Es: $A = \begin{pmatrix} 2 & -1 & -2 & 3 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}^{4\times 4}$

- determ. $FCJ(A)$;
- determ C che realizza similitudine.

Sol: • $p_A(x) = (-x)(1-x)^3 \Rightarrow \sigma(A) = \{0,1\}$,

λ	m.a	m.g
0	1	1
1	3	1

$\delta_{11} = 1$, $\delta_{21} = 3$

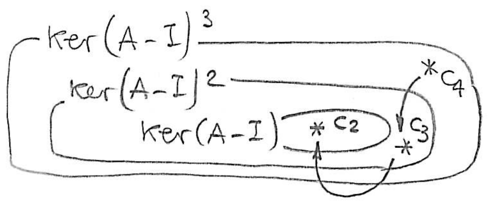
$\dim \ker (A - I) = \dim \ker \begin{pmatrix} 1 & -1 & -2 & 3 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 1$

$$\Rightarrow FCJ(A) = \begin{pmatrix} 0 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

- $C = (c_1, c_2, c_3, c_4) \in \mathbb{C}^{4\times 4}$ invertibile t.c. $AC = C\, FCJ(A)$

$Ac_1 = 0 \iff c_1 \in \ker(A)$

$Ac_2 = c_2$	$c_2 \in \ker(A-I)$	$c_2 \in \ker(A-I)$
$Ac_3 = c_2 + c_3$	$(A-I)c_3 = c_2$	$c_3 \notin \ker(A-I)$ ma $\in \ker(A-I)^2$
$Ac_4 = c_3 + c_4$	$(A-I)c_4 = c_3$	$c_4 \notin \ker(A-I)^2$ ma $\in \ker(A-I)^3$



* scelgo $c_1 \neq 0$ in $\ker(A)$

* scelgo c_4 in $\ker(A-I)^3$ ma non in $\ker(A-I)^2$

* pongo $c_3 = (A-I)c_4 \Rightarrow c_3 \in \ker(A-I)^2$ ma $\notin \ker(A-I)$

* pongo $c_2 = (A-I)c_3 \Rightarrow \ldots$

Si ha:

$\ker(A) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$; $c_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\ker(A-I) = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$; $\ker(A-I)^2 = \ker \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\rangle$

$\ker(A-I)^3 = \ker \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle$

Q.di: $c_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $c_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $c_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow C = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Es: $M = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 3 \end{pmatrix}$

- determinare $FCJ(M)$;
- determinare C che realizza la similitudine.

Sol: • $p_M(x) = \det(M - xI) = (3-x)^4 \Rightarrow \sigma(M) = \{3\}$,

λ	m.a	m.g
3	4	2

$\dim \ker (M - 3I) = \dim \ker \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 2$

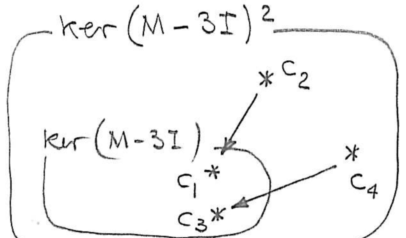
$\delta_{11} + \delta_{12} = 4$

$\dim \ker (M-3I)^2 = \dim \ker \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 4 \Rightarrow \delta_{11} = \delta_{12} = 2$

$$\Rightarrow FCJ(M) = \begin{pmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$$

- $C = (c_1, c_2, c_3, c_4) \in \mathbb{C}^{4\times 4}$ invertibile t.c. $MC = C\, FCJ(M)$

$Mc_1 = 3c_1$	$c_1 \in \ker(M-3I)$	$c_1 \in \ker(M-3I)$
$Mc_2 = c_1 + 3c_2$	$(M-3I)c_2 = c_1$	$c_2 \in \ker(M-3I)^2$, $\notin \ker(M-3I)$
$Mc_3 = 3c_3$	$c_3 \in \ker(M-3I)$	$c_3 \in \ker(M-3I)$
$Mc_4 = c_3 + 3c_4$	$(M-3I)c_4 = c_3$	$c_4 \in \ker(M-3I)^2$, $\notin \ker(M-3I)$



$\ker(M-3I) = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

$\ker(M-3I)^2 = \left\langle \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{generatori di } \ker(M-3I)!}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$

* SCELGO: $c_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $c_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (sono i due "nuovi generatori"...)

* PONGO: $c_1 = (M-3I)c_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $c_3 = (M-3I)c_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; si constata che sono lin indip.

Oss: Scelta di c_2 e c_4 diverse dai "nuovi generatori" NON garantiscono che $(M-3I)c_2$, $(M-3I)c_4$ risultino lin indip!

$$\Rightarrow \quad \boxed{C = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.}$$

Es: $c_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow c_1' = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ ... non sono lin indip!

$c_4' = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow c_3' = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

Es: Siano $A = \left[\begin{array}{c|ccc} 1 & & & \\ \hline & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{array}\right]$, $B = \left[\begin{array}{ccc|c} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ \hline & & & 1 \end{array}\right]$, $P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

- verificare che $P^T P = I$ (ovvero che $P^T = P^{-1}$);
- verificare che $AP = PB$ (ovvero che A e B sono simili);
- posto $G = (g_1, g_2, g_3, g_4) \in \mathbb{C}^{4\times 4}$, calcolare GP in termini di $g_1, \dots, g_4$ e descrivere a parole come si ottiene GP da G (Oss: P è un esempio di matrice di permutazione).

Oss: "funziona sempre così"... ad Es: la matrice $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ realizza la similitudine di...

$$\left[\begin{array}{cc|cc} 2 & 1 & & \\ & 2 & & \\ \hline & & 0 & 1 \\ & & & 0 \end{array}\right] \quad e \quad \left[\begin{array}{cc|cc} 0 & 1 & & \\ & 0 & & \\ \hline & & 2 & 1 \\ & & & 2 \end{array}\right]$$