$\boxed{\text{Pb: data } A \in \mathbb{C}^{m\times k}\text{, determ una decompos ai val sing}}$

Oss: $A\in\mathbb{C}^{m\times k}$, (U,Σ,V) decomp ai v.s (q.di $A = U\Sigma V^H$)

- $A^HA = V(\Sigma^H\Sigma)V^H$
  - * $\Sigma^H\Sigma \in \mathbb{C}^{k\times k}$ è diag ad elem in $\mathbb{R}$;
  - * V è unitaria $\Rightarrow V^H = V^{-1}$

Ovvero: è una diagonalizz di A^HA con matrice unitaria.

Oss: A^HA è hermitiana $\Rightarrow \sigma(A^HA)\subset\mathbb{R}$ ed è diagonalizz con matr unitaria

- $AV = U\Sigma$: $Av_1 = \sigma_1 u_1, \dots, Av_p = \sigma_p u_p$ $\quad (p = \min(m,k))$

Procedura per determ una decompos ai valori sing:

$(U,\Sigma,V) = SVD(A)$

dati: $A\in\mathbb{C}^{m\times k}$

- determ $V\in\mathbb{C}^{k\times k}$ unitaria, $\lambda_1,\dots,\lambda_k\in\mathbb{R}$ tali che $\begin{cases} A^HA = V\,\mathrm{diag}(\lambda_1,\dots,\lambda_k)\,V^H \\ \lambda_1\ge\dots\ge\lambda_k\ge 0\end{cases}$
- $\sigma_1=\sqrt{\lambda_1},\dots,\sigma_p=\sqrt{\lambda_p}$
- per k t.c. $\sigma_k\neq 0$: $u_k = \frac{1}{\sigma_k}Av_k$
- completare fino ad ottenere $u_1,\dots,u_m$ base o.n di $\mathbb{C}^m$

Es:

(I) $A = \begin{bmatrix} i & 1\\ 0&0\\ 0&0\end{bmatrix}\in\mathbb{C}^{3\times 2}$ $\quad [U\in\mathbb{C}^{3\times 3},\ \Sigma\in\mathbb{C}^{3\times 2},\ V\in\mathbb{C}^{2\times 2}]$

- $A^HA = \begin{bmatrix}1&-i\\ i&1\end{bmatrix}$ (Es: verif che è hermitiana!)

* polinomio caratt: $\det(A^HA - xI) = (1-x)^2 - 1 = (1-x+1)(1-x-1)$ $\quad \lambda_1 = 2,\ \lambda_2 = 0$

* $\begin{cases}\ker(A^HA-2I) = \left\langle \begin{bmatrix}-i\\1\end{bmatrix}\right\rangle = \left\langle \frac{1}{\sqrt2}\begin{bmatrix}-i\\1\end{bmatrix}\right\rangle \\ \ker(A^HA) = \left\langle \begin{bmatrix}i\\1\end{bmatrix}\right\rangle = \left\langle \frac{1}{\sqrt2}\begin{bmatrix}i\\1\end{bmatrix}\right\rangle\end{cases}$

* $A^HA = \begin{bmatrix}-i/\sqrt2 & i/\sqrt2\\ 1/\sqrt2 & 1/\sqrt2\end{bmatrix}\begin{bmatrix}2 & \\ & 0\end{bmatrix}\begin{bmatrix}\ast\end{bmatrix}^H$ $\qquad V = \frac{1}{\sqrt2}\begin{bmatrix}-i & i\\ 1 & 1\end{bmatrix}$

Oss: $\nexists\, V$ ad elem reali che diagonalizza A^HA.

- $p = 2$; $\sigma_1=\sqrt2$, $\sigma_2 = 0$ $\qquad \Sigma = \begin{bmatrix}\sqrt2 & 0\\ 0&0\\ 0&0\end{bmatrix}$

Oss: $\lambda_1,\dots,\lambda_p$ sono i primi p autovalori di A^HA q.di sono univocam determinati.

- $u_1 = \frac{1}{\sqrt2}\begin{bmatrix} i & 1\\ 0&0\\ 0&0\end{bmatrix}\frac{1}{\sqrt2}\begin{pmatrix}-i\\1\end{pmatrix} = \frac12\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$
- $u_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $u_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

q.di: $A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\sqrt2&0\\0&0\\0&0\end{bmatrix}\frac{1}{\sqrt2}\begin{bmatrix}i & 1\\ -i & 1\end{bmatrix}$

(II) $B = \begin{bmatrix}1&0&1\\1&0&-1\end{bmatrix}\in\mathbb{C}^{2\times 3}$ $\quad [U\in\mathbb{C}^{2\times 2},\ \Sigma\in\mathbb{C}^{2\times 3},\ V\in\mathbb{C}^{3\times 3}]$

- $B^HB = \begin{bmatrix}2&0&0\\0&0&0\\0&0&2\end{bmatrix}$ $\quad$ * p. caratt: $(2-x)^2(-x)$ $\quad \lambda_1=\lambda_2=2,\ \lambda_3 = 0$

* $\begin{cases}\ker(B^HB-2I) = \left\langle \begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\rangle \\ \ker(B^HB) = \left\langle \begin{bmatrix}0\\1\\0\end{bmatrix}\right\rangle\end{cases}$

* $B^HB = \boxed{\begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix}}\boxed{\begin{matrix}2&&\\&2&\\&&0\end{matrix}}\boxed{\#^H}$ $\qquad V = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$

<u>Oss</u>: V è ad elem reali (perché lo è B!); un'altra V che va bene è $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$.

- $p=2$; $\sigma_1=\sqrt{2}$, $\sigma_2=\sqrt{2}$ $\qquad \Sigma = \begin{pmatrix}\sqrt{2}&0&0\\0&\sqrt{2}&0\end{pmatrix}$
- $u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1\\1&0&-1\end{pmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\quad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1\\1&0&-1\end{pmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

Q.di: $A = \frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}\sqrt{2}&0&0\\0&\sqrt{2}&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$

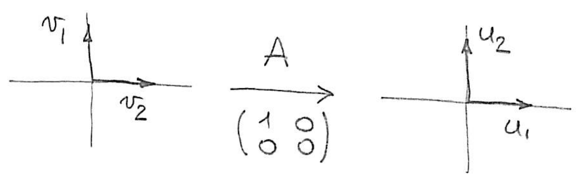
<u>Es</u>: $A = \begin{bmatrix}0&1\\0&0\end{bmatrix} \in \mathbb{C}^{2\times 2}$ (non diagonalizz!)

- $A^HA = \begin{bmatrix}0&0\\0&1\end{bmatrix}$ $\quad$ * p.c.: $(1-x)(-x) < \begin{matrix}\lambda_1=1\\\lambda_2=0\end{matrix}$

  * $\ker(A^HA - I) = \langle\binom{0}{1}\rangle$, $\ker(A^HA) = \langle\binom{1}{0}\rangle$

  * $A^HA = \boxed{\begin{matrix}0&1\\1&0\end{matrix}}\boxed{\begin{matrix}1&\\&0\end{matrix}}\boxed{\#^H}$ $\qquad V = \begin{bmatrix}0&1\\1&0\end{bmatrix}$
- $p=2$; $\sigma_1=1$, $\sigma_2=0$ $\qquad \Sigma = \boxed{\begin{matrix}1&\\&0\end{matrix}}$
- $u_1 = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$
- $u_2 = \begin{bmatrix}0\\1\end{bmatrix}$

Q.di: $A = \boxed{\begin{matrix}1&0\\0&1\end{matrix}}\boxed{\begin{matrix}1&\\&0\end{matrix}}\boxed{\begin{matrix}0&1\\1&0\end{matrix}}$



<u>Oss</u>: $A \in \mathbb{C}^{m\times k}$, (U, Σ, V) decompos val sing di A.

- $x \in \ker(A) \Leftrightarrow V^H x \in \ker(\Sigma)$ q.di $\boxed{\dim\ker(A) = \dim\ker(\Sigma)}$
- utilizz il teo che lega la dim del nucleo e la dim dell'immagine si ottiene: $\boxed{rk(A) = rk(\Sigma)}$

Posto γ = numero di valori singolari <u>non nulli</u>:

- $\dim\ker(A) = k-\gamma$ e $v_{\gamma+1}, \dots, v_k$ base on di $\ker(A)$
- $rk(A) = \gamma$ e $u_1, \dots, u_\gamma$ base on di $im(A)$