Interpolazione

$k \in \mathbb{N}$

$x_0, \dots, x_k$ reali distinti

$y_0, \dots, y_k$ reali

dati $(x_0, y_0), \dots, (x_k, y_k)$

det $p \in \mathbb{P}_k(\mathbb{R})$ t.c.

$$\underbrace{p(x_0) = y_0, \dots, p(x_k) = y_k}_{\text{condiz di INTERPOLAZIONE}}$$

Teo (esiste unicità)

$\forall k \in \mathbb{N}$, $x_0, \dots, x_k$ reali distinti

$y_0, \dots, y_k$ reali,

$\exists!\, p \in \mathbb{P}_k(\mathbb{R})$

che verifica le condiz di interpolazione.

Base di LAGRANGE

$$\begin{cases} l_0(x) = \dfrac{(x-x_1)\cdots(x-x_k)}{(x_0-x_1)\cdots(x_0-x_k)} \\ \vdots \\ l_k(x) = \dfrac{(x-x_0)\cdots(x-x_{k-1})}{(x_k-x_0)\cdots(x_k-x_{k-1})} \end{cases}$$

$$l_j(x_i) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$p(x) = y_0 l_0(x) + \cdots + y_k l_k(x)$$

"FORMA di LAGRANGE" del polinomio interpolante

Es: $(-1,0), (0,1), (2,-2) \leftarrow$ dati

det $p \in P_2(\mathbb{R})$ che interp i dati.

Sol: $l_0(x) = \dfrac{x(x-2)}{(-1)(-3)} = \dfrac{x^2-2x}{3}$

$$(-1)\,(-3) \qquad \underline{3}$$

$$l_1(x) = \frac{(x+1)(x-2)}{-2} = \frac{x^2 - x - 2}{-2}$$

$$l_2(x) = \frac{(x+1)\,x}{6} = \frac{x^2 + x}{6}$$

Il polin interpolante è

$$\boxed{\begin{aligned} p(x) = {} & 0 \cdot l_0(x) + 1 \cdot l_1(x) \\ & - 2 \cdot l_2(x) \end{aligned}}$$

- $P_2(\mathbb{R}) = \operatorname{span}(1, x, x^2)$

$$p(x) = a_0 + a_1 x + a_2 x^2$$

$$p(-1) = a_0 + a_1(-1) + a_2(-1)^2 = 0$$

$$p(0) = a_0 + a_1(0) + a_2(0)^2 = 1$$

$$p(2) = a_0 + a_1(2) + a_2(2)^2 = -2$$

$$\begin{array}{c c} -1 & \begin{bmatrix} 1 & -1 & (-1)^2 \\ \end{bmatrix} \begin{bmatrix} a_0 \\ \end{bmatrix} \quad \begin{bmatrix} 0 \\ \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} & \\ 0 & \\ 2 & \\ \end{array}\begin{bmatrix} 1 & 0 & (0)^2 \\ 1 & 2 & (2)^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{array}{ccc} 1 & x & x^2 \end{array}$$

$$V \qquad \alpha \qquad b$$

"matrice di VANDERMONDE"

$1, x, x^2$ base di Vandermonde

$$V = \overset{c_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \overset{r_1}{\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}} + V_2$$

$$V_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & 3 \end{bmatrix} = \underset{c_2}{\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}} \overset{r_2}{\begin{pmatrix} 0 & 1 & -1 \end{pmatrix}} + V_3$$

$$V_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \underset{c_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \underset{r_3}{\begin{pmatrix} 0 & 0 & 6 \end{pmatrix}}$$

$$V = \underset{S}{\begin{bmatrix}1&0&0\\1&1&0\\1&3&1\end{bmatrix}}\;\underset{D}{\begin{array}{c}\begin{array}{ccc}&&c_3\end{array}\\ \begin{bmatrix}1&-1&1\\0&1&-1\\0&0&6\end{bmatrix}\end{array}}$$

$$c = SA(S, b) \qquad \begin{bmatrix}1&0&0\\1&1&0\\1&3&1\end{bmatrix} x = \begin{bmatrix}0\\1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\-2-3\end{bmatrix} = \begin{bmatrix}0\\1\\-5\end{bmatrix}$$

$$\alpha = SI(D, c) \qquad \begin{bmatrix}1&-1&1\\0&1&-1\\0&0&6\end{bmatrix}\alpha = \begin{bmatrix}0\\1\\-5\end{bmatrix}$$

$$= \begin{bmatrix}1/6 + 5/6\\1 - 5/6\\-5/6\end{bmatrix} = \begin{bmatrix}1\\1/6\\-5/6\end{bmatrix}$$

$$\boxed{p(x) = 1\cdot 1 + \frac{1}{6}\cdot x - \frac{5}{6}\cdot x^2}$$

"FORMA di VANDERMONDE"

del polin interpolante.

(verif che l'oggetto descritto nelle due forme è lo stesso)

Oss: · f di LAGRANGE
$(\ell_0(x), \dots, \ell_k(x))$ I

· f di VANDERMONDE
$(1, \dots, x^k)$ V

· forma di NEWTON

Base di Newton di $P_k(\mathbb{R})$

$1,\ x-x_0,\ (x-x_0)(x-x_1),$

$(x-x_0)(x-x_1)(x-x_2), \dots$

$\dots, (x-x_0)(x-x_1)\dots(x-x_{k-1})$

k=2

$$p(x) = b_0 \cdot 1 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$$

· $p(x_0) = b_0 \cdot 1 + b_1(x_0-x_0) + b_2(x_0-x_0)(x_0-x_1)$

$$\bullet\ p(x_0) = b_0 \cdot 1 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1) = y_0$$

$$\bullet\ p(x_1) = b_0 \cdot 1 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1) = y_1$$

$$\bullet\ p(x_2) = b_0 \cdot 1 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) = y_2$$

$$\begin{array}{cc} x_0 & \\ x_1 & \\ x_2 & \\ & \end{array}
\begin{array}{c}
\begin{bmatrix} 1 & 0 & 0 \\ 1 & x_1 - x_0 & 0 \\ 1 & x_2 - x_0 & (x_2 - x_0)(x_2 - x_1) \end{bmatrix} \\
\begin{matrix} 1 & x - x_0 & (x - x_0)(x - x_1) \end{matrix}
\end{array}
\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

<u>Es</u> (continua)

base di Newton di $P_2(\mathbb{R})$

$$1,\ x+1,\ (x+1)x$$

$$\begin{array}{c} -1 \\ 0 \end{array}
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}
\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{cc} 0 \\ 2 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$1 \quad x+1 \quad (x+1)x$$

$$b_0 = 0$$

$$b_1 = 1$$

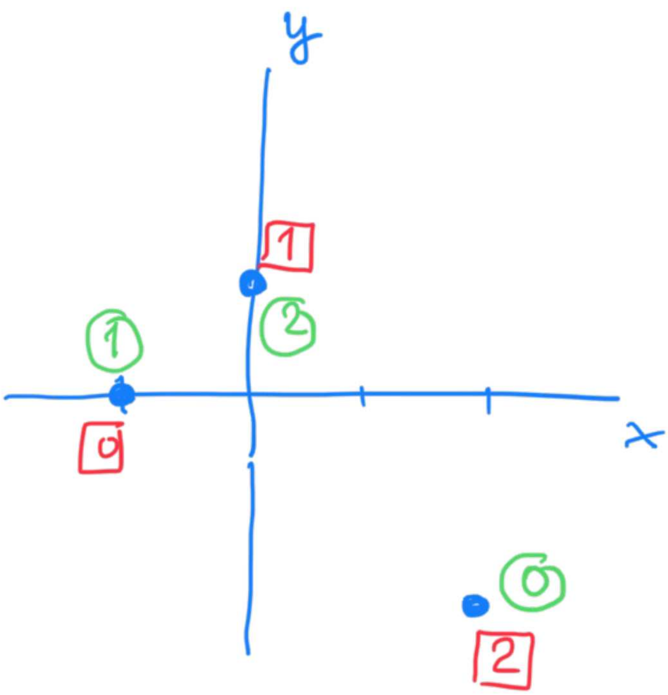
$$b_2 = \frac{1}{6}(-2-3) = -5/6$$

$$\boxed{p(x) = 0 \cdot 1 + 1 \cdot (x+1) - \frac{5}{6}(x+1)x}$$

f. di NEWTON del polin interpolante.

Oss:

$P_2(\mathbb{R})$



$(2,-2), (-1,0), (0,1)$

N.B. le basi di LAGRANGE e NEWTON dipendono dalle ascisse dei dati.

L'<u>oggetto</u> che trovo è <u>lo stesso</u> di prima: l'<u>ordine</u> dei dati non influisce sull'oggetto <u>ma</u> solo sulla <u>forma</u> di scriverlo!

Problema LINEARE di interpolazione.

$k \in \mathbb{N}$;

$\mathcal{F}$ ssv delle f continue su $[a,b]$; dim $\mathcal{F} = j < +\infty$;

$L_0, \dots, L_k : \mathcal{F} \to \mathbb{R}$, lineari

$y_0, \dots, y_k \in \mathbb{R}$

<u>det</u> $f \in \mathcal{F}$ t.c.

$$L_0(f) = y_0, \dots, L_k(f) = y_k$$

<u>Es1</u>: • dato $\bar{x} \in [a,b]$,

$L : \mathcal{F} \to \mathbb{R}$ t.c. $L(f) = f(\bar{x})$

<u>è lineare</u>:

$$\forall \alpha \in \mathbb{R}, \forall f \in \mathcal{F}: \; L(\alpha f) = \alpha L(f)$$

$$\forall f, g \in \mathcal{F}: \; L(f+g) = L(f) + L(g)$$

• $\alpha \in \mathbb{R}, f \in \mathcal{F}$

$$L(\alpha f) \equiv (\alpha f)(\bar{x}) = \alpha f(\bar{x})$$

$$= \alpha L(f)$$

• $f, g \in \mathcal{F}$

$$L(f+g) \equiv (f+g)(\bar{x}) =$$

$$= f(\bar{x}) + g(\bar{x}) =$$

$$= L(f) + L(g)$$

- dato $\bar{x} \in [a,b]$

se $\forall f \in \mathcal{F}$, f è DERIVABILE:

$$L : \mathcal{F} \to \mathbb{R} \text{ t.c. } L(f) = f'(\bar{x})$$

- $\alpha \in \mathbb{R}$, $f \in \mathcal{F}$,

$$L(\alpha f) \equiv (\alpha f)'(\bar{x}) =$$

$$= \alpha f'(\bar{x}) = \alpha L(f)$$

- $\forall f, g \in \mathcal{F}$;

$$L(f+g) \equiv (f+g)'(\bar{x}) =$$

$$= f'(\bar{x}) + g'(\bar{x})$$

$$= L(f) + L(g)$$

è lineare!

- $[c,d] \subset [a,b]$

$\forall f \in \mathcal{F}$, f è <u>integrabile</u> su $[c,d]$;

$$L : \mathcal{F} \to \mathbb{R} \text{ t.c. } L(f) = \int_c^d f(t)\,dt$$

[<u>Es</u>: verif che L è lineare.]

<u>Oss</u>: $\mathcal{F} = \text{span}\left(q_1(x), \dots, q_j(x)\right)$

con $q_1, \dots, q_j$ lin indip.

- $f \in \mathcal{F}$

$$f(x) = a_1\, q_1(x) + \dots + a_j\, q_j(x)$$

- $L_0(f) =$

$$= L_0\left(a_1 q_1(x) + \dots + a_j q_j(x)\right)$$

$$= a_1 \underline{L_0(q_1(x))} + \dots + a_j \underline{L_0(q_j(x))}$$

$$\underbrace{\qquad\qquad}_{\in \mathbb{R}} \qquad\qquad \underbrace{\qquad\qquad}_{\in \mathbb{R}}$$

$$= y_0$$

$\vdots$

- $$L_k(f) = a_1 \overbrace{L_k(q_1(x))}^{\in \mathbb{R}} + \dots + a_j \overbrace{L_k(q_j(x))}^{\in \mathbb{R}}$$

$$= y_k$$

$$\begin{array}{c c}
\begin{array}{c} L_0 \\ \vdots \\ L_k \end{array} &
\begin{bmatrix} L_0(q_1) & L_0(q_2) & \cdots & L_0(q_j) \\ \vdots & \vdots & & \vdots \\ L_k(q_1) & L_k(q_2) & \cdots & L_k(q_j) \end{bmatrix}
\begin{bmatrix} a_1 \\ \vdots \\ a_j \end{bmatrix}
=
\begin{bmatrix} y_0 \\ \vdots \\ y_k \end{bmatrix} \\
& \begin{array}{cccc} q_1 & q_2 & \cdots & q_j \end{array}
\end{array}$$

$$\underset{\mathbb{R}^{(k+1)\times j}}{\underset{\cup}{M}} \qquad \underset{\mathbb{R}^{j}}{\underset{\cup}{\alpha}} = \underset{\mathbb{R}^{k+1}}{\underset{\cup}{y}}$$

soluz sist $M\alpha = y$

$$\Updownarrow$$

soluz pb lin interp

Es ① Pb int polinomiale

$k \in \mathbb{N}$

$x_0, \dots, x_k$ dist

$y_0, \dots, y_k$

det $p \in \mathbb{P}_k(\mathbb{R})$

t.c.

$$\left.\begin{array}{l} p(x_0) = y_0 \\ \vdots \\ p(x_k) = y_k \end{array}\right\}$$

$k \in \mathbb{N}$

$\mathcal{F} = \mathbb{P}_k(\mathbb{R})$

$L_0 : \mathcal{F} \to \mathbb{R} \mid L_0(p) = p(x_0)$

$\vdots$

$L_k : \mathcal{F} \to \mathbb{R} \mid L_k(p) = p(x_k)$

$$\underbrace{L_i(p) = p(x_i) = y_i \quad i = 0, \dots, k}$$

② Interpolaz di HERMITE

dati $k \in \mathbb{N}$; $x_0, \dots, x_k$ reali dist

$y_0, \dots, y_k$, $y_0', \dots, y_k'$ reali

det $p \in \mathbb{P}_{2k+1}(\mathbb{R})$ t.c.

$$\underbrace{\begin{array}{c|c} p(x_0)=y_0 & p'(x_0)=y'_0 \\ \vdots & \vdots \\ p(x_k)=y_k & p'(x_k)=y'_k \end{array}}_{2(k+1)}$$

$$\dim P_{2k+1}(\mathbb{R}) = 2k+2$$

E' pb lin di interpolazione:

$k \in \mathbb{N}, \quad \mathcal{F} = P_{2k+1}(\mathbb{R})$

$$L_0 : \mathcal{F} \to \mathbb{R} \mid L_0(p) = p(x_0)$$

$$\vdots$$

$$L_k : \mathcal{F} \to \mathbb{R} \mid L_k(p) = p(x_k)$$

$$L_{k+1} : \mathcal{F} \to \mathbb{R} \mid L_{k+1}(p) = p'(x_0)$$

$$\vdots \qquad\qquad \vdots$$

$$L_{2k+2} : \mathcal{F} \to \mathbb{R} \mid L_{2k+2}(p) = p'(x_k)$$

Oss (interpr geometrica)

<u>Teo</u> (esist e unicità):

$\forall k$, $x_0, \dots, x_k$ distinti,

$y_0, \dots, y_k$, $y'_0, \dots, y'_k$ reali

$\exists!\ p \in P_{2k+1}(\mathbb{R})$ che

risolve il Pb di interp di Hermite.

Base di $P_5(\mathbb{R})$ $\quad (k=2)$

$\underbrace{1,\ x-x_0,\ (x-x_0)(x-x_1),}$

$(x-x_0)(x-x_1)(x-x_2),\ \leftarrow \underline{\text{IV}}$

$(x-x_0)^2(x-x_1)(x-x_2),\ \leftarrow \underline{\text{V}}$

$(x-x_0)^2(x-x_1)^2(x-x_2)\ \leftarrow \underline{\text{VI}}$

$$
\begin{array}{c}
L_0 \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & x_1-x_0 & 0 & 0 & 0 & 0 \\
1 & \times & (x_2-x_0)(x_2-x_1) & 0 & 0 & 0 \\
0 & \times & \times & (x_0-x_1)(x_0-x_2) & 0 & 0 \\
0 & \times & \times & \times & (x_1-x_0)^2(x_1-x_2) & 0 \\
0 & \times & \times & \times & \times & \bullet
\end{bmatrix}
$$

$(x_2-x_0)^2(x_2-x_1)^2$

$$\left[(x-x_0)(x-x_1)(x-x_2)\right]' =$$

$$= (x-x_1)(x-x_2) + (x-x_0)(x-x_2)$$
$$+ (x-x_0)(x-x_1)$$

$$\left[(x-x_0)^2(x-x_1)(x-x_2)\right]' =$$

$$= 2(x-x_0)(x-x_1)(x-x_2) +$$
$$+ (x-x_0)^2(x-x_2) +$$
$$(x-x_0)^2(x-x_1)$$

$$\left[(x-x_0)^2(x-x_1)^2(x-x_2)\right]' =$$

$$= 2(x-x_0)(x-x_1)^2(x-x_2) +$$

$$= 2(x-x_0)(x-x_1)^2(x-x_2) +$$

$$+ 2(x-x_0)^2(x-x_1)(x-x_2) +$$

$$+ (x-x_0)^2(x-x_1)^2$$

La matrice ha sulla diag elem $\neq 0$ $\Rightarrow$ invertibile!

Dunque il sist ha 1 soluz

$\Rightarrow$ il pb di interp di Hermite ha 1 soluzione!