, w) \end{pmatrix}$$

Assume that (1) is false. Then $\exists w \in V$ s.t. $\delta G(x_0, w) \neq 0$.

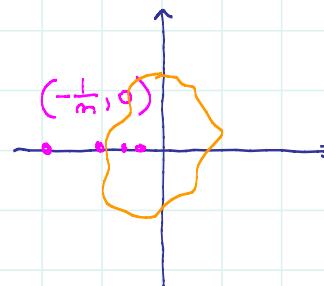
Assume that (2) is false. Then $J\Phi(0, 0)$ has rank = 2.

Note also that $\Phi(0, 0) = (0, 0)$.

By the inverse function theorem the image of Φ contains a neighborhood of the origin, in

particular $(-\frac{1}{n}, 0)$ are in the image of Φ when n is large enough.

This is a pbw. because if Ψ denotes the inverse of Φ , then



$\Psi(-\frac{1}{n}, 0)$ corresponds to a point that tends to x_0 belongs to Z_G , and has
 $F(x) = F(x_0) - \frac{1}{n}$.
 — o — o —

THE EXAMPLE (Poincaré ineq. in an interval).

$$\min \left\{ \underbrace{\int_0^l u(x)^2 dx}_{F(u)} : u(0) = u(l) = 0, \underbrace{\int_0^l u(x)^2 dx = 1}_{G(u)} \right\}$$

Existence via direct method weak formulation in H^1

Compactness $M \geq F(u_n) \Rightarrow \|u_n\|_{L^2(0,l)} \leq M'$

+ DBC $\Rightarrow \|u_n\|_{L^\infty(0,l)} \leq M''$

$$\Rightarrow \begin{aligned} \exists u_{n_k} &\rightarrow u_0 \quad \text{unif. in } [0, l] \\ u_{n_k} &\rightarrow u_0 \quad \text{weakly in } L^2(0, l) \end{aligned}$$

Both DBCs and integral constraint pass to the limit due to unif. conv.

LSC : trivial because of LSC of the norm.

Regularity Due to Lagrange multipliers we have two possibilities

(1) $\delta G(u_0, v) = 0 \quad \forall v \in C_c^\infty(0, l)$
 $\uparrow$
 minimizer

$$0 = \delta G(u_0, v) = 2 \int_0^l u_0(x) v(x) dx$$

FLCV $\Rightarrow u_0(x) \equiv 0 \Rightarrow$ DOES NOT SATISFY $\int_0^l u_0^2 = 1$

$$(2) \quad \delta F(u_0, v) = \lambda \delta G(u_0, v)$$

$$2 \int_0^l \ddot{u}_0(x) \dot{v}(x) dx = 2\lambda \int_0^l u_0(x) v(x) dx$$

In the usual way we obtain

$$(\ddot{u}_0(x))' = -\lambda u_0(x)$$

and we discover that $u_0 \in C^\infty$. At this point ELE becomes

$$\left\{ \begin{array}{l} \ddot{u}_0 = -\lambda u_0 \\ u_0(0) = 0 \\ u_0(l) = 0 \\ \int_0^l u_0(x)^2 dx = 1 \end{array} \right\} \quad \text{3 conditions and 3 parameters}$$

Let us determine the sign of λ . Multiply by u_0 and integrate

$$-\lambda \underbrace{\int_0^l u_0(x)^2 dx}_1 = \underbrace{\int_0^l \ddot{u}_0(x) u_0(x) dx}_{\substack{\uparrow \\ \text{boundary terms} = 0}} = -\int_0^l \dot{u}_0(x)^2 dx$$

$\Rightarrow \lambda$ is positive.

The general solution is $u_0(x) = a \sin(\sqrt{\lambda} x) + b \cos(\sqrt{\lambda} x)$

$$u_0(0) = 0 \Rightarrow b = 0$$

$$u_0(l) = 0 \Rightarrow \underbrace{a}_{\neq 0} \sin(\sqrt{\lambda} l) = 0 \Rightarrow \sqrt{\lambda} l = k\pi \quad k \geq 1$$

$$\Rightarrow \lambda = \frac{k^2 \pi^2}{l^2}$$

The only possibilities are with

$$u_0(x) = a_k \sin\left(\frac{k\pi}{l} x\right) \quad \text{where } a_k \text{ is s.t. } \int_0^l u_0(x)^2 dx = 1$$

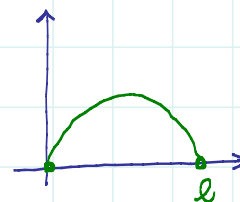
$$\begin{aligned}
 1 &= \int_0^l u_0(x)^2 dx = a_k^2 \int_0^l \sin^2\left(\frac{k\pi}{l}x\right) dx & x &= \frac{ly}{k\pi} \\
 & & dx &= \frac{l}{k\pi} dy \\
 &= a_k^2 \int_0^{k\pi} \sin^2(y) \frac{l}{k\pi} dy \\
 &= \frac{l}{k\pi} a_k^2 \underbrace{\int_0^{k\pi} \sin^2(y) dy}_{\frac{k\pi}{2}} = \frac{l}{2} a_k^2
 \end{aligned}$$

Conclusion

$$u_0(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{k\pi}{l}x\right)$$

$$u_1(x) = \sqrt{\frac{2}{l}} \frac{k\pi}{l} \cos\left(\frac{k\pi}{l}x\right)$$

Conclusion: the minimum is for $k=1$



The minimum prob. is equivalent to an inequality of the form

$$\int_0^l u(x)^2 dx \leq P_l \int_0^l u'(x)^2 dx \quad \forall u \in H^1((0,l))$$

with $u(0) = u(l) = 0$

$\frac{1}{P_l}$ = min that we studied before

Note: the previous computation shows that $P_l \sim l^2$

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Istituzioni di Analisi

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LECTURE 24

Note Title

21/10/2021

Example 1 Discuss existence / uniq. / reg. of functions $u: \mathbb{R} \rightarrow \mathbb{R}$ that are periodic (no prescribed period) such that

$$\ddot{u} = u^5 + \cos x$$

Road map ① If a T -periodic solution exists, then $T = 2k\pi$ for some integer $k \geq 1$.

$$\cos x = \ddot{u}(x) - u^5(x) = T\text{-periodic function} \\ \Rightarrow T = 2k\pi$$

② There exists a solution 2π -periodic

③ There are no other solutions.

Proof of ② $\min \left\{ \int_0^{2\pi} \frac{1}{2} \dot{u}^2 + \frac{1}{6} u^6 + \cos x \cdot u \, dx : u(0) = u(2\pi) \right\}$

Weak Formulation in $H^1((0, 2\pi))$

Compactness

$$L(x, s, p) \geq \frac{1}{2} p^2 + \frac{1}{7} s^6 - A$$

$$M \geq F(u_n) \Rightarrow \int_0^{2\pi} \dot{u}_n(x)^2 \, dx \leq M'$$

$$\Rightarrow \int_0^{2\pi} u_n(x)^6 \, dx \leq M'' \Rightarrow |u_n(x_n)| \leq M'''$$

$$\begin{aligned} \leadsto u_{n_k} &\rightarrow \infty \text{ unif.} \\ \dot{u}_{n_k} &\rightarrow \dot{u}_\infty \text{ weakly} \end{aligned}$$

LSC : as usual (BCs pass to the limit)

Regularity: using $v \in C_c^\infty((0, 2\pi))$ we obtain that any minimizer $u_0 \in H^1$ satisfies

$$(u_0)' = u_0^5 + \cos x$$

$$\leadsto u_0(x) \in C^\infty.$$

Then using perturbations v such that $v(0) = v(2\pi) \neq 0$ we obtain the additional BC

$$u_0(0) = u_0(2\pi) [= 0] \quad \uparrow \begin{cases} \text{Questo NO } \emptyset \\ \text{Aggiunto dopo video} \end{cases}$$

In particular, any minimizer solves

$$\begin{cases} u_0''(x) = u_0(x)^5 + \cos x \\ u_0(0) = u_0(2\pi) \\ u_0'(0) = u_0'(2\pi) \end{cases}$$

Remark Up to now, everything work also with 4π instead of 2π .

Let us extend u_0 to $\mathbb{R}$ by 2π -periodicity.

The extension is of class C^1 , but also of class C^2 (here we need that 2π is 2π), and also of class C^3 , and so on

$\uparrow$ use the equation
in order to compute u_0''' and on

Again, since 2π is a period of $\cos x$, the extension solves the equation for every $x \in \mathbb{R}$.

- ③ Uniqueness In the interval $[0, 2\pi]$ the solution is unique because of the usual argument
- $\rightarrow$ any solution is a minimizer
 - $\rightarrow$ the minimizer is unique due to strict convexity of the Lagrangian.

Assume now that $u_1(x)$ is another solution with period 100π . Then $u_1(x)$ solves the equ. in $[0, 100\pi]$ and therefore it is the unique minimiser in $[0, 100\pi]$.

On the other hand, in $[0, 100\pi]$ we know a solution to the equ, which is the old $u_0(x)$.

By uniqueness in $[0, 100\pi]$ they need to coincide.

Example 2 $\min \left\{ \int_0^1 \dot{u}(x)^6 - u(x)^4 + \arctan u(x) : u(0) = u(1) = 0 \right\}$

The min exists and it is negative.

Take any $u_0(x) \leq 0$ in the interval. consider $u_\varepsilon(x) = \varepsilon u_0(x)$ with $\varepsilon > 0$ small. Then $F(u_\varepsilon) \sim \varepsilon \cdot \text{negative constant}$.

More formally: If $\min = 0$, then $u_0(x) \equiv 0$ is a minimiser, but it is NOT a solution to ELE

$$(6\dot{u}(x)^5)' = -4u(x)^3 + \frac{1}{1+u(x)^2}$$

Existence follows from the direct method. The key point is that

$$L(x, s, p) \geq A|p|^\alpha - B|s|^\beta - C$$

with $A > 0$ and $\beta < \alpha$. Indeed

$M \geq F(u_m)$ we want to obtain $\|u_m\|_{L^\alpha}$. How to obtain?

Due to DBC we observe that

$$|u_m(x)| \leq C_1 + C_2 \cdot \|\dot{u}_m\|_{L^\alpha}$$

$$|u_n(x)| \leq |u_n(a)| + \left| \int_a^x u_n(t) dt \right|$$

$$\leq c_1$$

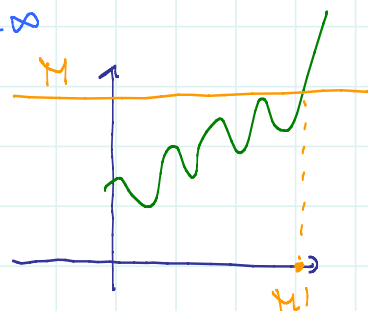
$$|x-a|^{\frac{1}{\alpha}} \cdot \|u_n\|_{L^\alpha}$$

$$\frac{1}{\alpha} + \frac{1}{\alpha} = 1$$

At this point

$$\chi \geq F(u_n) \geq A \int_a^b |u_n(x)|^d dx - B(b-a) \{ c_1 + c_2 \|u_n\|_{L^\alpha} \}^B - c(b-a)$$

function of $\|u_n\|_{L^\alpha}$ that tends to $+\infty$ as $\|u_n\|_{L^\alpha} \rightarrow +\infty$



Example 3 $\int_0^\pi \{ u^2 - c u^2 + \arctan u \} dx$

minimize with $u(0) = u(\pi) = 0$.

We know that $\int_0^\pi u(x)^2 dx \leq 1 \cdot \int_0^\pi u'(x)^2 dx$ if $u(0) = u(\pi) = 0$

If $c < 1$, then we have existence of a minimizer

If $c > 1$, then $\inf = -\infty$

Proof of second claim Consider $u_n(x) = n \sin x$

$$F(u_n) = n^2 \underbrace{\int_0^\pi \cos^2 x dx}_{\frac{\pi}{2}} - c n^2 \underbrace{\int_0^\pi \sin^2 x dx}_{\frac{\pi}{2}} + \int_0^\pi \arctan(\dots)$$

$$= n^2 (1-c) \frac{\pi}{2} + \text{useless guy}$$

Proof if $c < 1$

$$(1-c) \int_0^1 u^2 dx + \underbrace{c \int_0^1 (u^2 - u^2) dx}_{\geq 0} + \int_0^1 \arctan(u) dx$$

$$M \geq F(u_n) \Rightarrow M \geq (1-c) \int_0^1 (u_n^2 + \arctan(u_n)) dx$$

Question: what about $c = 1$

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