Esame di ALGEBRA LINEARE - anno accademico 2022/2023
Corso di laurea in Ingegneria Gestionale
Prova scritta del 20/12/2022
TEMPO A DISPOSIZIONE: 120 minuti

(Cognome) (Nome) (Numero di matricola)

PRIMA PARTE

PUNTEGGIO : risposta mancante = 0 ; risposta esatta = +1 risposta sbagliata = -1
calcoli e spiegazioni non sono richiesti

- Calcolare i^{31} = $\boxed{-i}$

- Sia $z = 1 - i\sqrt{3}$. Scrivere z nella rappresentazione trigonometrica $z = \varrho \cdot e^{i\vartheta}$: $z = \boxed{2e^{i(-\frac{\pi}{3})}}$

Dati W e Z i seguenti sottospazi di $\mathbb{R}^3$:

$$W = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \; : \; x_1 + 4x_2 - 5x_3 = 0 \} , \; Z = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle .$$

- Allora $\mathbb{R}^3 = W \bigoplus Z$ | vero | falso |

- Determinare una base di W :

- $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \Longrightarrow \text{rg}(A) = \boxed{3} \qquad \dim(Ker(\mathcal{L}_A)) = \boxed{1}$

- $\det \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & -3 & 1 & 0 \end{pmatrix} = \boxed{5}$

- $A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 5 & 0 & 1 & 5 \end{pmatrix} \Longrightarrow m.g.(5) = \boxed{2}$

- $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \Longrightarrow A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$

- Sia $f : \mathbb{R}^2 \to \mathbb{R}^2$ lineare. Sapendo che $f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $f \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$, allora $f \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

20/12/2022

Traccia soluzioni

(1)
$$\begin{cases} (z-1)^4 = -32\,|z-1|^2 \\ |e^{iz}| \geq 1 \end{cases}$$

(i) Poniamo $w = z-1$ $\quad (z = w+1)$

eq $\Leftrightarrow$ $w^4 = -32\,|w|^2$

$w = \rho \cdot e^{i\vartheta} \Rightarrow$ eq. : $\rho^4 e^{i4\vartheta} = 32 \cdot e^{i\pi} \cdot \rho^2$

$$\Leftrightarrow \begin{cases} \rho^4 = 32\,\rho^2 \\ 4\vartheta = \pi + 2k\pi \quad , k \in \mathbb{Z} \end{cases}$$

sol. distinte:

$\rho = 0$
cioè
$w = 0$

$$\begin{cases} \rho = \sqrt{32} = 4 \cdot \sqrt{2} \\ \vartheta = \dfrac{\pi}{4} + \dfrac{2k\pi}{4} \quad k = 0, 1, 2, 3 \end{cases}$$

$$w_0 = 4 + i4 \qquad\qquad z_0 = 5 + i4$$

$$w_1 = -4 + i4 \qquad\qquad z_1 = -3 + i4$$

$$w_2 = -4 - i4 \quad \longrightarrow \quad z_2 = -3 - i4$$

$$w_3 = 4 - i4 \qquad\qquad z_3 = 5 - i4$$

$$w_4 = 0 \qquad\qquad z_4 = 1$$

(ii) $$\left| e^{iz} \right| = \left| e^{i(x+iy)} \right| =$$

$$\left| e^{ix-y} \right| = \left| e^{ix} \right| \cdot \left| e^{-y} \right| = e^{-y}$$

poiché $\left| e^{ix} \right| = \sqrt{\cos^2 x + \sin^2 x} = 1$

$e^{-y} > 0$

Quindi $\left| e^{iz} \right| \geq 1 \iff e^{-y} \geq 1$

$$\iff \begin{cases} y \leq 0 \\ x \text{ qualsiesi} \end{cases}$$

SOLUZIONE SISTEMA: $z_4 = 1 \qquad z_2 = -3 - i4 \qquad z_3 = 5 - i4$

2 $A_t = \begin{pmatrix} 3 & t & 1 \\ 4 & 1 & 3 \\ t & 0 & 2 \end{pmatrix}$ matrice 3×3

i) $\det(A_t) = 3t^2 - 9t + 6$

$\det(A_t) = 0 \iff t = 1, 2$

$\Rightarrow$ Per $t \neq 1, 2$ $\det(A_t) \neq 0$

$$\begin{cases} rg = 3 \\ \dim(\ker) = 0 \end{cases}$$

$t = 1, 2$: $M = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ $\det \neq 0$

$$\Rightarrow \begin{cases} rg = 2 \\ \dim(\ker) = 3 - 2 = 1 \end{cases}$$

ii) $A_t \cdot X = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix}$

$t \neq 1, 2$ $rg(A_t) = 3$

$(A_t | b)$ matrice 3×4 $\Rightarrow$ $rg \leq 3$

Quindi per $t \neq 1, 2$

$$rg(A_t) = 3 \leq rg(A_t | b) \leq 3$$

CIOE' $\quad rg(A_t) = 3 = rg(A_t | b)$

$\exists !$ SOLUZIONE

$t = 1$: $\quad (A_{t=1} | b) = \left(\begin{array}{ccc|c} 3 & 1 & 1 & -2 \\ 4 & 1 & 3 & -2 \\ 1 & 0 & 2 & 0 \end{array}\right)$

$$b = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -2 \cdot (\text{colonne } 2)$$

$\Rightarrow \quad rg(A_t | b) = 2 = rg(A)$

$\Rightarrow \quad \exists \; \infty$ SOLUZIONI

Le SOLUZIONI costituiscono spazio affine di $\dim = 3 - 2 = 1$

$t = 2$: $\quad (A_{t=2} | b) = \left(\begin{array}{ccc|c} 3 & 2 & 1 & -2 \\ 4 & 1 & 3 & -2 \\ 2 & 0 & 2 & 0 \end{array}\right)$

$$\det \begin{pmatrix} 3 & 2 & -2 \\ 4 & 1 & -2 \\ 2 & 0 & 0 \end{pmatrix} = -4 \neq 0 \quad \Rightarrow \quad rg(A_t | b) = 3$$

Quindi: $\quad rg(A_t | b) = 3 > 2 = rg(A) \Rightarrow$ non $\exists$ SOL.

iii $t = 2$

$$rg(A_t) = \dim \operatorname{Im}(f_{A_t}) = 2$$

BASE $\operatorname{Im}(f_{A_t}) = \left\{ \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

eq. intrinseche $= 3-2=1$ equazioni

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$

$v_1 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \in \operatorname{Im} \to a_1 \cdot 3 + a_2 \cdot 4 + a_3 \cdot 2 = 0$

$v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{Im} \to a_1 \cdot 2 + a_2 \cdot 1 + a_3 \cdot 0 = 0$

$$\begin{cases} 3a_1 + 4a_2 + 2a_3 = 0 \\ 2a_1 + a_2 = 0 \end{cases}$$

SOL: $a_1 = t \qquad a_2 = -2t \qquad a_3 = \frac{5}{2} t$

$t=2 \to$ eq: $\quad 2x_1 - 4x_2 + 5x_3 = 0$

3 $\quad f: \mathbb{R}^3 \to \mathbb{R}^3$ lineare

$$f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}, \quad f\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix}, \quad f\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\6\\2\end{pmatrix}$$

$f\begin{pmatrix}1\\0\\0\end{pmatrix} =$ colonne 1 di A

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix} \Rightarrow f\begin{pmatrix}1\\0\\0\end{pmatrix} = f\begin{pmatrix}1\\1\\1\end{pmatrix} - f\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}-1\\6\\2\end{pmatrix} - \begin{pmatrix}1\\3\\1\end{pmatrix} = \begin{pmatrix}-2\\3\\1\end{pmatrix}$$

$f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix} =$ colonne 2 di A

$f\begin{pmatrix}0\\0\\1\end{pmatrix} =$ colonne 3 di A

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} \Rightarrow f\begin{pmatrix}0\\0\\1\end{pmatrix} = f\begin{pmatrix}0\\1\\1\end{pmatrix} - f\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\3\\1\end{pmatrix} - \begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

4

$$A = \begin{pmatrix} -2 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 3 & 1 \end{pmatrix}$$

$$p_A(\lambda) = \det(A - \lambda Id) = \ldots = \lambda^2 \cdot (\lambda+1)^2$$

AUTOVALORI: $0 \quad m.a. = 2$

$-1 \quad m.a. = 2$

$$m.g.(0) = \dim(\mathrm{Ker}(A)) = 4 - rg(A) = 1$$

$$rg(A) = 3 \quad \text{poiché} \quad \det \begin{pmatrix} -2 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \neq 0$$

$$m.g.(-1) = \dim(\mathrm{Ker}(A + 1 \cdot Id)) = 1$$

$$rg(A + 1 \cdot Id) = 3 \qquad A + 1 Id = \begin{pmatrix} -1 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 2 \end{pmatrix} \qquad \det \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \neq 0$$

$V_0 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ autospazio relativo a $\lambda_0 = 0$

$V_{-1} = \left\langle \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ autospazio relativo a $\lambda_1 = -1$

A è triang.le perché le radici di $p_A(\lambda) \in \mathbb{R}$

A non è diag.le perché $m.a.(0) = 2 \neq 1 = m.g.(0)$