ON T-DIVISORS AND INTERSECTIONS IN THE MODULI SPACE OF STABLE SURFACES $\overline{\mathfrak{M}}_{1,3}$

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Abstract. The moduli space of stable surfaces with $K_X^2 = 1$ and $\chi(X) = 3$ has at least two irreducible components that contain surfaces with T-singularities. We show that the two known components intersect transversally in a divisor. Moreover, we exhibit other new boundary divisors and study how they intersect one another.

Contents

1. Introduction

Surfaces of general type and their moduli spaces is a classical subject of study that goes back to the beginning of the 20th century. The moduli spaces $\mathfrak{M}_{K^2,\chi}$ classifying canonical models of surfaces of general type were constructed by Gieseker [Gie77] and we now have a modular compactification $\overline{\mathfrak{M}}_{K^2,\chi}$, the moduli space of stable surfaces, due mainly to work of Kollár, Shepherd-Barron, and Alexeev [KSB88, Ale94].

In this paper we continue the investigation of I-surfaces, that is, stable surfaces with $K_X^2 = 1$ and $\chi(X) = 3$, refining a conjectural picture developed by the last four authors in [FPRR21], where I-surfaces with one T-singularity were studied. Note that T-singularities are precisely the quotient singularities that can occur in the closure of the Gieseker components of $\overline{\mathfrak{M}}_{K^2,\chi}$.

Classical I-surfaces, i.e. with canonical singularities, or more generally I-surfaces with K_X Cartier, are very well understood. They are

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double covers of the quadric cone in $\mathbb{P}^3$ branched over a quintic section and the vertex [FPR17]. Nonetheless, progress on understanding their degenerations, or better all I-surfaces has been slow.

A philosophically sound method consists of imposing the existence of one T-singularity (i.e. locally of the form $\frac{1}{dn^2}(1, dna-1)$). If the singularity imposes independent conditions on the stable surface then we get a stratum whose codimension coincides with the dimension of the local deformation space of the singularity.

For example, there is a T-divisor in $\overline{\mathfrak{M}}_{1,3}$ consisting of the stratum of I-surfaces with a singularity of type $\frac{1}{4}(1,1)$, obtained by allowing the branch locus of the double cover to pass through the vertex. In [FPRR21] it was shown that the other I-surfaces with one T-singularity do not conform to this simple pattern.

Our starting point is the observation [FPRR21, Cor. 1.2] that $\overline{\mathfrak{M}}_{1,3}$ has at least two irreducible components, both of dimension 28: the Gieseker component $\mathfrak{M}_{1,3}$ and the component $\mathfrak{M}_{\mathrm{RU}}$ whose general element is an I-surface with a unique $\frac{1}{25}(1,14)$ singularity constructed by Rana and Urzúa in [RU19].

Confirming a suspicion from [FPRR21] we show that the two components intersect in a divisor parametrising RU-surfaces of cuspidal type. We go on to investigate the intersections of the various T-divisors obtained so far.

The schematic picture in Figure 1 illustrates our results, where we label each stratum by the singularities of its general element.

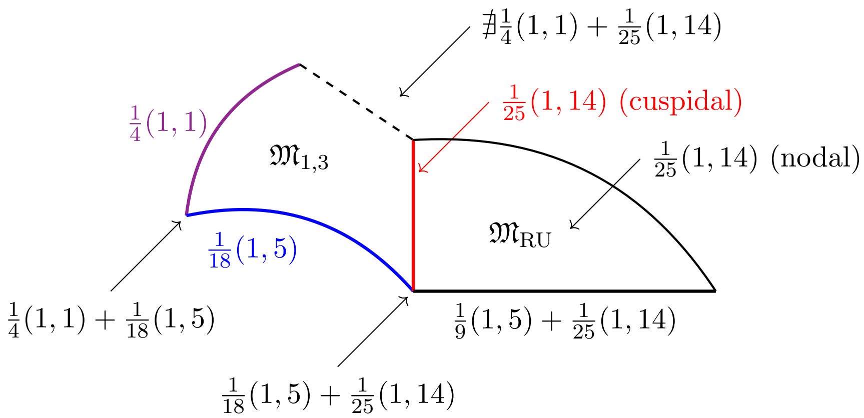

FIGURE 1. Schematic picture of (known parts of) the moduli space of I-surfaces

More precisely, we prove the following:

(1) I-surfaces with a $\frac{1}{25}(1,14)$ singularity and of cuspidal type are smoothable; they form a divisor which is the intersection of the two components. (Section 4)

(2) While the $\frac{1}{9}(1,5)$ singularity does not occur in the closure of the Gieseker component, there is a divisor in the RU-component of surfaces with T-singularities $\frac{1}{9}(1,5)+\frac{1}{25}(1,14)$. (Section 5.3)
(3) The divisor of surfaces with an $\frac{1}{18}(1,5)$ singularity, known to be in the closure of the Gieseker component, intersects the aforementioned divisors in a subset of codimension two parametrising surfaces with singularities $\frac{1}{18}(1,5)+\frac{1}{25}(1,14)$. (Prop. 5.12 and Ex. 5.13)
(4) The divisors parametrising surfaces with a $\frac{1}{4}(1,1)$ singularity respectively a $\frac{1}{18}(1,5)$ singularity intersect, as expected, in a subset of codimension two, whose general element is a surface with singularities $\frac{1}{4}(1,1)+\frac{1}{18}(1,5)$. (Section 5.4)
(5) We suspect the remaining intersection to be empty, but can only show that the combination $\frac{1}{4}(1,1)+\frac{1}{25}(1,14)$ cannot occur on an I-surface (Section 5). If the two divisors intersect at all, then they do so in considerably more singular surfaces, and presumably in high codimension.

Two complementary approaches are used to establish these claims: geometric study of the minimal resolution and algebraic study of the canonical ring. Both points of view give us descriptions of the general surface in each stratum. Canonical rings are needed to establish deformations and (partial) smoothings via explicit equations, and the study of configurations of rational curves on the minimal resolution gives the non-existence result.

Having established a complicated format for the canonical ring of the RU-surface in Corollary 3.11, we include a very surprising alternative description of the same surface as a hypersurface in weighted projective space in Section 3.7.

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2. Preliminaries

We work over the complex numbers. Linear equivalence is denoted by $\sim$.

For a Hirzebruch surface $\mathbb{F}_n$, we denote by σ_∞ the negative section and by Γ the class of a ruling, so that a section σ_0 disjoint from σ_∞ is linearly equivalent to $n\Gamma+\sigma_\infty$.

An I-surface X is a stable surface with $K_X^2 = 1$, $p_g = 2$, $q = 0$ (see [FPRR21]).[1]

A curve is an effective divisor, not necessarily irreducible or reduced. For a positive integer n, an irreducible $(-n)$-curve D on a smooth surface is a smooth rational curve with $D^2 = -n$.

A T-singularity Q is either a rational double point or a 2-dimensional quotient singularity of type $\frac{1}{dn^2}(1, dna-1)$, where $n > 1$ and $d, a > 0$ are integers with a and n coprime. These are precisely the quotient singularities that admit a $\mathbb{Q}$-Gorenstein smoothing, that is, that can occur on smoothable stable surfaces (cf. [KSB88, §3]).

The exceptional divisor of the minimal resolution of a T-singularity $\frac{1}{dn^2}(1, dna-1)$ is a so-called *T-string*, a string of rational curves with self-intersections $-b_1, -b_2, \dots, -b_r$ given by the Hirzebruch–Jung continued fraction expansion $[b_1, b_2, \dots, b_r]$ of $\frac{dn^2}{dna-1}$ (see, e.g. [CLS11, Chapter 10]). The index of X at Q is n.

3. Computing the canonical ring of the RU-surface

For the reader's convenience we recall the construction of a general RU-surface.

Example 3.1. *Let Y be an elliptic surface with $p_g(Y) = 2$, $q(Y) = 0$ such that:*

- *Y has a (-3)-section A*
- *all the elliptic fibers are irreducible.*

By [FPRR21, Lemma 3.8]*, the surface Y is a double cover $\pi\colon Y \to \mathbb{F}_6$ branched on a smooth divisor $D \in |\sigma_\infty + 3\sigma_0|$.*

Let F_1 be a singular fiber and let q be its singular point: blow up F_1 at q and then at a point q_1 infinitely near to q and lying on the strict transform of F_1 to get a surface $\tilde{Y}$. The strict transform of F_1 is a (-5)-curve B, the strict transform of A (that we still denote by A) is a (-3)-curve, the strict transform of the curve of the first blow up is a (-2)-curve, which we call C, so that A, B, C is a string of type $[3, 5, 2]$. *Note that this is true both for F_1 nodal and for F_1 cuspidal. Then the string A, B, C can be blown down to obtain an I-surface with unique singularity of type $\frac{1}{25}(1, 14)$, compare Figure 2.*

Since we assumed the fiber F_1 to be irreducible it is either a nodal curve (type I_1) or a cuspidal curve (type II) and we call X a nodal or cuspidal RU-surface respectively.

We know from [FPRR21, Prop. 3.13] that nodal RU-surfaces form an open subset of an irreducible component $\mathfrak{M}_{\mathrm{RU}}$ of the moduli space. In order to understand the interaction of this component with the Gieseker

[1] To exclude some more pathological (non-smoothable) examples (compare [Rol21]) we could be more specific and fix the Hilbert series of the canonical divisor to be $h(t) = \frac{1-t^{10}}{(1-t)^2(1-t^2)(1-t^5)}$.

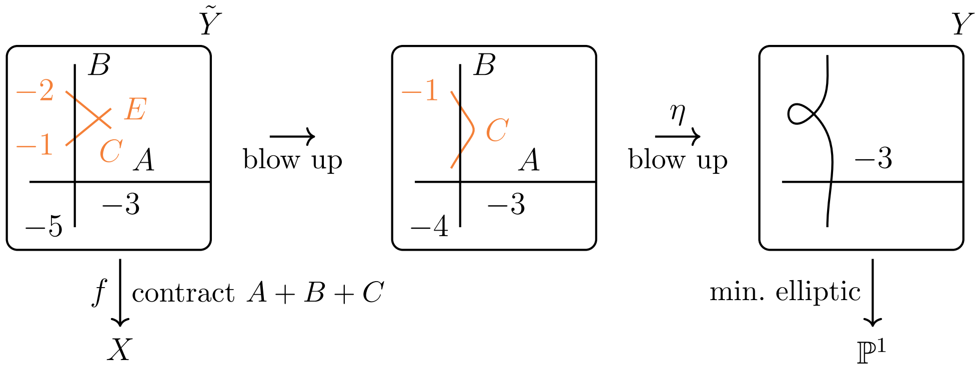
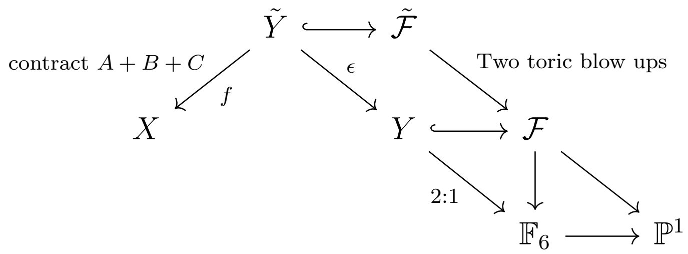


FIGURE 2. Construction of an RU-surface, nodal case

component, we study RU-surfaces from the point of view of canonical rings.

3.1. **Strategy.** Let us explain the strategy underlying the algebraic computations along the following diagram.



(1) Construct Y in a toric bundle $\mathcal{F}$ over $\mathbb{P}^1$.
(2) Blow up to $\tilde{Y}$ in a new toric variety $\tilde{\mathcal{F}}$.
(3) Construct the canonical model of X by writing $R(X, K_X)$ as a subring of the Cox ring of $\tilde{Y}$.
(4) Study $\mathbb{Q}$-Gorenstein deformations of X by deforming the canonical ring.

The deformations in Step (4) were originally found by considering all deformations over the base $\operatorname{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^k)$ for $k = 2$ and extending them to $k = 3, 4$ etc. according to ideas of Reid [Rei90]. Rather than reproducing these unwieldy computations, we express the final results using formats for Gorenstein rings.

3.2. **The elliptic surface.** Instead of using the double cover $Y \to \mathbb{F}_6$, which leads to an elliptic surface with fibers polarised in degree 2, we construct Y via the halfpolarisation A, so that the fibers have degree 1. Consider the toric variety $\mathcal{F}$ with Cox ring

$$\begin{pmatrix} t_0 & t_1 & s_1 & s_0 & \zeta \\ 1 & 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$

and irrelevant ideal $I = (t_0, t_1) \cap (s_0, s_1, \zeta)$. Geometrically, this is a $\mathbb{P}(1_{s_1}, 2_{s_0}, 3_\zeta)$-bundle over $\mathbb{P}^1_{t_0,t_1}$. Alternatively, $\mathcal{F} \cong (\mathbb{C}^5 \setminus V(I))/(\mathbb{C}^*)^2$ where the action is determined by the columns of the above matrix. The subvariety $\zeta = 0$ is isomorphic to $\mathbb{F}_6$. Write Γ for the fiber of $\mathbb{F}_6 \to \mathbb{P}^1$ with base coordinates t_0, t_1. The fiber coordinates are s_0, s_1^2; $(s_0 = 0)$ being the positive section σ_0 and $(s_1^2 = 0)$ being the negative section σ_∞. The elliptic surface Y is a relative sextic in $\mathcal{F}$ and the halfpolarisation A is a section cutting out one Weierstrass point on each fiber.

Lemma 3.2. *Let Y be a double cover of $\mathbb{F}_6$ branched over a smooth element D of $|-6\Gamma + 4\sigma_0|$. Then Y is a hypersurface in $\mathcal{F}$ of bidegree $(0, 6)$. One can choose coordinates so that Y has a nodal or cuspidal fiber B over $(0, 1)$ with singularity at the point with coordinates $(0, 1; 0, 1, 0)$. The equation of Y can be written in the form:*

$$(\zeta - \theta t_1^3 s_0 s_1)\zeta = s_0^3 + t_0 k'_{11} s_0 s_1^4 + t_0(t_0 l'_{16} + \tau t_1^{17}) s_1^6,$$

where $k'_{11}(t_0, t_1), l'_{16}(t_0, t_1)$ are general polynomials in t_0, t_1 of respective degrees $11, 16$ and θ, τ are parameters. In particular, the special fiber is nodal if $(\theta, \tau \neq 0)$, cuspidal if $(\theta = 0)$, type I_2 if $(\tau = 0)$ and type III if $(\theta = \tau = 0)$.[2]

Proof. The linear system $|-6\Gamma + 4\sigma_0|$ on $\mathbb{F}_6$ decomposes into $A + |3\sigma_0|$. By the above discussion, the branching over A is inherited from the structure of the toric variety, and thus Y has equation:

$$\zeta^2 = s_0^3 + j_6(t_0, t_1) s_0^2 s_1^2 + k_{12}(t_0, t_1) s_0 s_1^4 + l_{18}(t_0, t_1) s_1^6$$

where ζ is the double cover variable and j, k, l are general polynomials in t_0, t_1 of respective degrees $6, 12, 18$ so that the right hand side of the equation cuts out a general element of $|3\sigma_0|$. The coefficient of s_0^3 is non-zero because otherwise two components of D would intersect giving a singularity. We normalise this coefficient to be 1 and then we use the Tschirnhausen transformation $s_0 \mapsto s_0 + \frac{1}{3} j_6 s_1^2$ to remove j_6.

The elliptic fibration $Y \to \mathbb{P}^1$ has singular fibers when the discriminant $\Delta := 4k^3 + 27l^2$ vanishes. Assume that the fiber $B\colon (t_0 = 0)$ is singular, so that $\Delta|_B$ vanishes. Note that $\Delta|_B = 4\alpha^3 + 27\beta^2$, where α (resp. β) is the coefficient of t_1^{12} in k (resp. t_1^{18} in l). Hence there exists ε with $\alpha = -3\varepsilon^2$ and $\beta = 2\varepsilon^3$.

Next we use the coordinate change $s_0 \mapsto s_0 + \varepsilon t_1^6 s_1^2$ to move the singularity of B onto the section $\sigma_0\colon (s_0 = 0)$, and the equation of $\tilde{Y}$ becomes

$$\zeta^2 = s_0^3 + 3\varepsilon t_1^6 s_0^2 s_1^2 + t_0 k'_{11}(t_0, t_1) s_0 s_1^4 + t_0(t_0 l'_{16}(t_0, t_1) + \tau t_1^{17}) s_1^6.$$

[2]More precisely, in the latter two cases, the surface Y is singular and its minimal resolution has a fiber of the given type.

If ε is zero then the fiber B is cuspidal. We write now $\theta^2 := 3\varepsilon$ and use the coordinates change $\zeta \mapsto \zeta + \theta t_1^3 s_0 s_1$ to move the $3\varepsilon t_1^6$-term to the left side. This gives the claimed equation.

Close to $(0,1;0,1,0)$ we can set $t_1 = s_1 = 1$ so that the equation becomes

$$\begin{aligned}\zeta^2 &= s_0^3 + 3\varepsilon s_0^2 + t_0 k'_{11}(t_0,1)s_0 + t_0(t_0 l'_{16}(t_0,1) + \tau)\\ &= \tau t_0 + \left(3\epsilon s_0^2 + t_0 s_0 k'_{11}(0,1) + t_0^2 l'_{16}(0,1)\right) + \text{h.o.t.},\end{aligned}$$

which defines a smooth surface if $\tau \neq 0$. Otherwise, if $\tau = 0$ (resp. $\varepsilon = \tau = 0$), we get a fiber of type I_2, respectively *III* after resolving the A_1 surface singularity. □

Note that the choice of square-root θ corresponds to a choice of branch for the second blowup of the nodal fiber (cf. Figure 2). The curious change of coordinates on ζ ensures that the blowups are at torus fixed points and thus in the next Section they can be expressed in terms of toric geometry.

3.3. **The resolution $\tilde{Y}$.** Consider now the toric variety $\tilde{\mathcal{F}}$ with Cox ring

$$(1)\qquad \begin{pmatrix} t_0 & t_1 & s_1 & s_0 & \zeta & c & e\\ 1 & 1 & -3 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 2 & 3 & 0 & 0\\ 2 & 0 & 0 & 1 & 1 & -1 & 0\\ 1 & 0 & 0 & 0 & 1 & 1 & -1\end{pmatrix}$$

and irrelevant ideal

$$\begin{aligned}(t_0,t_1)\cap(s_0,s_1,\zeta)\cap(c,t_1)\cap(c,s_1)\cap(t_0,s_0,\zeta)\\ \cap(e,t_1)\cap(e,s_0)\cap(e,s_1)\cap(t_0,\zeta,c).\end{aligned}$$

This is the toric blow up of $\mathcal{F}$ at the subscheme $t_0^2 = s_0 = \zeta = 0$ with exceptional divisor $(c = 0) \cong \mathbb{P}(1,1,2)$, followed by the toric blowup at the subscheme $t_0 = \zeta = c = 0$ with exceptional divisor $(e = 0) \cong \mathbb{P}^2$. Indeed, consider the subvariety $(e = 0)$. The structure of the irrelevant ideal implies that t_1, s_0, s_1 are non-zero. Thus we apply part of the $(\mathbb{C}^*)^4$-action (first three rows of the matrix (1)) to rescale these three coordinates $t_1 = s_0 = s_1 = 1$. We are left with the $\mathbb{C}^*$-quotient of $\mathbb{C}^3_{t_0,\zeta,c}$ induced by the last row with irrelevant ideal (t_0,ζ,c). In the same way, $(c = 0)$ is the toric subvariety with Cox ring

$$\begin{pmatrix} t_0 & s_0 & \zeta & e\\ 2 & 1 & 1 & 0\\ 1 & 0 & 1 & -1\end{pmatrix}$$

and irrelevant ideal $(s_0,e)\cap(t_0,\zeta)$. That is, the blowup of $\mathbb{P}(2_{t_0}, 1_{s_0}, 1_\zeta)$ at the point $(0,1,0)$.

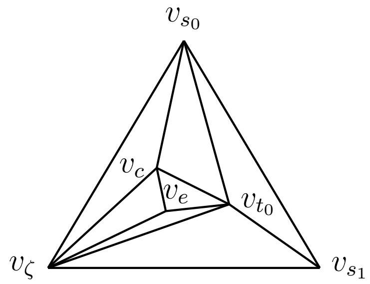


FIGURE 3. A cross section of the fan $\Sigma(\tilde{\mathcal{F}})$, with the origin and v_{t_1} behind the page.

Remark 3.3. *A reference for this approach to toric varieties is Chapter 14 of the book* [CLS11]. *The generators of the Cox ring (columns of* (1)*) correspond to primitive generators of the rays in the fan Σ of $\tilde{\mathcal{F}}$ via Gale duality, and the irrelevant ideal encodes the cones of Σ. For example, the generator c gives a ray inside the cone σ spanned by primitive vectors v_{t_0}, v_{s_0}, v_ζ with primitive generator $v_c = 2v_{t_0} + v_{s_0} + v_\zeta$ determined by the third row. The components of the irrelevant ideal involving c ensure that the cones of Σ include the barycentric subdivision of σ, with respect to the ray generated by v_c.*

Lemma 3.4. *The double blowup $\tilde{Y}$ of Y is a hypersurface in $\tilde{\mathcal{F}}$ of multidegree $(0, 6, 2, 1)$. The equation of $\tilde{Y}$ can be written in the form:*

$$(e\zeta - \theta t_1^3 s_0 s_1)\zeta = cs_0^3 + cet_0 k'_{11} s_0 s_1^4 + t_0(c^2e^3t_0 l'_{16} + \tau t_1^{17})s_1^6 \tag{2}$$

where $k'_{11} = k'_{11}(c^2e^3t_0, t_1)$, $l'_{16} = l'_{16}(c^2e^3t_0, t_1)$

Proof. The first blow up is at the subscheme $t_0^2 = s_0 = \zeta = 0$ which is supported at the singularity of B. We add one new variable c to the Cox ring of $\mathcal{F}$ and read off the blowup map using the third row of the Cox grading (1). We use the same labels for the new coordinates:

$$t_0 \mapsto c^2t_0,\ t_1 \mapsto t_1,\ s_0 \mapsto cs_0,\ s_1 \mapsto s_1,\ \zeta \mapsto c\zeta$$

Note the weighting on t_0. The equation of the first blowup $Y^{(1)}$ is obtained by pulling back the equation of Y under this map and dividing by c^2:

$$(\zeta - \theta t_1^6 s_0 s_1)\zeta = cs_0^3 + ct_0 k'_{11}(c^2t_0, t_1)s_0 s_1^4 + t_0(c^2t_0 l'_{16}(c^2t_0, t_1) + \tau t_1^{17})s_1^6.$$

The second blowup is at $\zeta = c = t_0 = 0$ and the map is read off from the fourth row of the Cox grading (1):

$$t_0 \mapsto et_0,\ t_1 \mapsto t_1,\ s_0 \mapsto s_0,\ s_1 \mapsto s_1,\ \zeta \mapsto e\zeta,\ c \mapsto ec$$

The equation of the second blowup $\tilde{Y} = Y^{(2)}$ is then as claimed in (2). Note that the multidegree of this equation is $(0, 6, 2, 1)$ since e.g. the monomial cs_0^3 has such degree (see matrix (1)). □

Remark 3.5. *According to the discussion before Lemma 3.4, the exceptional curve $C = \tilde{Y} \cap (c = 0)$ is obtained by substituting $c = 0$ and $t_1 = s_1 = 1$ into equation (2):*

$$C\colon\ ((e\zeta - \theta s_0)\zeta = \tau t_0) \subset \mathrm{Bl}_{(0,1,0)}\,\mathbb{P}(2_{t_0}, 1_{s_0}, 1_\zeta).$$

Recall that the parameter τ is the coefficient of t_1^{17} in l'_{17}.

Thus C is usually an irreducible rational curve. If $\theta = 0$, then nothing especially interesting happens to C. On the other hand, if $\tau = 0$, then C breaks into two rational curves meeting in a singularity on $\tilde{Y}$. In this situation, Y has an A_1-singularity at the node of the fiber B. This case is treated in more detail later. If $\theta = \tau = 0$ then C is a nonreduced double curve and Y has an A_1 singularity at the node of B.

In a similar way, we find that the second exceptional curve is $E\colon (e = 0)$, which implies $t_1 = s_0 = s_1 = 1$ with equation

$$E\colon\ (\theta\zeta = c + \tau t_0) \subset \mathbb{P}^2_{t_0,\zeta,c}.$$

3.4. **Canonical rings from Cox rings.** We denote the Cox ring of $\tilde{\mathcal{F}}$ by S. This is a $\mathbb{Z}^4$-graded polynomial ring with generators t_0, t_1, s_0, s_1, ζ, c, e. There is a $\mathbb{Z}$-linear coordinate change on $A_1(\tilde{\mathcal{F}})$, i.e., applied to the degrees, which shifts the $\mathbb{Z}^4$-grading on S to:

$$\begin{pmatrix} s_1 & t_0 & c & e & t_1 & s_0 & \zeta \\ 1 & & & & 0 & 2 & 3 \\ & 1 & & & 1 & 6 & 9 \\ & & 1 & & 2 & 11 & 17 \\ & & & 1 & 3 & 17 & 25 \end{pmatrix} \tag{3}$$

This can be obtained from matrix (1) by multiplication on the left with the 4×4 matrix

$$\begin{pmatrix} -3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & -1 & 0 \\ 3 & 9 & -1 & -1 \end{pmatrix}$$

and a permutation of the columns.

By Lemma 3.4, for $d \in \mathbb{Z}^4$ we have maps

$$S_d \to H^0(\widetilde{Y}, d_1A + d_2B + d_3C + d_4E)$$

where the divisor classes are

$$A\colon (s_1 = 0),\ B\colon (t_0 = 0),\ C\colon (c = 0),\ E\colon (e = 0),\ \Gamma\colon (t_1 = 0)$$

on $\tilde{Y}$. Note that $(s_1 = 0)$ cuts out A because A is pulled back from the irreducible component $(s_1^2 = 0)$ of the branch locus of the double cover $Y \to \mathbb{F}_6$. Thus the grading records the various linear equivalences on $\tilde{Y}$:

$$B \sim \Gamma - 2C - 3E,$$
$$(s_0 = 0) \sim 2A + 6B + 11C + 17E,$$
$$(\zeta = 0) \sim 3A + 9B + 17C + 25E.$$

The pullback of K_X to $\tilde{Y}$ is

$$\begin{aligned}\Gamma + C + 2E + \tfrac{1}{5}(3A + 4B + 2C) &\sim B + 3C + 5E + \tfrac{1}{5}(3A + 4B + 2C)\\ &\sim 5E + \tfrac{1}{5}(3A + 9B + 17C).\end{aligned}$$

3.5. **Generators.** We want to construct the canonical ring $R(X, K_X)$ as the image of a map from the Cox ring S. To avoid having to keep track of corrections to multiplication maps in $R(X, K_X)$ due to rounding, we introduce formal 5-th roots of s_1, t_0, c: $\alpha^5 = s_1$, $\beta^5 = t_0$, $\gamma^5 = c$. The extension $S[\alpha, \beta, \gamma]$ is then a $(\frac{1}{5}\mathbb{Z})^3 \oplus \mathbb{Z}$-graded polynomial ring generated by α, β, γ, e (note that e belongs to S). In what follows, we consider the ring homomorphism

$$\iota\colon \bigoplus_{n\geq 0} S[\alpha, \beta, \gamma]_{(\frac{3}{5}n, \frac{9}{5}n, \frac{17}{5}n, 5n)} \to \bigoplus_{n\geq 0} H^0(\tilde{Y}, nf^*K_X) \cong R(X, K_X)$$

The proof that this ring homomorphism is in fact surjective, is Corollary 3.12.

Let R' be the image of ι and $X' = \operatorname{Proj}(R')$.

Lemma 3.6. *The graded ring*

$$R' \subseteq \bigoplus_{n\geq 0} H^0\left(\tilde{Y}, n(5E + \tfrac{1}{5}(3A + 9B + 17C)\right)$$

is generated by

$$\begin{aligned}
x_0 &= \alpha^3\beta^9\gamma^{17}e^5 && \textit{deg 1}\\
x_1 &= \alpha^3\beta^4\gamma^7e^2t_1 && \textit{deg 1}\\
y &= \alpha^6\beta^3\gamma^4et_1^3 && \textit{deg 2}\\
w &= \alpha^9\beta^2\gamma t_1^5 && \textit{deg 3}\\
u_0 &= \alpha^2\beta^6\gamma^{13}e^3s_0 && \textit{deg 4}\\
u_1 &= \alpha^2\beta\gamma^3t_1s_0 && \textit{deg 4}\\
z &= \zeta && \textit{deg 5}\\
t &= \alpha\beta^3\gamma^9es_0^2 && \textit{deg 7}\\
g &= \alpha\beta^3\gamma^{14}s_0^5 && \textit{deg 17}
\end{aligned}$$

Proof. The generators can be determined algorithmically as follows. The grading on $S[\alpha, \beta, \gamma]$ is induced by the map $\delta\colon \mathbb{Z}^7 \to (\frac{1}{5}\mathbb{Z})^3 \oplus \mathbb{Z}$,

$$\delta(n_1, \ldots, n_7) = \left(\tfrac{n_1}{5}, \tfrac{n_2}{5}, \tfrac{n_3}{5}, n_4, n_5, n_6, n_7\right)\begin{pmatrix}1 & & & & 0 & 2 & 3\\ & 1 & & & 1 & 6 & 9\\ & & 1 & & 2 & 11 & 17\\ & & & 1 & 3 & 17 & 25\end{pmatrix}^t$$

via $\deg(\alpha^{n_1}\beta^{n_2}\gamma^{n_3}e^{n_4}t_1^{n_5}s_0^{n_6}\zeta^{n_7}) = \delta(n_1, \ldots, n_7)$. Let $\Sigma = \mathbb{Z}^+\cdot(\frac{3}{5}, \frac{9}{5}, \frac{17}{5}, 5)$ be the cone in $(\frac{1}{5}\mathbb{Z})^3 \oplus \mathbb{Z}$ generated by $\frac{1}{5}(3A + 9B + 17C) + 5E$, the pullback of K_X. Then the intersection of the preimage $\delta^{-1}\Sigma$ with the positive octant $(\frac{1}{5}\mathbb{Z}^+)^3 \oplus \mathbb{Z}^+$, is the cone of monomials in R'. The primitive generators of this cone can be found using standard Hilbert basis algorithms for lattice cones (see e.g. Chapter 7 of [MS05]) and these are the generators of R'. □

Remark 3.7. *It follows from Lemma 3.9 below, that the generator g in degree 17 is eliminated by relations.*

Remark 3.8. *At this stage, we already note that the fixed part of the canonical linear system is the image of the curve E, because both $x_0 = \alpha^3\beta^9\gamma^{17}e^5$ and $x_1 = \alpha^3\beta^4\gamma^7e^2t_1$ are divisible by powers of e. The other common factors correspond to curves A, B, C which are contracted to the $\frac{1}{25}(1,14)$-point. A more precise description of the fixed part can be found in Remark 3.14.*

3.6. **Relations.** There are 10 binomial relations between the generators found in Lemma 3.6 (excluding those involving g). These define a cone over the degree 5 generator z in $\mathbb{P}(1,1,2,3,4,4,5,7)$:

$$R_1\colon x_0y - x_1^3 = 0 \qquad R_2\colon x_0w - x_1^2y = 0$$
$$R_3\colon x_1w - y^2 = 0 \qquad R_4\colon x_0u_1 - x_1u_0 = 0$$
$$R_5\colon x_1^2u_1 - yu_0 = 0 \qquad R_6\colon x_1yu_1 - wu_0 = 0$$
$$R_7\colon x_0t - u_0^2 = 0 \qquad R_8\colon x_1t - u_0u_1 = 0$$
$$R_9\colon x_1u_1^2 - yt = 0 \qquad R_{10}\colon yu_1^2 - wt = 0$$

The equation (2) defining $\tilde{Y}$ induces several relations which cut out X' inside this cone:

Lemma 3.9. *There are four relations induced by (2) and these can be written as:*

$$R_{11}\colon \qquad x_0P + x_1^2(\theta u_1z + \tau w^3) + u_0t = 0$$
$$R_{12}\colon \qquad x_1P + y(\theta u_1z + \tau w^3) + u_1t = 0$$
$$R_{13}\colon \qquad yP + w(\theta u_1z + \tau w^3) + u_1^3 = 0$$
$$R_{14}\colon \qquad u_0P + x_1u_1(\theta u_1z + \tau w^3) + t^2 = 0$$

where $P_{10} = z^2 + \dots$ is a general homogeneous form of degree 10 and θ, τ are parameters. The generator g of degree 17 is eliminated by a fifth relation $tP = \cdots + g$.

Proof. The leading term of the equation (2) cutting out $\tilde{Y}$ in $\tilde{\mathcal{F}}$ is $e\zeta^2$. By Lemma 3.6 the unique generator involving ζ is z, and e appears in x_0, x_1, y, u_0, t, thus we expect five relations induced by (2).

We first study the relation R_{11} more precisely. Using Lemma 3.6 we can write in $S[\alpha,\beta,\gamma]$

$$x_0z^2 = \left(\alpha^3\beta^9\gamma^{17}e^5\right)\zeta^2 = \left(\alpha^3\beta^9\gamma^{17}e^4\right)\left(e\zeta^2\right),$$

and multiplying (2) with the excess monomial $\alpha^3\beta^9\gamma^{17}e^4$, we get for the left hand side

$$\begin{aligned}\alpha^3\beta^9\gamma^{17}e^4\left(e\zeta-\theta t_1^3s_0s_1\right)\zeta &= x_0z^2-\theta\left(\alpha^3\beta^4\gamma^7e^2t_1\right)^2\left(\alpha^2\beta\gamma^3t_1s_0\right)\zeta\\ &= x_0z^2-\theta x_1^2u_1z\end{aligned}$$

and for the right hand side

$$\begin{aligned}&\alpha^3\beta^9\gamma^{17}e^4\left(cs_0^3+cet_0k'_{11}s_0s_1^4+t_0(c^2e^3t_0l'_{16}+\tau t_1^{17})s_1^6\right)\\ &=\left(\alpha^2\beta^6\gamma^{13}e^3s_0\right)\left(\alpha\beta^3\gamma^9es_0^2\right)+\tau\left(\alpha^3\beta^4\gamma^7e^2t_1\right)^2\left(\alpha^9\beta^2\gamma t_1^5\right)^3\\ &\quad+\left(\alpha^3\beta^9\gamma^{17}e^5\right)\left(s_1^2\left(ct_0k'_{11}s_0s_1^2+t_0^2c^2e^2l'_{16}s_1^4\right)\right)\\ &=u_0t+\tau x_1^2w^3+x_0\tilde{P}\end{aligned}$$

Setting the two expressions equal and rearranging the terms we get a relation of the form

$$x_0P+x_1^2(\theta u_1z+\tau w^3)+u_0t=0.$$

Note that the monomial u_0t corresponds to the term cs_0^3 from (2), and most of the terms (including x_0z^2 itself) are wrapped up in the general element P of degree 10.

We can repeat this procedure to get the remaining relations, but it is easier to derive them from R_{11}. Using R_1 and R_4 we have: $x_0:x_1=x_1^2:y=u_0:u_1$. Thus writing $R_{12}=\frac{x_1}{x_0}R_{11}$ and applying these ratios gives the claimed relation.

The relation $tP=\dots$ always eliminates g, because g corresponds to the term cs_0^3 in (2), which always appears with nonzero coefficient. ☐

Remark 3.10. *Recall the following characterisation of T-singularities via their index 1 canonical covers:*

$$\frac{1}{dn^2}(1,dna-1)\cong\left[(xy-z^{dn})\subset\frac{1}{n}(1_x,-1_y,a_z)\right],$$

see e.g. [KSB88, Prop. 3.10] *or* [Hac16, Ex. 2.1.8].

Corollary 3.11. *The 14 relations defining $X'\subset\mathbb{P}(1,1,2,3,4,4,5,7)$ fit into a Pfaffian presentation as follows:*

$$\operatorname{Pf}_4 M = MV = 0$$

where

$$M=\begin{pmatrix}0 & 0 & x_0 & x_1^2 & u_0\\ & 0 & x_1 & y & u_1\\ & & u_0 & x_1u_1 & t\\ & & & t & -(\tau w^3+\theta u_1z)\\ -sym & & & & P\end{pmatrix},\quad V=\begin{pmatrix}0\\-u_1^2\\0\\w\\-y\\0\end{pmatrix},$$

and we omit the diagonal zero entries of the 6×6 skew matrix M. For general choices of P_{10} and τ the singular locus of X' is one point of type $\frac{1}{25}(1,14)$.

Proof. Write out the matrix product and the Pfaffians and compare this with the list of relations.

For the singular locus, X' has a covering by orbifold affine charts $U_m\colon (m \neq 0)$ centred at each coordinate point P_m of the ambient space. Despite the number of equations, it is straightforward to show the nonsingularity (except U_z) of each chart. Here are a couple of sample computations. The chart U_y is contained in $U_{x_1} \cap U_w$ because $R_3|_{y=1}$ gives $x_1 w = 1$, thus we ignore U_y. Infact, similar considerations show that $U_{x_1} = U_y \subset U_{x_0} \cap U_w$ and $U_{u_0} \subset U_{x_0} \cap U_t$. Hence we only need to check nonsingularity of four charts U_{x_0}, U_w, U_{u_1} and U_t. For U_{x_0}, we use R_1, R_2, R_4, R_7 to eliminate y, w, u_1, t respectively, giving a hypersurface in $\mathbb{C}^4_{x_1,u_0,z}$ induced by any one of $R_{11}, \dots, R_{14}$. It is then easy to show that this chart is nonsingular. The other three charts work in a similar way.

Since we know that the other charts of X' are nonsingular, we only need to consider the orbifold chart U_z in an analytic neighbourhood of $P_z \in X$. We use $R_{11}, R_{12}, R_{13}, R_{14}$ to eliminate x_0, x_1, y, u_0 respectively. Thus the local coordinates near P_z are w, u_1, t and R_{10} defines X' locally as the hypersurface $u_1^5 - wt + \text{h.o.t} = 0$ in $\frac{1}{5}(3_w, 4_{u_1}, 2_t) \overset{\cdot 2}{\cong} \frac{1}{5}(1,3,4)$ after substituting $y = u_1^3 + \text{h.o.t.}$ using R_{13}. By Remark 3.10, this is a $\frac{1}{25}(1,14)$ singularity. □

Corollary 3.12. *The coordinate ring R' of $X' \subset \mathbb{P}(1,1,2,3,4,4,5,7)$ described in Corollary 3.11 is the canonical ring $R(X, K_X)$ and $X' \cong X$ is an RU-surface of nodal type if $\theta \neq 0$ and of cuspidal type if $\theta = 0$.*

Proof. A standard computer calculation (for instance the command `MinimalFreeResolution` in MAGMA applied to the equations given by Cor. 3.11) shows that the minimal free resolution of $\mathcal{O}_{X'}$ as an $\mathcal{O}_{\mathbb{P}}$-module is

$$0 \to \mathcal{O}(-28) \to \mathcal{O}(-28)\otimes\mathcal{L}_1^\vee \to \mathcal{O}(-28)\otimes\mathcal{L}_2^\vee \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{O} \to \mathcal{O}_{X'},$$

where $\mathcal{L}_1 = \bigoplus_{d\in L_1} \mathcal{O}(-d)$ with

$$L_1 = (3, 4^2, 5, 6, 7, 8^2, 9, 10, 11^2, 12, 14)$$

and $\mathcal{L}_2 = \bigoplus_{d\in L_2} \mathcal{O}(-d)$ with

$$L_2 = (5, 6, 7^2, 8^3, 9^2, 10^3, 11^4, 12^3, 13^4, 14^3, 15^3, 16^2, 17, 18^2, 19).$$

Thus the coordinate ring R' is Cohen–Macaulay by the Auslander–Buchsbaum formula [BH93, §1.3]. Combining this with regularity in codimension 1, which follows from Corollary 3.11, we see that X' is projectively normal by Serre's criterion for normality. The dualising sheaf of X' is $\omega_{X'} = \mathcal{E}xt^5_{\mathcal{O}}(\mathcal{O}_{X'}, \omega_{\mathbb{P}}) = \mathcal{O}_{X'}(28 - 1 - 1 - 2 - 3 - 4 - 4 - 5 - 7) = \mathcal{O}_{X'}(1)$ where $\mathcal{E}xt^5_{\mathcal{O}}(\mathcal{O}_{X'}, \mathcal{O}) = \mathcal{O}(28)$ can be read off from the resolution above [BH93, §3.6]. By projective normality, we obtain $H^0(X, nK_X) \cong H^0(X', \mathcal{O}_{X'}(n))$. Hence we have $X' \cong X$, $p_g(X) =$

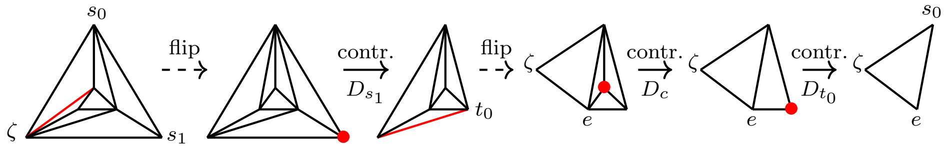


FIGURE 4. A birational map from $\tilde{\mathcal{F}}$ to $\mathbb{P}(1,3,17,25)$. The origin and v_{t_1} are behind the page.

$h^0(\mathcal{O}_{X'}(1)) = 2$, and $K_X^2 = 1$ follows from $P_2(X) = h^0(\mathcal{O}_{X'}(2)) = 4$ and the Riemann–Roch formula of Blache [Bla95]. □

Remark 3.13. *The vanishing of θ corresponds to a cuspidal singular fiber as explained above. The vanishing of τ imposes an extra $\frac{1}{9}(1,5)$ point on X' (these are two independent conditions in moduli).*

Remark 3.14. *As promised in Remark 3.8, we describe the fixed part F of $|K_X|$. Indeed, e divides all generators except w, u_1, z. Thus the image of E is obtained by restricting the relation R_{13} to the locus $x_0 = x_1 = y = u_0 = t = 0$:*

$$F\colon \left(w(\theta u_1 z + \tau w^3) + u_1^3 = 0\right) \subset \mathbb{P}(3_w, 4_{u_1}, 5_z).$$

For general θ, τ, the curve F passes through the index 5 point and has a node there. If $\theta = 0$ then F is a cone with vertex P_z. If $\tau = 0$ then F has two components each passing through P_w and P_z. If $\theta = \tau = 0$ then F is the triple line joining P_w and P_z.

3.7. **A hypersurface in weighted projective space.** We give an alternative description of the general RU-surface, which is algebraically much simpler. The following result is inspired by the observation that $\tilde{\mathcal{F}}$ is birational to $\mathbb{P}(1,3,17,25)$, as can be read of from the last row of the weight matrix (3). The decomposition of this map into extremal contractions is briefly described in Figure 4, where the red loci denote exceptional locus of each factor (for more details on how to compute this, see [CLS11, Chapter 15]). Indeed, modulo flips, the birational map is a composition of three divisorial contractions D_{s_1}, D_c and D_{t_0} whose restrictions to $\tilde{Y}$ contract the T-chain $A + B + C$.

Proposition 3.15. *A general quasismooth hypersurface S of weighted degree 51 in $\mathbb{P}(1,3,17,25)$ is an RU-surface (here quasismooth means that the affine cone is smooth outside of the vertex).*

Proof. By adjunction we have $K_S = \mathcal{O}_S(51-25-17-3-1) = \mathcal{O}_S(5)$ which has two global sections and $K_S^2 = (51 \cdot 5^2)/(1 \cdot 3 \cdot 17 \cdot 25) = 1$.

Write e, t_1, s_0, ζ for the coordinates on the weighted projective space then the equation of degree 51 can be expressed as

$$eP_{50} + \tau t_1^{17} + \theta t_1^3 s_0 \zeta + s_0^3 \tag{4}$$

where $P_{50}(e, t_1, s_0, \zeta)$ is a polynomial of weighted degree 50 and θ, τ are parameters. Note that there are only three forms of degree 51 that are not divisible by e. It is easy to check by hand that S is quasismooth.

Now S contains the index 5 coordinate point $(0, 0, 0, 1)$ and we get a $\frac{1}{25}(3, 17) \cong \frac{1}{25}(1, 14)$ singularity there because since P_{50} contains the monomial ζ^2. If τ is nonzero then S does not contain the singular point of index 3. We assume that the coefficient of s_0^3 is nonzero $(= 1)$ to avoid the index 17 point.

By [RU19, Theorem 3.2 (A1)] the surface constructed is an RU-surface. □

Remark 3.16. *In coordinates, the map $\tilde{\mathcal{F}} \dashrightarrow \mathbb{P}(1, 3, 17, 25)$ is given by:*

$$(s_1, t_0, c, e, t_1, s_0, \zeta) \mapsto (1, 1, 1, e, t_1, s_0, \zeta),$$

and this maps the equation (2) defining $\tilde{Y}$, to the equation (4) defining S_{51}. The degree 51 can also be read off from the multidegree $(6, 18, 34, 51)$ of $\tilde{Y}$ with respect to the weight matrix (3). Thus we have a generically 1-1 map between the moduli spaces parametrising hypersurfaces $\tilde{Y} \subset \tilde{\mathcal{F}}$ and hypersurfaces $S_{51} \subset \mathbb{P}(1, 3, 17, 25)$. In particular, the general element in $\mathfrak{M}_{RU}$ can be realised as such a hypersurface S_{51}.

This agrees with the naive parameter count. In $\mathbb{P}(1, 3, 17, 25)$, the linear system $\mathbb{P}(H^0(\mathcal{O}(51)))$ has dimension 50 and the automorphism group has dimension 22 (an automorphism φ of the weighted projective space $\mathbb{P}(1, 3, 17, 25)$ is given, up to $\mathbb{C}^*$ action, by $\varphi = (P_0, P_1, P_2, P_3)$ where $\deg(P_0) = 1, \deg(P_1) = 3, \deg(P_2) = 17, \deg(P_3) = 25$), which suggests that quasismooth hypersurfaces of weighted degree 51 have 28 moduli.

We now show that the parameters occurring in (4) play the same role as in Section 3.

Proposition 3.17. *The RU-hypersurface with $\tau = 0$ (respectively $\tau = \theta = 0$) has an additional $\frac{1}{9}(1, 5)$ (resp. $\frac{1}{18}(1, 5)$) singularity.*

Proof. Near the index 3 point $(0, 1, 0, 0)$ the local analytic form of S is

$$(\theta s_0 \zeta + e^3 + \text{h.o.t.}) = 0 \subset \frac{1}{3}(1_e, 2_{s_0}, 1_\zeta),$$

if we assume that the monomial $e^2 t_1^{18}$ appears in P_{50}. By Remark 3.10, this is a $\frac{1}{9}(1, 5)$ singularity.

Moreover, since $t_1^{11} s_0$, ζ^2, and $e t_1^8 \zeta$ appear in P_{50}, if $\theta = 0$ we get $(e s_0 + e\zeta^2 + e^2\zeta + \text{h.o.t.} = 0) \subset \frac{1}{3}(1, 2, 1)$ which is a $\frac{1}{18}(1, 5)$ singularity. □

Proposition 3.18. *The canonical model of an RU-hypersurface is the same as the one described in Corollary 3.11.*

Proof. As shown above, $K_S = \mathcal{O}_S(5)$ so we can write out generators and relations for the canonical ring directly:

$$x_0 = e^5,\ x_1 = e^2t_1,\ y = et_1^3,\ w = t_1^5,\ u_0 = e^3s_0,\ u_1 = t_1s_0,$$
$$z = \zeta,\ t = es_0^2,\ g = s_0^5$$

in degrees $1, 1, 2, 3, 4, 4, 5, 7, 17$ respectively.

The easy monomial relations between these generators are the same as in Section 3.6 and the equation of degree 51 can be expressed in terms of these new generators in five different ways by multiplying it with each of $e^4, et_1, e^3t_1^2, t_1^3, s_0^2$ (cf. Lemma 3.9). The last of these five equations involves s_0^5 and therefore we can use it to eliminate the spurious generator g of degree 17 in the same way as Lemma 3.9.

Finally, we can fit these relations into the skew-matrix format of Corollary 3.11. □

Remark 3.19. *After an appropriate coordinate change, P_{50} involves only even powers of ζ. Thus the general RU-hypersurface has an involution $(e, t_1, s_0, \zeta) \mapsto (e, t_1, s_0, -\zeta)$ if and only if θ vanishes. This is another interpretation of the obstruction to $\mathbb{Q}$-Gorenstein smoothing of the general RU-surface cf.* [FPRR21, Prop. 3.18].

4. Cuspidal RU-surfaces are $\mathbb{Q}$-Gorenstein smoothable

In this section we assume that $\theta = 0$, that is, we consider the cuspidal RU-surfaces. We exhibit a $\mathbb{Q}$-Gorenstein smoothing of the general cuspidal RU-surface. Since the relations $R_{11}, \ldots, R_{14}$ are only determined modulo $R_1, \ldots, R_{10}$, we use R_3 to rewrite R_{14} as

$$R_{14}\colon u_0P + \tau y^2w^2u_1 + t^2 = 0.$$

The new relation no longer fits into the previous Pfaffian format. This choice of R_{14} is crucial in finding the $\mathbb{Q}$-Gorenstein smoothing:

Proposition 4.1. *Consider the family $\mathcal{X}/\Lambda$ defined by relations*

$$\tilde{R}_1\colon x_0y - x_1^3 + \lambda^3\tau w = 0 \qquad \tilde{R}_2\colon x_0w - x_1^2y + \lambda u_0 = 0$$
$$\tilde{R}_3\colon x_1w - y^2 + \lambda u_1 = 0 \qquad \tilde{R}_4\colon x_0u_1 - x_1u_0 + \lambda^2\tau yw = 0$$
$$\tilde{R}_5\colon x_1^2u_1 - yu_0 + \lambda^2\tau w^2 = 0 \qquad \tilde{R}_6\colon x_1yu_1 - wu_0 - \lambda t = 0$$
$$\tilde{R}_7\colon x_0t - u_0^2 + \lambda\tau x_1y^2w = 0 \qquad \tilde{R}_8\colon x_1t - u_0u_1 + \lambda\tau yw^2 = 0$$
$$\tilde{R}_9\colon x_1u_1^2 - yt - \lambda\tau w^3 = 0 \qquad \tilde{R}_{10}\colon yu_1^2 - wt + \lambda\tilde{P} = 0$$

$$\tilde{R}_{11}\colon x_0\tilde{P} + \tau x_1^2w^3 + u_0t - \lambda^2\tau wu_1^2 = 0$$
$$\tilde{R}_{12}\colon x_1\tilde{P} + \tau yw^3 + u_1t = 0$$
$$\tilde{R}_{13}\colon y\tilde{P} + \tau w^4 + u_1^3 = 0$$
$$\tilde{R}_{14}\colon u_0\tilde{P} + \tau y^2w^2u_1 + t^2 = 0$$

where λ is the coordinate on $0 \in \Lambda \subset \mathbb{C}$ and $\tilde{P}$ is a general polynomial of degree 10 satisfying $\tilde{P}|_{\lambda=0} = P$. If $\tau \neq 0$ then the central fiber $\mathcal{X}_0$ is a cuspidal Rana–Urzúa surface with a single $\frac{1}{25}(1,14)$ point, and $\mathcal{X}/\Lambda$ is a $\mathbb{Q}$-Gorenstein smoothing of $\mathcal{X}_0$.

Proof. By construction, the fiber over $\lambda = 0$ is a Rana–Urzúa surface because the relations match those of §3.6. By Lemma 4.2 below, the general fiber with $\lambda \neq 0$ is a smooth surface. Hence $\mathcal{X}/\Lambda$ is flat.

The $\mathbb{Q}$-Gorenstein condition is only relevant near the singular point of $\mathcal{X}$ at P_z over $\lambda = 0$. Substituting $z = 1$ into the equations $\tilde{R}_{10}$, $\tilde{R}_{11}$, $\tilde{R}_{12}$, $\tilde{R}_{13}$, $\tilde{R}_{14}$ allows us to eliminate x_0, x_1, y, u_0 respectively in the same way as the proof of Corollary 3.11, so that we are left with $(u_1^5 - wt + \lambda + \text{h.o.t.} = 0)$ in $\frac{1}{5}(1,3,4) \times \Lambda$. Thus the family $\mathcal{X}/\Lambda$ induces a $\mathbb{Q}$-Gorenstein smoothing of the $\frac{1}{25}(1,14)$ point on the fiber over $\lambda = 0$ (see e.g. [Hac16, Ex. 2.1.8]). □

Lemma 4.2. *The family $\mathcal{X}/\Lambda$ fits into the matrix format*

$$\operatorname{Pf}_4 M_1 = \operatorname{Pf}_4 M_2 = M_1 V_1 = M_2 V_2 = 0,$$

where

$$M_1 = \begin{pmatrix} \lambda & x_1 & w & u_1 \\ & u_0 & yu_1 & t \\ & & t & -\tau y w^2 \\ -sym & & & \tilde{P} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \lambda & x_1 & y & t \\ & y & w & u_1^2 \\ & & -u_1 & \tau w^3 \\ -sym & & & -\tilde{P} \end{pmatrix},$$

$$V_1 = {}^t\left(-\lambda\tau y^2 w, \quad u_0, \quad -x_1 y, \quad x_0, \quad 0\right),$$

$$V_2 = {}^t\left(u_0, \quad -\lambda^2\tau w, \quad x_1^2, \quad -x_0, \quad 0\right).$$

If $\lambda \neq 0$ then the fiber $\mathcal{X}_\lambda$ is isomorphic to a nonsingular hypersurface of weighted degree 10 in $\mathbb{P}(1,1,2,5)$.

Proof. Up to sign, the Pfaffians of M_1 are $\tilde{R}_{10}$, $\tilde{R}_{14}$, $\tilde{R}_{12}$, $\tilde{R}_6$, $\tilde{R}_8$ and the Pfaffians of M_2 are $\tilde{R}_{10}$, $\tilde{R}_{13}$, $\tilde{R}_{12}$, $\tilde{R}_3$, $\tilde{R}_9$. The product $M_1 V_1$ gives $\tilde{R}_2$, $y\tilde{R}_4$, $\tilde{R}_7$, $y\tilde{R}_8$, $\tilde{R}_{11} + \tau w(x_1 w - \lambda u_1)\tilde{R}_3$ and $M_2 V_2$ gives $\tilde{R}_1$, $\tilde{R}_2$, $\tilde{R}_4$, $\tilde{R}_5$, $\tilde{R}_{11}$. Thus taken all together these generate the ideal defining $\mathcal{X}/\Lambda$.

Assume now that $\lambda \neq 0$. We perform row and column operations on M_i preserving antisymmetry, and apply the complementary row operations to V_i so that the products $M_i V_i$ are preserved. This gives new matrices M_i' and V_i':

$$M_1' = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & \frac{\tilde{R}_6}{\lambda} & \frac{\tilde{R}_8}{\lambda} \\ -\mathrm{sym} & & & \frac{\tilde{R}_{10}}{\lambda} \end{pmatrix}, \quad M_2' = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & \frac{\tilde{R}_3}{\lambda} & \frac{\tilde{R}_9}{\lambda} \\ -\mathrm{sym} & & & \frac{\tilde{R}_{10}}{\lambda} \end{pmatrix}$$

$$V_1' = {}^t\left(\frac{y\tilde{R}_4}{\lambda}, \quad \frac{\tilde{R}_2}{\lambda}, \quad -x_1 y, \quad x_0, \quad 0\right), \quad V_2' = {}^t\left(\frac{\tilde{R}_2}{\lambda}, \quad \frac{\tilde{R}_1}{\lambda}, \quad x_1^2, \quad -x_0, \quad 0\right).$$

Thus the format reduces to $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_6, \tilde{R}_8, \tilde{R}_9, \tilde{R}_{10}, y\tilde{R}_4$.

Moreover, the assumption $\lambda \neq 0$ enables us to rewrite w, u_0, u_1, t using relations $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_6$ respectively:

$$x_0 y - x_1^3 = -\lambda^3 \tau w \qquad x_0 w - x_1^2 y = -\lambda u_0$$
$$x_1 w - y^2 = -\lambda u_1 \qquad x_1 y u_1 - w u_0 = \lambda t$$

Doing this transforms $\tilde{R}_8, \tilde{R}_9, y\tilde{R}_4$ into identities rather than relations. For example $\tilde{R}_4$ reduces to the identity:

$$\begin{aligned} & x_0 u_1 - x_1 u_0 + \lambda^2 \tau y w \\ \equiv\ & x_0 \left(\frac{x_1 w - y^2}{-\lambda} \right) - x_1 \left(\frac{x_0 w - x_1^2 y}{-\lambda} \right) + \lambda^2 \tau y \left(\frac{x_0 y - x_1^3}{-\lambda^3 \tau} \right) \\ \equiv\ & \frac{1}{\lambda} \left(-x_0 x_1 w + x_0 y^2 + x_1 x_0 w - x_1^3 y - x_0 y^2 + x_1^3 y \right) \\ \equiv\ & 0. \end{aligned}$$

The remaining relation is $\tilde{R}_{10}$. By repeatedly using $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_6$ to eliminate w, u_0, u_1, t as above, we are left with a relation between generators x_0, x_1, y, z, which we display here, multiplied by λ^{11} for readability:

$$x_0(x_0y - x_1^3)^3 - 3\lambda^3 x_1^2 y (x_0 y - x_1^3)^2 + 3\lambda^6 \tau^{-1} x_1 y^3 (x_0 y - x_1^3) + \lambda^9 y^5 + \lambda^{12} \tilde{P} = 0.$$

Since $\tilde{P}$ is general, this is a nonsingular surface. □

The last equation in the proof can be interpreted as follows:

Corollary 4.3. *Let $\mathcal{X} \to \Lambda$ be the 1-parameter smoothing of a cuspidal RU-surface X_0 constructed above. Then there is a diagram*

$$\begin{array}{ccccc} \mathcal{X} & \overset{\phi}{\dashrightarrow} & \tilde{\mathcal{X}} & \overset{\deg 10}{\hookrightarrow} & \mathbb{P}(1,1,2,5) \times \Lambda \\ & \searrow & \downarrow & \swarrow & \\ & & \Lambda & & \end{array}$$

where ϕ is birational and an isomorphism outside the central fibers X_0 and $\tilde{X}_0 = (x_0(x_1^3 - x_0 y)^3 = 0)$.

This phenomenon could be taken as a starting point for a comparison of the closure of the Gieseker component and a GIT moduli space of hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$, which could possibly be constructed using the techniques of [BDHK18].

5. I-surfaces with two T-singularities

In this Section, we apply and extend the results of Section 3 to understand how T-divisors in $\overline{\mathfrak{M}}_{1,3}$ intersect each other. We recall that for an I-surface X with a unique non-canonical T-singularity Q, the point Q is as in Table 1 by [FPRR21, Thm. 1.1].

Table 1. T-singularities $\frac{1}{dn^2}(1, dna-1)$ occurring individually

Cartier index n	d	T-singularity	T-string
2	$d \leq 32$	$\frac{1}{4d}(1, 2d-1)$	$[4]$ or $[3,3]$ or $[3,2,\ldots,3]$
3	2	$\frac{1}{18}(1,5)$	$[4,3,2]$
5	1	$\frac{1}{25}(1,14)$	$[2,5,3]$

Table 2. Codiscrepancy divisors

T-singularity	Δ_j
$\frac{1}{4}(1,1)$	$\frac{1}{2}A_j$ with $A_j^2 = -4$
$\frac{1}{18}(1,5)$	$\frac{2}{3}A_j + \frac{2}{3}B_j + \frac{1}{3}C_j$ with $A_j^2 = -4, B_j^2 = -3, C_j^2 = -2$
$\frac{1}{25}(1,14)$	$\frac{3}{5}A_j + \frac{4}{5}B_j + \frac{2}{5}C_j$ with $A_j^2 = -3, B_j^2 = -5, C_j^2 = -2$

Suppose X is an I-surface with two distinct T-singularities Q_1 and Q_2 of index i_1, resp. i_2 (with $i_1 \neq i_2$) belonging to the above list, where for simplicity of exposition we restrict to the case $d = 1$ in the index 2 case. In particular, these surfaces should correspond to general points in the intersection of two T-divisors in $\overline{\mathfrak{M}}_{1,3}$.

We let $f\colon \widetilde{Y} \to X$ be the minimal desingularization and $\epsilon\colon \widetilde{Y} \to Y$ be the morphism to a minimal model:

$$X \xleftarrow[\text{resolution}]{\;f\;} \widetilde{Y} \xrightarrow[\text{sequence of blow-ups}]{\;\epsilon\;} Y \tag{5}$$

Here ϵ is the composition of k blow-ups at $P_1, \ldots, P_k$ (possibly infinitely near). We will denote the exceptional divisor of ϵ by $E = \sum_{i=1}^k E_i$, where the E_i's are the (-1)-curves of each blow-up (possibly not irreducible, nor reduced). In particular, $K_{\widetilde{Y}} = \epsilon^* K_Y + E$.

We write

$$f^* K_X = K_{\tilde{Y}} + \Delta$$

where $\Delta = \Delta_1 + \Delta_2$ is the codiscrepancy divisor of f with Δ_j supported on $f^{-1}(Q_j)$. These are chains of smooth rational curves with self-intersections and coefficients as in Table 2.

Since X has only rational singularities then we have $q(Y) = q(\widetilde{Y}) = q(X) = 0$ and $p_g(Y) = p_g(\widetilde{Y}) = p_g(X) = 2$, so the Kodaira dimension of $\widetilde{Y}$ is positive and the minimal model Y is unique. Arguing as in [FPRR21, §4] we see that Y is a properly elliptic surface and moreover, by [Lee99, Prop. 20], we have

$$K_{\widetilde{Y}}^2 = 1 + (\Delta_1)^2 + (\Delta_2)^2 = d_1 - r_1 + d_2 - r_2 - 1 \tag{6}$$

where r_j is the length of the T-string and d_j is as in Table 1.

5.1. **Useful results.** We summarize some results that will be used throughout this section. First we recall some well known facts about the structure of the ϵ-exceptional divisors.

Remark 5.1. *Let Γ, Γ' be two distinct irreducible (-1)-curves on $\widetilde{Y}$. Then $\Gamma \cap \Gamma' = \emptyset$ since $\widetilde{Y}$ has Kodaira dimension 1.*

Remark 5.2. *Let $E = \sum_{i=1}^k E_i$ be the exceptional divisor of ϵ, where the E_i's are the (-1)-curves of each blow-up. Then every E_i contains at least one irreducible (-1)-curve Γ_i. Moreover if $D \subset E$ is an ϵ-exceptional irreducible $(-n)$-curve with $n \geq 2$, then $E - D \geq \sum m_h \Gamma_h$, with Γ_h's irreducible (-1)-curves and $\sum m_h \geq k$.*

Now we show some properties of the components of the T-strings (i.e. f-exceptional divisors) and the ϵ-exceptional divisors.

From now on, we abuse notation and denote the pull back of K_Y to $\widetilde{Y}$ by K_Y, and the pull back of K_X to $\widetilde{Y}$ by K_X. With this convention, we can write

$$K_X = K_{\widetilde{Y}} + \Delta_1 + \Delta_2 = K_Y + E + \Delta_1 + \Delta_2.$$

Remark 5.3. *Let $\Gamma \subset \widetilde{Y}$ be an ϵ-exceptional irreducible (-1)-curve. Then $K_X \Gamma > 0$.*

Lemma 5.4. *Let $D \subset \widetilde{Y}$ be an irreducible (-2)-curve.*

Then $K_Y D = ED = 0$. In particular:

(i) if $D \subset \widetilde{Y}$ is not contracted by ϵ, then $\epsilon(D) \subset Y$ is again a (-2)-curve and $D \cap E = \emptyset$;

(ii) if $D \subset \widetilde{Y}$ is contracted by ϵ, then D intersects only one (-1)-curve $\Gamma \subset E$ and $D\Gamma = 1$.

Proof. Since $D^2 = -2$, by adjunction we have

$$0 = K_{\widetilde{Y}} D = K_Y D + ED$$

and $K_Y D \geq 0$ because K_Y is nef.

If D is not ϵ-exceptional then the second summand is non-negative, giving $ED = 0$ and the first item.

If D is ϵ-exceptional then the first summand is zero, so also the second. To conclude the proof of *(ii)* assume that D intersects two distinct (-1)-curves Γ, Γ'. Contracting Γ and Γ', D becomes a curve with non-negative self intersection and negative intersection with the canonical divisor. This is absurd since Y is an elliptic surface. If

$D\Gamma \geq 2$ then contracting Γ we obtain a curve which is not ϵ-exceptional and has negative intersection with the canonical divisor, impossible on the minimal surface Y. □

Lemma 5.5. *Let $D \subset \widetilde{Y}$ be an irreducible (-3)-curve that is not ϵ-exceptional. Then $\epsilon(D) \subset Y$ is either a (-3)-curve or a (-2)-curve.*

Proof. Since $D^2 = -3$, by adjunction we have

$$1 = K_{\widetilde{Y}}D = K_Y D + ED,$$

$K_Y D \geq 0$ because K_Y is nef and $ED \geq 0$ since D is not exceptional. If $ED = 0$, then $\epsilon(D)$ is again a -3-curve. So assume $K_Y D = 0$ and $ED = 1$ and write $D = \epsilon^*(\epsilon(D)) - \sum m_i E_i$, with $m_i \geq 0$. We have $1 = DE = \sum m_i$, so D is obtained by blowing up $\epsilon(D)$ once at a smooth point and $\epsilon(D)$ is a -2-curve. □

Lemma 5.6. *We have*

$$K_X K_Y = (\Delta_1 + \Delta_2)K_Y \geq \frac{1}{2}, \quad K_X E \leq \frac{1}{2}$$

Proof. We have

$$K_X K_Y = (K_Y + E + \Delta_1 + \Delta_2)K_Y = (\Delta_1 + \Delta_2)K_Y$$

since $K_Y^2 = K_Y E = 0$. Moreover, we have $K_X K_Y > 0$ because K_Y moves and K_X is positive outside the support of the f-exceptional locus. Looking at the description of the codiscrepancy divisors we note that the divisors with coefficient less than $\frac{1}{2}$ are (-2)-curves. Then by the above Lemma 5.4 we obtain $(\Delta_1 + \Delta_2)K_Y \geq \frac{1}{2}$. The second inequality follows since $1 = K_X^2 = K_X(K_Y + E)$. □

Proposition 5.7. *Let $D \subset E \subset \widetilde{Y}$ be an ϵ-exceptional irreducible $(-n)$-curve, with $n \geq 2$. Then D is also f-exceptional.*

Proof. Suppose not, then we have $0 < K_X D = K_{\widetilde{Y}}D + (\Delta_1 + \Delta_2)D = n - 2 + (\Delta_1 + \Delta_2)D$. If $n \geq 3$ then we get $K_X D \geq 1$, which is absurd since $K_X D \leq K_X E \leq \frac{1}{2}$ by Lemma 5.6.

Therefore we may assume $n = 2$. We first consider the case where Q_1, Q_2 are T-singularities of index $i_1 = 5$, resp. $i_2 = 2$, so that by equation (6), $\epsilon\colon \widetilde{Y} \to Y$ is the composition of three blow-ups. In this case $K_X D = (\Delta_1 + \Delta_2)D \geq \frac{2}{5}$ (see the codiscrepancies shown above). By Remark 5.2 and Remark 5.3, since we have three blow-ups and for every irreducible (-1)-curve $\Gamma \subset E$ it is $K_X\Gamma \geq \frac{1}{10}$, we obtain $K_X(E - D) \geq \frac{3}{10}$. Whence $K_X E = K_X D + K_X(E - D) \geq \frac{2}{5} + \frac{3}{10} > \frac{1}{2}$, which contradicts Lemma 5.6.

The cases $(i_1, i_2) = (5, 3)$ and $(i_1, i_2) = (3, 2)$ are similar. □

Corollary 5.8. *Let $D \subset E \subset \widetilde{Y}$ be an ϵ-exceptional irreducible $(-n)$-curve. If $n \geq 2$ then $K_X D = 0$; if $n = 1$ then $K_X D \geq \frac{1}{i_1 i_2}$.*

5.2. **I-surfaces with a singularity of type $\frac{1}{25}(1,14)$ and a singularity of index 2 do not exist.** In this section we are going to prove the following

Proposition 5.9. *There are no I-surfaces with a singularity of type $\frac{1}{25}(1,14)$, a singularity of type $\frac{1}{4}(1,1)$ and smooth elsewhere.*

Remark 5.10. *A generalisation of the below proof shows that there are no I-surfaces with a singularity $\frac{1}{25}(1,14)$, a singularity of type $\frac{1}{4d}(1,2d-1)$ and smooth elsewhere. We do not include the details here but it involves keeping track of the possible intersections with the index 2 T-chain. Moreover, we do not know if there is an I-surface with more general singularities of index 5 and index 2.*

Proof. Assume by a contradiction the existence of such a surface and consider the diagram 5 where the resolution of the two singular points yields a string of type $[3,5,2]$ and a string of type $[4]$.

The strategy of the proof consists in studying the possible configuration of ϵ-exceptional irreducible (-1)-curves.

First note that by equation (6), $\epsilon\colon \widetilde{Y} \to Y$ is a composition of three blow-ups because $K^2_{\widetilde{Y}} = -3$. Let Q_1 be the point of index 5 and Q_2 be the point of index 2. The codiscrepancy divisor corresponding to Q_1 (respectively Q_2) is $\Delta_1 = \frac{3}{5}A_1 + \frac{4}{5}B_1 + \frac{2}{5}C_1$ (resp. $\Delta_2 = \frac{1}{2}A_2$). Thus we can write

$$K_X = K_Y + \sum_{i=1}^{3} E_i + \Delta_1 + \Delta_2.$$

By Remark 5.2, Lemma 5.6, Corollary 5.8 we have $\frac{3}{10} \le K_X E \le \frac{1}{2}$ and $\frac{1}{2} \le K_X K_Y \le \frac{7}{10}$. Now

$$K_X K_Y = (\Delta_1 + \Delta_2) K_Y = (\frac{3}{5}A_1 + \frac{4}{5}B_1 + \frac{2}{5}C_1 + \frac{1}{2}A_2) K_Y,$$

hence the only possibilities for $K_X K_Y$ are $\frac{1}{2}, \frac{3}{5}$ and for $K_X E$ we get:

$$K_X E = \frac{1}{2} \quad \text{or} \quad K_X E = \frac{2}{5}. \tag{7}$$

Now let Γ be an ϵ-exceptional irreducible (-1)-curve. We have $K_X \Gamma \ge \frac{1}{10}$ since X has index 10. Now, since $K_X(E - \Gamma) \ge \frac{2}{10}$ we obtain $K_X \Gamma \le \frac{3}{10}$. Hence since $K_X = K_{\widetilde{Y}} + \Delta_1 + \Delta_2$ and $K_{\widetilde{Y}}\Gamma = -1$, we get $\frac{11}{10} \le (\Delta_1 + \Delta_2)\Gamma \le \frac{13}{10}$.

We exclude the cases where $\Gamma C_1 = 2, 3$ and $\Gamma A_1 = 2$ since they contradict Lemma 5.4(i) and Lemma 5.5. Therefore we are left with

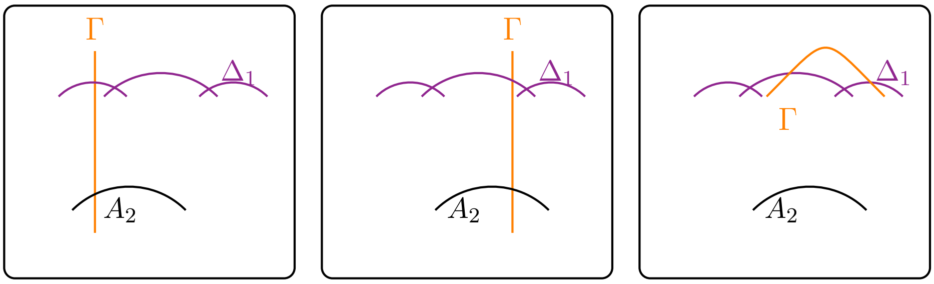


FIGURE 5. Configurations of type (*I*), (*III*) and (*II*)

the following possibilities:

$$(\Delta_1+\Delta_2)\Gamma=\tfrac{3}{5}+\tfrac{1}{2} \;=\; \tfrac{3}{5}A_1\Gamma+\tfrac{1}{2}A_2\Gamma \;;\; K_X\Gamma=\tfrac{1}{10} \qquad (I)$$

$$(\Delta_1+\Delta_2)\Gamma=\tfrac{4}{5}+\tfrac{2}{5} \;=\; \tfrac{4}{5}B_1\Gamma+\tfrac{2}{5}C_1\Gamma \;;\; K_X\Gamma=\tfrac{2}{10} \qquad (II)$$

$$(\Delta_1+\Delta_2)\Gamma=\tfrac{4}{5}+\tfrac{1}{2} \;=\; \tfrac{4}{5}B_1\Gamma+\tfrac{1}{2}A_2\Gamma \;;\; K_X\Gamma=\tfrac{3}{10} \qquad (III)$$

We will show that any of the above configurations gives a contradiction. Note that, since in view of Prop. 5.7 none of the Γ's gives a three step contraction on its own, we need to combine them.

Configuration (II). Assume that there is a curve Γ of type (*II*). Since $K_X(\sum_{i=1}^3 E_i)=\frac{1}{2}$ or $\frac{2}{5}$, there is a second exceptional curve Γ' of type (*I*).

First blow down Γ. Then the image of C_1 is a (-1)-curve which intersects the image of B_1 in two points. Blowing this down we obtain a nodal curve which intersects the image of A_1 in 1 point.

Now blow down Γ'. We obtain a minimal elliptic surface Y with $\epsilon(A_1)K_Y=\epsilon(B_1)K_Y=0$, which implies that $\epsilon(A_1)+\epsilon(B_1)$ is contained in a fiber, contradicting Kodaira's list.

So a type (*II*) configuration does not exist.

Configuration (*III*). Consider a type (*III*) curve Γ. Since a type (*II*) configuration does not exist, then by equation (7) there is a second exceptional curve Γ' of type (*I*).

Then blowing down Γ and Γ' we see that there are no (-1)-curves arising from $\Delta_1+\Delta_2$. Hence there exists a third curve Γ'' of type (I). Blowing it down, we obtain another (-1)-curve, which is absurd.

So a type (*III*) configuration does not exist.

Configuration (*I*). We are left with the case where all the exceptional curves are of type (*I*). If there exist two curves Γ,Γ' of type (*I*), then blowing them down and arguing as in the previous case we obtain a curve having negative intersection with the canonical divisor, which is absurd. Since, as we noticed earlier, there are at least two irreducible -1-curves on $\widetilde{Y}$, the proof is complete. □

5.3. **The divisor of surfaces with one singularity of index 3.** In this subsection we study the divisor of surfaces with an additional singularity of index 3, and show that it fits into the original Pfaffian format of §3.6.

Lemma 5.11. *In the notation of §3.6, if $\tau = 0$ and P_{10}, θ are general then X has a $\frac{1}{9}(1,5)$ singularity in addition to the $\frac{1}{25}(1,14)$ singularity.*

Proof. The index 3 coordinate point P_w is contained in X because w^4 no longer appears in R_{13} when $\tau = 0$. In a neighbourhood of P_w, the relations R_2, R_3, R_6, R_{10} eliminate x_0, x_1, u_0, t respectively. Thus the local coordinates at P_w are y, u_1, z. Since $\theta \neq 0$ in general, R_{13} locally defines X as $u_1 z = y^3 + \dots$ in $\frac{1}{3}(1_{u_1}, 2_z, 2_y)$ (because in general the monomial y^2w^2 appears in P_{10}). By Remark 3.10, this is a $\frac{1}{9}(1,5)$-singularity. □

Proposition 5.12. *Suppose that $\tau = 0$ and consider the surfaces $X_{\lambda,\theta}$ in $\mathbb{P}(1,1,2,3,4,4,5,7)$ defined by*

$$\operatorname{Pf}_4 \tilde{M} = \tilde{M}\tilde{V} = 0$$

where

$$\tilde{M} = \begin{pmatrix} 0 & 0 & x_0 & x_1^2 & u_0 \\ & 0 & x_1 & y & u_1 \\ & & u_0 & x_1u_1 & t \\ & & & t & \theta u_1 z \\ & & & & P \\ -sym & & & & \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} 0 \\ -u_1^2 \\ 0 \\ w \\ -y \\ \lambda \end{pmatrix}$$

and λ, θ are parameters satisfying $\lambda\theta = 0$.

If $\lambda = \theta = 0$ then $X_{0,0}$ is a surface with one $\frac{1}{25}(1,14)$-singularity and one $\frac{1}{18}(1,5)$-singularity.

If $\lambda \neq 0$, $\theta = 0$ then $X_{\lambda,0}$ is a surface $X_{3,10} \subset \mathbb{P}(1,1,2,3,5)$ with one $\frac{1}{18}(1,5)$ singularity.

If $\lambda = 0$, $\theta \neq 0$ then $X_{0,\theta}$ is an RU-surface with an extra $\frac{1}{9}(1,5)$ singularity as in Lemma 5.11.

Proof. Clearly, if $\theta \neq 0$ then we are in the situation described by Lemma 5.11.

If $\lambda \neq 0$, we can adapt the proof of Proposition 4.1. This time, since $\tau = 0$, the generator w can not be eliminated. It turns out that the general fiber $X_{\lambda,0}$ has equations

$$X_{\lambda,0}\colon (x_0y - x_1^3 = \lambda^3 P + y^5 - 3x_1y^3w + 3x_1^2yw^2 - x_0w^3 = 0) \subset \mathbb{P}(1,1,2,3,5).$$

That is, $X_{\lambda,0}$ has a $\frac{1}{18}(1,5)$ singularity (see [FPRR21]).

If $\theta = \lambda = 0$ then we get a RU-surface with an extra singularity of index 3. Near the coordinate point P_w we use R_2, R_3, R_6, R_{10} to eliminate x_0, x_1, u_0, t respectively. Then R_{13} cuts out a hypersurface

$$(yu_1 + yz^2 + y^2z + \cdots = 0) \subset \frac{1}{3}(1_{u_1}, 2_y, 2_z),$$

because the monomial w^2u_1 appears in P_{10}. This is local analytically a $\frac{1}{18}(1,5)$-singularity. □

We have thus shown that both the Gieseker component and the RU-component contain in their closures a divisor parametrising surfaces with an additional T-singularity of index 3 and these divisors meet the intersection divisor, i.e., the divisor of cuspidal RU-surfaces in an irreducible subset of codimension two parametrising surfaces with one $\frac{1}{25}(1,14)$-singularity and one $\frac{1}{18}(1,5)$-singularity.

These correspond to the central fiber of the family $(\lambda = \tau = \theta = 0)$ and their minimal resolution is an elliptic surface with a (-3)-section and a singular fiber of type III (see Lemma 3.2). Thus geometrically, these surfaces can be obtained as follows.

Example 5.13. *We consider an elliptic surface with $p_g = 2$, a fiber of type III, and a (-3)-section.*

We blow-up the singular point p_1 and its infinitesimal point p_2 given by the intersection of the two branches of the singular fiber. We blow-up two more points as shown in Figure 6.

With this procedure we obtain a string $[2,5,3]$ and a string $[4,3,2]$ connected by a (-1)-curve. Blowing down the two strings we obtain an I-surface with a singularity of type $\frac{1}{18}(1,5)$ and a singularity of type $\frac{1}{25}(1,14)$.

There is a second way to construct an I-surface with a singularity of type $\frac{1}{18}(1,5)$ and a singularity of type $\frac{1}{25}(1,14)$. Algebraically, we assume that $\lambda = \tau = 0$ and the coefficient of t_1^{16} in $l'_{16}(t_0,t_1)$ vanishes (again, see Lemma 3.2). Then the elliptic surface Y has an I_3 fiber over $(0,1;0,1,0)$. Since θ is generic here, this example is not smoothable as can also be deduced from the non-existence of an involution.

Example 5.14. *We consider an elliptic surface with $p_g = 2$, a fiber of type I_3, and a (-3)-section.*

We blow-up the singular points p_1, p_2 of the singular fiber, and two infinitesimal point p_3, p_4 as shown in Figure 7.

With this procedure we obtain a string $[2,5,3]$ and a string $[4,3,2]$ connected by two (-1)-curves. Blowing down the two strings we obtain an I-surface with a singularity of type $\frac{1}{18}(1,5)$ and a singularity of type $\frac{1}{25}(1,14)$.

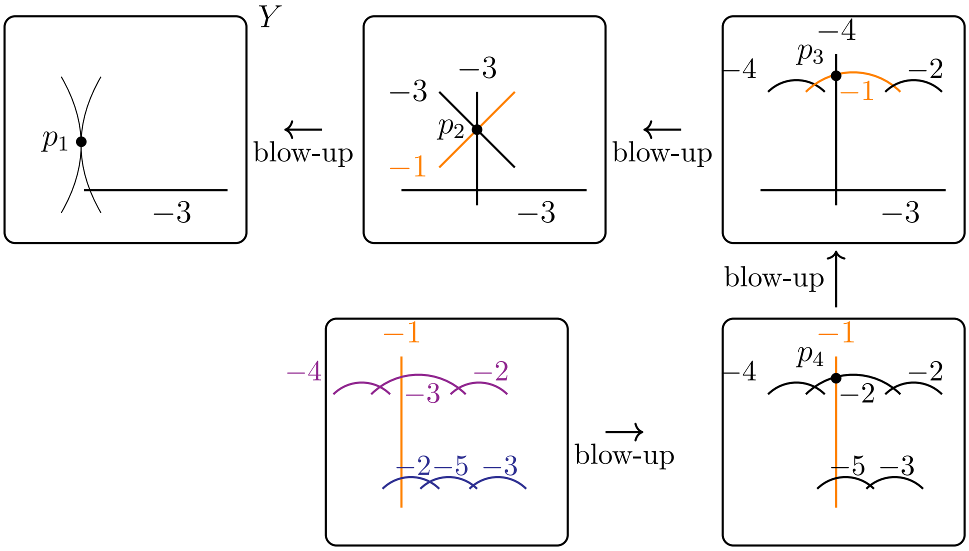


FIGURE 6. Construction of an I-surface with a singularity of type $\frac{1}{18}(1,5)$ and a singularity of type $\frac{1}{25}(1,14)$.

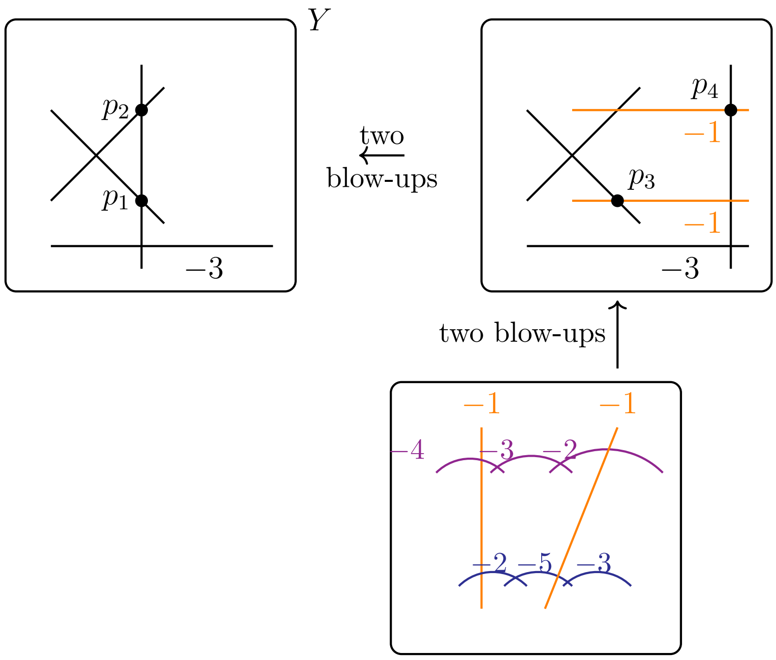


FIGURE 7. Construction of an I-surface with a singularity of type $\frac{1}{18}(1,5)$ and a singularity of type $\frac{1}{25}(1,14)$.

Remark 5.15. *Let X be an I-surface with a singularity of type $\frac{1}{18}(1,5)$ and a singularity of type $\frac{1}{25}(1,14)$. Arguing as in Proposition 5.9 one can see that X is as in the above Examples 5.13, 5.14.*

5.4. **I-surfaces with a singularity of index 2 and a singularity of index 3.** Let us write the results of [FPRR21] in a slightly different form to exhibit clearly the intersection of the index 2 and index 3 divisors: consider the surfaces

$$X = X_{\mu,\nu,\tilde{f}} \colon \begin{pmatrix} x_0 y - x_1^3 - \mu u = 0 \\ z^2 - \nu y^5 - \tilde{f}_{10}(x_0, x_1, y, u) = 0 \end{pmatrix} \subset \mathbb{P}(1,1,2,3,5),$$

where μ, ν are parameters and $\tilde{f}_{10}$ is sufficiently general but not containing the monomial y^5. This is an admissible family of stable surfaces and

- for $\mu\nu \neq 0$ we can eliminate the variable u via the first equation and get a classical I-surface;
- for $\mu \neq 0$, $\nu = 0$, we again eliminate u but the branch divisor passes through the vertex of the cone and we acquire an $\frac{1}{4}(1,1)$ point;
- for $\mu = 0$, $\nu \neq 0$, we get a general member in the divisor parametrising I-surfaces with an $\frac{1}{18}(1,5)$ point;
- for $\mu = \nu = 0$ we get an I-surface with both an $\frac{1}{4}(1,1)$ point and an $\frac{1}{18}(1,5)$ point. Indeed, at P_y, the second equation reduces to $z^2 - u^2 + x_1^2 + \text{h.o.t.} = 0$ in $\frac{1}{2}(1_{x_1}, 1_u, 1_z)$ after eliminating x_0 using x_1^3 via the first equation. This is a $\frac{1}{4}(1,1)$ singularity by Remark 3.10.

A small dimension count thus confirms that these two T-divisors in the closure of the Gieseker components intersect as expected in an irreducible subset of codimension two whose general element is a surface as in the last item.

We complement this algebraic discussion by an explicit geometric construction of surfaces in the intersection.

Example 5.16. *We consider an elliptic surface with $p_g = 2$, an I_2 fiber, an I_r fiber ($r \geq 2$) and a (-3)-section.*

We blow-up the singular points p_1, p_2 of the fiber of type I_2.

With this procedure we obtain a string $[4]$ and a string $[4,3,2]$ connected by two (-1)-curves (see Figure 8). Blowing down the two strings we obtain an I-surface with a singularity of type $\frac{1}{18}(1,5)$ and a singularity of type $\frac{1}{4}(1,1)$.

Remark 5.17. *Let X be an I-surface with a singularity of type $\frac{1}{18}(1,5)$ and a singularity of type $\frac{1}{4}(1,1)$. Arguing as in Proposition 5.9 one can see that X is as in the above Example 5.16.*

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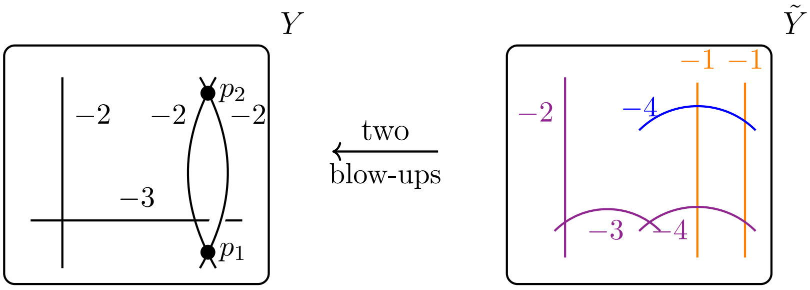

FIGURE 8. Construction of an I-surface with a singularity of type $\frac{1}{18}(1,5)$ and a singularity of type $\frac{1}{4}(1,1)$.

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