Oss: $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$ invertibili.

* $F(x) = Ax\cdot x$ (f. quadr associata ad A)
* $G(x) = C^TACx\cdot x$ (f. quadr associata a C^TAC)

- $\boxed{x = Cx' \Rightarrow F(x) = G(x')}$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "Ax\cdot x"] & \mathbb{R} \\ \mathbb{R}^n \arrow[u, "C"] \arrow[ru, "C^TACx'\cdot x'"'] & \end{tikzcd}$$

Q.di: $\boxed{\text{classificare } A \text{ è } \underline{\text{equivalente}} \text{ a classificare } C^TAC}$

Pb: $\exists\, C$ invertibile $\left\langle \begin{array}{l} \text{facile da calcolare} \\ C^TAC \text{ facile da classificare} \end{array}\right.$?

Risp: sì! Gli esempi seguenti descrivono una procedura FACILE che det C invert t.c. C^TAC è diagonale.

Es: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$;

$$\begin{array}{c|ccc} A & 1 & 0 & 1 \\ & 0 & 0 & 1 \\ & 1 & 1 & 1 \\ \hline I & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}
\quad \underset{(\mathrm{I})}{\boxed{\begin{array}{c} R1 \\ R2 \\ R3-R1 \\ \hline R1 \\ R2 \\ R3 \end{array}}}
\quad \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}
\underset{(\mathrm{II})}{\Big|}
\begin{array}{ccc} C1 & C2 & C3-C1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}
\quad \underset{(\mathrm{III})}{\boxed{\begin{array}{c} R1 \\ R2+R3 \\ R3 \\ \hline R1 \\ R2 \\ R3 \end{array}}}
\quad \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}
\underset{(\mathrm{IV})}{\Big|}
\begin{array}{ccc} C1 & C2+C3 & C3 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array}
\quad \underset{(\mathrm{V})}{\boxed{\begin{array}{c} R1 \\ R2 \\ 2R3-R2 \\ \hline R1 \\ R2 \\ R3 \end{array}}} \rightarrow$$

$$\rightarrow \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array}
\underset{(\mathrm{VI})}{\Big|}
\begin{array}{ccc} C1 & C2 & 2C3-C2 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \\ \hline 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}
\Bigg| \quad \begin{array}{c} \mathrm{diag}(1,2,-2) \\ \\ C \end{array}
\qquad \boxed{\mathrm{diag}(1,2,-2) = C^TAC}$$

(I) $H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ $\Big|$ $\dfrac{H_1A}{I}$

(II) $\dfrac{H_1AH_1^T}{H_1^T}$

(III) $H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\Big|$ $\dfrac{H_2H_1AH_1^T}{H_1^T}$

(IV) $\dfrac{H_2H_1AH_1^TH_2^T}{H_1^TH_2^T}$

(V) $H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$ $\Big|$ $\dfrac{H_3H_2H_1AH_1^TH_2^T}{H_1^TH_2^T}$

(VI) $\left.\begin{array}{l} \dfrac{H_3H_2H_1AH_1^TH_2^TH_3^T}{H_1^TH_2^TH_3^T} \begin{array}{l} = \mathrm{diag}(1,2,-2) \\ = C \end{array} \end{array}\right\}$ $\mathrm{diag}(1,2,-2) = C^TAC$.

$\boxed{\text{Oss: le } H_k \text{ sono triangolari con elem } \neq 0 \text{ nella diagonale, ovvero } \underline{\text{invertibili}} \Rightarrow C \text{ invertibile!}}$

- $\boxed{\mathrm{diag}(1,2,-2) \text{ indef} \Rightarrow A \text{ indef}}$

Es: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$;

$$\begin{array}{c|ccc} A & 0 & 0 & 1 \\ & 0 & 0 & 1 \\ & 1 & 1 & 0 \\ \hline I & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}
\quad \overset{1}{\boxed{\begin{array}{c} R1+R3 \\ R2 \\ R1 \\ \hline - \\ - \\ - \end{array}}}
\quad \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}
\overset{2}{\Big|}
\begin{array}{ccc} C1+C3 & C2 & C3 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}
\quad \overset{3}{\boxed{\begin{array}{c} - \\ 2R2-R1 \\ 2R3-R1 \\ \hline - \\ - \\ - \end{array}}}
\quad \begin{array}{ccc} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}
\overset{4}{\Big|}
\begin{array}{ccc} - & 2C2-C1 & 2C3-C1 \\ 2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \\ \hline 1 & -1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{array}
\quad \overset{5}{\boxed{\begin{array}{c} - \\ - \\ R3+R2 \\ \hline - \\ - \\ - \end{array}}} \rightarrow$$

$$\rightarrow \begin{array}{ccc} 2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{array}
\overset{6}{\Big|}
\begin{array}{ccc} - & - & C3+C2 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ \hline 1 & -1 & -2 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{array}
\Bigg| \quad \begin{array}{c} \mathrm{diag}(2,-2,0) \\ \\ C \end{array}
\qquad \boxed{\mathrm{diag}(2,-2,0) = C^TAC}$$

(1) $H_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\Big|$ $\dfrac{H_1A}{I}$

(2) $\dfrac{H_1AH_1^T}{H_1^T}$

(3) $H_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ $\Big|$ $\dfrac{H_2H_1AH_1^T}{H_1^T}$

(4) $\dfrac{H_2H_1AH_1^TH_2^T}{H_1^TH_2^T}$

(5) $H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\Big|$ $\dfrac{H_3H_2H_1AH_1^TH_2^T}{H_1^TH_2^T}$

(6) $\left.\begin{array}{l} \dfrac{H_3H_2H_1AH_1^TH_2^TH_3^T}{H_1^TH_2^TH_3^T} \begin{array}{l} = \mathrm{diag}(2,-2,0) \\ = C \end{array} \end{array}\right\}$ ...

- $\boxed{\mathrm{diag}(2,-2,0) \text{ indef} \Rightarrow A \text{ indef}}$

Es: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$;

A	$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} - \\ II + III \\ - \end{matrix}$	$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} - \\ - \\ 2III - II \end{matrix}$	$\begin{matrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{matrix}$	diag $(0,2,-2)$
I	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} - \\ - \\ - \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} - \\ - \\ - \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{matrix}$	C

diag $(0,2,-2) = C^T A C$ $\Rightarrow$ A indefinita

Oss: la prima colonna di A è già "diagonale", perciò si inizia ad operare per rendere "diagonale" la seconda...

def (matrici congruenti): $A, B \in \mathbb{R}^{n \times n}$ simmetriche;
se $\exists\, C \in \mathbb{R}^{n \times n}$ invertibile t.c. $B = C^T A C$,
A e B si dicono <u>CONGRUENTI</u>

Oss: la congruenza è una relaz di equivalenza tra le matrici simmetriche di ordine n (ad elem in $\mathbb{R}$).

def (segnatura): $A \in \mathbb{R}^{n \times n}$ simmetrica; $\lambda_1, \dots, \lambda_n$ $(\in \mathbb{R})$ autovalori di A.

Posto: $N = \# \lambda_k < 0$, $Z = \# \lambda_k = 0$, $P = \# \lambda_k > 0$
la terna (N, Z, P) si dice <u>INERZIA</u> di A.

Es: Sia $A \in \mathbb{R}^{n \times n}$ simmetrica con autovalori $-3, 1, 0, 1, 0, -2$.
l'inerzia di A è $(2, 2, 2)$.

TEO (Sylvester): L'inerzia è invariante per congruenza.
(<u>dim</u>: no)

Oss: la procedura presentata negli Es precedenti determina una sequenza di matrici <u>congruenti</u> l'ultima delle quali è diagonale.

Es: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$; A congruente a diag $(2,-2,0)$

- inerzia di A = inerzia di diag $(2,-2,0) = (1,1,1)$
- autovalori di A: radici di $p_A(x) = -x^3 + 2x$

ovvero: $0, \sqrt{2}, -\sqrt{2}$.

Oss: Per il Teo di Sylvester, <u>tutte</u> le matrici diagonali congruenti ad una stessa $A \in \mathbb{R}^{n \times n}$ simmetrica hanno lo stesso numero di elem negativi, nulli e positivi sulla diagonale.